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PROCEEDINGS

OF

THE PHYSICAL SOCIETY

OF LONDON.

I. *Observations on the Electric Arc.*

By WALTER L. UPSON, *E.E., M.S., Princeton**.

[Plates I. & II.]

ALTHOUGH numerous investigators have studied the phenomena of the electric arc in air, between carbon terminals, and also between metal terminals, yet the use of the electric arc in the production of electric oscillations raises some new questions with regard to it.

While I was working in the Pender Electrical Laboratory of University College, London, during the present session, Prof. J. A. Fleming, F.R.S., suggested therefore to me that it would be interesting to make a further examination of the arc between metal and carbon terminals, in air and hydrogen.

I am indebted to him for the facilities for carrying out the work, and also for many suggestions during its progress. The following apparatus was first constructed with the kind assistance of Prof. W. C. Clinton, B.Sc.

DESCRIPTION OF APPARATUS.

Two upright brass tubes are connected across the top by a brass casting through which is bored a hole, to provide a means of causing cold water to circulate through a metal electrode.

* Read June 14, 1907.

From the centre of the casting, extending downwards, is a cylindrical piece of copper, which ends in a tip about $\frac{1}{2}$ inch diameter, on to which may be fitted the upper metallic terminal of the arc. The brass tubes are clamped to a slate base, and extend through it. Thus, by connecting the end of one of them to a water supply, a steady flow is obtained, and this is directed on to the very tip of the upper metallic terminal holder (see fig. 1, p. 3).

Through the centre of the slate base passes a screw of four inches' length, which is capable of being turned by means of a milled ebonite head, fixed at its lower end, and thus being made to approach or move away from the upper terminal holder. This screw likewise ends in a tip, by which the lower arc terminal may be held.

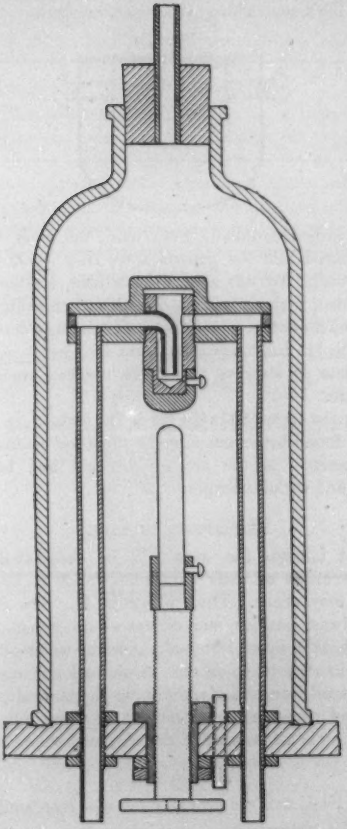
Surrounding the whole apparatus above the slate base was placed a bell jar, in the neck of which was a rubber cork holding a tube whose end was covered with wire gauze. There was also a tube let into the slate base, by which connexion could be made to a gas reservoir of any kind. The gas thus admitted was expelled through the tube fixed in the cork, and could, in the case of hydrogen or coal-gas, be made to burn at the end.

The two terminals of the arc were connected with the source of electric supply—usually the street mains at 110 volts, continuous current—the upper, through one of the tubes of the water circulation, and the lower through the central screw. Potential leads were connected to the other tube of the water circulation, and to the terminal leading to the central screw.

The electrodes used were ordinary solid carbons, 0·47, 0·4, and 0·37 inch in diameter, cored carbons 0·47 inch diameter; also metal terminals of copper, iron and aluminium, either as solid rods about 5 inches long, set into the lower holder, or pieces of the shape shown in fig. 2 (p. 4), fastened to the upper holder. The solid rods used were of the following diameters:—copper $\frac{3}{8}$ in., iron $\frac{1}{2}$ in., aluminium $\frac{1}{2}$ in. and $\frac{3}{8}$ in.

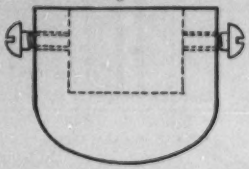
The light of the arc was passed through a double convex lens, and projected onto a sheet of squared paper, at such a distance that the arc length could easily be read off from

Fig. 1.



the image, in fractions of an inch. The arc was magnified to ten diameters. Its length could be kept constant by means of the screw holding the ower terminal.

Fig. 2.



When carbon terminals were used, the ends were previously shaped into the general form they would after use have assumed. By this means much time was saved, as it appeared that, with a given arc-length and current, terminals once shaped do not give variable volt readings according to the time the arc has burned, but may be considered in their normal state as soon as they have reached their normal temperature.

When metal terminals were used, the ends were rounded, but it was found better not to make them too pointed, owing to the tendency of the arc to bow out and become of uncertain and variable length.

Description of Arcs.

In what follows, the arcs will be designated by the chemical symbols of their electrodes, followed by the gas in which they occur. Thus, Cu–C in H. The first electrode is always positive, and, except where carbon, or otherwise stated, is the upper terminal, cooled by water circulation. Observations were taken on arcs in air and hydrogen, using carbon, copper, iron, and aluminium in all combinations, both positive and negative; also several of them in coal-gas. The following is a summary of the observed phenomena.

Arcs in Air.

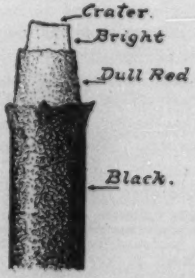
Cu–C and Fe–C arcs are probably more or less familiar.

Al–C. The characteristic aluminium flame is pale blue, being hardly distinguishable from that of iron. The Al

was oxidized rapidly, but was only slightly eaten away. Very little deposit was left on the globe and framework. A strong odour was emitted, however, but a match applied at the end of the tube did not light the gas which should have been escaping. The arc could be lengthened to 0·5 in. with 10 amperes.

C–Cu. The arc was very steady; of a purple colour, except at the point of junction with the copper, where there was a small region of green colour. A small deposit of carbon quickly formed on the copper. If this were not continually removed, the arc soon assumed the characteristics of carbon–carbon. The deposit on the apparatus was very slight. The carbon terminal burned away quite rapidly, assuming a three-stage form, as shown in fig. 3.

Fig. 3.



C–Fe. This arc continually splutters and hisses, and is very hard to measure. It is very bright; more blue (from iron) than purple (from carbon). When the iron has reached a certain temperature, bubbles form—probably magnetite—and the energy consumption increases. The arc, however, now becomes steady. It was drawn out to 0·5 in. with 5 amperes and 80 volts across the arc.

C–Al. This arc started more readily than when its poles were reversed. There appears not to be any very heavy

formation of aluminium oxide acting as an insulator on the end of the Al electrode. But once started, the arc was not very stable, and would not be drawn out to more than 0·3 in. with a current of from 10 to 12 amperes. The Al electrode was deeply pitted and very hard, containing a deposit of carbon embedded in it. The apparatus was coated with a light grey deposit of Al, but not so thickly as in the case of the Al–C arc in hydrogen. The end of the carbon was rounded, and showed indications of a crater form. A pungent odour was emitted as in the case of the same electrodes with current reversed.

Cu–Cu. A steady green arc could be drawn out to 0 2 in. with 2·5 amperes. The terminals were very little affected, being somewhat oxidized. A shallow crater was formed in the negative, and a small hard bubble of oxide on the positive.

Fe–Fe. This arc is very steady at low currents and small arc-lengths. It is blue-coloured, with a yellow aureole. As the arc-length and current are increased, yellow fumes are given off in considerable quantity, and a deposit of yellow-brown powder, no doubt iron oxide, covers the apparatus. The terminals, also, begin to boil and the arc becomes less steady. When the bubbles make their appearance the arc begins to hiss, and there appears to be an increase in the amount of power consumed in the arc. The boiling iron disturbs the arc, making it difficult to get accurate readings. There is also a hissing when the arc changes its point of attachment on the positive electrode. This is accompanied, as in the carbon-hissing arc, by a fall in voltage, but is quite distinct from the hiss which accompanies the boiling iron.

Al–Al. This arc was found to be very unstable. The terminals became quickly oxidized where the arc is formed, causing the latter to travel around continually, seeking unoxidized portions, and tending to lengthen out and run up the sides of the terminals, finally going out. The arc is a clear blue colour with a thin yellow aureole.

On each pole is a bright spot where the arc enters. A small grey deposit was formed on the framework.

Cu–Fe. The arc is bluish, with a yellow aureole. No copper-green is visible. The iron bubbles form a conducting oxide (magnetite), which, when cool, may be knocked off the end of the terminal. Also a bubble, or deposit of iron is left on the copper if the arc has been burned for a little time. There is also some brown iron-oxide deposit on the electrodes.

Fe–Cu. The copper was the upper cooled electrode. The arc showed a greenish colour, getting stronger at the copper electrode. Toward the iron it was blue. On knocking off the cap of oxide which covered the end of the iron, a heavy deposit of pure copper was found buried in the iron underneath. Quite a deep crater was burnt into the copper, while in the reverse experiment practically no effect was to be observed on it. The whole copper terminal was covered with a brown sooty deposit.

Cu–Al. The aluminium was the upper cooled electrode. This arc is more persistent than with the electrodes reversed in polarity. It is easily started, but is very unsettled in character, emitting constantly a hissing and spitting sound, and travelling about, over the surface of the electrodes, tending to lengthen out. But it does not go out as easily as the arc between the same terminals when the current is reversed. The flame is principally green from the copper. Sometimes the whole crater will be green with merely a thin aureole of blue. The aluminium electrode was covered with a golden-brown deposit from the copper, and its end was hard and pitted.

Al–Cu. The arc was blue in colour from the aluminium terminal, and green from the copper terminal, about equally divided. It was very unstable, and acted much like the Al–Al arc, being impeded in starting by the aluminium oxide. The arc would continually lengthen out until it ruptured itielf.

Fe–Al. The aluminium was above and cooled. The arc is as to be expected, blue with a yellow aureole. It

shows the characteristic aluminium tendency to glide about and play around the edges of the electrodes. It is less steady than the copper arcs. The poles presented much the same appearance as those of the Fe–Cu arc. The Al electrode was embedded with iron to a considerable depth.

Al–Fe. The aluminium was eaten into very slightly. The iron appeared as in the case of the Cu–Fe arc.

Arcs in Hydrogen.

C–C. The arc was pale blue in colour, with a faint purple core. Its length was difficult to measure on account of heavy carbon deposits. The carbons became shaped so that their ends formed parallel spherical surfaces, the positive being concave. Around the edge of the negative was built up a branching deposit of carbon, extending outward about a quarter of an inch. Compare arcs of carbon and iron.

Cu–C. The arc was small and pale, consisting of a central core of reddish-purple colour, enveloped by a region of green light. It persistently travelled around, finding no point of permanent attachment, and wearing a broad crater in the positive and shaping the negative to fit the crater. The end of the carbon became very hard, with a deposit of copper, while around the outside edge was soft carbon which easily fell away. The surrounding globe was blackened by a fine carbon deposit.

Fe–C. The arc is blue in colour with a reddish-purple core. The iron boils at its point of contact with the arc, its surface becoming irregular. A black deposit covered the apparatus. A kind of crater was eaten into the iron as in other metal positives, but was irregular, owing to the boiling which had taken place. The end of the iron was found to be so hard that a file would not cut it.

Al–C. This arc was more unstable than the other metal-carbon arcs. The globe and apparatus were heavily coated with a grey deposit of aluminium. The arc length was difficult to measure owing to the deep crater formed, and the shifting about of the arc. The aluminium wore

away rapidly, but did not boil. It looked the same as in air, but no non-conducting formation acted to prevent the re-establishment of the arc. The arc was greyish-blue in colour, with a rather faint purple core.

C–Cu. This arc appeared brighter than other arcs in hydrogen. The carbon was eaten away very rapidly, but there was no apparent deposit of material in the globe. A grey carbon deposit on the copper terminal tended to make the arc one between carbon electrodes. The arc was more stable than when the current was reversed. It did not travel round on the copper surface and was fairly steady on the carbon except as it ate the latter away. There was no positive crater formed, the end of the carbon being quite irregular.

C–Fe. There was an intermittent discharge of sparks from all over the end of the carbon, and no steady arc. The discharge could not be maintained for any length of time by the currents used, but would constantly go out. A carbon deposit gradually formed on the iron, and black cobwebs became suspended from the framework and globe. (See fig. 4, Pl. I.) On continuing to send the current, it became possible to form a steady arc, but it was evident that we now had a carbon–carbon arc. On the carbon electrode there was built up a deposit of feathery branches which extended outward horizontally for $\frac{3}{8}$ inch all round. The iron was also heavily coated but not with branching deposits. The centre of the carbon was quite deeply cratered, the interior being light grey in colour and probably converted into graphite by the heat.

C–Al. This arc was extremely unstable. It would not continue at less than seven amperes. A small lump of carbon deposit appeared immediately on the Al electrode, and the carbon electrode burned away rapidly, the arc always remaining attached to the carbon deposit on the aluminium. There appears to have been no increase in the white aluminium deposit on the globe after the arc had been changed from Al–C to C–Al. This may have been because the arc was not attached directly to the aluminium surface.

Cu–Cu. With clean terminals no arc could be maintained in hydrogen with 110 volts supply and 15 amperes at contact, not even the shortest. On drawing the electrodes apart, a shower of sparks, or arcs, appeared, but immediately went out. Previously an arc was obtained, using a copper rod on which some carbon had been left from a former experiment. The arc remained until the carbon was exhausted.

Fe–Fe. No permanent arc could be maintained. The spark shower lasted only a little longer than that with copper electrodes, but no length at all appreciable could be obtained between the electrodes while the sparking lasted. The sparks were numerous and gave the same hissing sound that usually accompanies a spark discharge in a vessel. The terminals were reversed, but with the same result. The sparks were pale blue in colour. The lower iron rod presented the same appearance as in the iron–copper arc below. The upper electrode was covered with a black deposit, but at one spot was a bare globule of metal where the contact had been made.

Al–Al. This gave better results than Fe–Fe, or than Al in combination with either Cu or Fe. The arc was reddish-purple or magenta coloured, and travelled about continuously, finding no point of permanent attachment. The metal was eaten into somewhat as in other arcs of Al in H. Attempts were made to steady the arc by shaping the lower electrode. The worst effect was obtained when this was pointed, for the arc stretched out to such lengths as to continually rupture itself. The current was also far more variable, in this case, than when the ends of the electrodes were flat and parallel. The colour of the arc was found to vary with its length, being reddish for short arcs and more bluish as the arc-length increased. There was considerable dull grey deposit from this arc, and the upper electrode seemed to be considerably built up with aluminium carried over from the lower.

Cu–Fe. This was very unstable. It consisted of a shower of sparks which was continually going out. There was some black deposit on the globe.

Fe–Cu. This was very little different from the last. The iron electrode showed a surface of pure metal surrounded by the black deposit which seems characteristic of iron arcs in hydrogen. It was only slightly eaten away, but in pits of some depth. Some slight traces of copper were present on the end of the iron. The copper electrode was covered with the black iron deposit. At one point only, a small drop of pure metal like iron showed where the arc had made contact. On removing the deposit quite an amount of iron was found to have been carried over onto the copper. Thus we have, as with most of the arcs, evidence of the double transfer of material.

Cu–Al. The arc was a mixture of green and yellowish-red colours, and was very unstable. There was no particular deposit on the apparatus. The electrodes became roughened up somewhat, and the aluminium was eaten into considerably. The Al electrode was the upper cooled terminal.

Al–Cu. This arc was about as stable and bright as the last. The electrodes were burned away considerably, and the globe was much blackened. On filing the end of the copper, the metal looked like brass in streaks. The aluminium was covered with a slightly reddish-black deposit, and its end was found, on filing, to be porous to a considerable depth.

Fe–Al. This is a better arc than Al–Fe, being more in one place than a general spark discharge over the surfaces of the electrodes. It continually hissed, as is general with the metal arcs in hydrogen, but there was no accompanying voltage drop. There seemed to be a general black deposit. On the end of the iron was a metallic globule. The Al electrode was above, and cooled.

Al–Fe. A rotating mirror revealed a broad band of distinct sparks. The direct appearance was as though a number of arcs were continually jumping across the gap; in fact it resembled a medium between the ordinary metal arc and the momentary rain of sparks from Cu–Cu in H.

The sparks were bluish in colour, accompanied by heat-glow from the metal. The iron appeared little affected but the aluminium boiled considerably, and on making contact at start the metals tended to fuse together. A kind of crater formed in the aluminium. There was quite a heavy black deposit on the apparatus.

Arcs in Coal Gas.

Cu–C. The arc was maintained with much more difficulty than in air, and required larger currents. Great quantities of carbon were deposited on the copper so quickly that it was impossible to measure accurately the arc-length. The arc was small and dull in appearance.

Cu–Cu. At about 28 volts and 8 amperes with an arc-length of approximately 0·01 in., a discharge took place, which in a moment went out. It could not be made continuous even with the smallest gap between the electrodes. On continuing the discharge for ten or fifteen minutes the arc suddenly started and remained steady to a length of 0·05 in. The current dropped considerably below 8 amperes, the arc continuing. The current was raised to 11 amperes and the arc burned quietly at 0·1 in. length, while a heavy deposit formed on both poles much as in the C–C arc in H. What really happens seems to be this:—With clean copper poles no arc will exist at currents up to 15 amperes and higher, as in hydrogen. But with a continued spark discharge carbon is separated from the coal-gas and deposited upon the poles. When there is sufficient carbon, the arc starts, and we really have a carbon–carbon arc. There was some copper deposit mixed with the carbon on the electrodes, but the latter were only very slightly burnt by the arc.

In general, it may be inferred that coal-gas is unsuitable for investigation on the actions of arcs between metal electrodes, for in a moment the carbon deposit makes the arc virtually one between carbon electrodes. The arc itself resembles those in hydrogen in brilliancy.

Maximum Arc Lengths.

When the arc took the form of a spark-discharge, as with various metal–metal arcs in hydrogen, a maximum possible length of 0·05 in. was reached for all such arcs, provided both terminals were not of the same metal. This limit of 0·05 in. was with 110 volts supply and current up to 15 amperes. When the terminals are of the same metal the results vary. Copper and iron give no definite length of arc, and aluminium reaches its limit at 0·05 in. When carbon is one of the electrodes we have always a continuous arc. If carbon is negative there is a constant limit of 0·07 in. in hydrogen. But if the arcs are carbon–metal, the limit varies as follows :—

Arcs in H.	Approximate limit of length (at 110 volts supply and 15 amp.).
Carbon–Iron	0·05 inch.
Carbon–Aluminium	0·07 „
Carbon–Copper	0·13 „
Carbon–Carbon	0·07 „

In coal-gas, Cu–C gave an arc-length of 0·1 in. at 15 amperes.

In air, the arcs may always be drawn out longer than in hydrogen or coal-gas.

Cu–Cu gave 0·2 in. arc-length with only 2·5 amperes, which cannot be obtained from carbons with 110 volts supply.

The lowest limits found were—

for C–Al, 0·3 in. at 10 to 12 amperes,
and C–Cu, 0·26 to 0·34 in. „ 5 „ 6 „

The following comparisons of maximum arc-length were made, showing the effect of reversing the polarity when carbon formed one of the electrodes :—

Arc.	Max. length, inch.	Current.	Voltage across Arc.
Al–C	0·5	10	—
C–Al	0·3	10 to 12	—
Cu–C	0·6 to 0·8	7 „ 9·5	70 to 53
C–Cu	0·26 „ 0·34	5 „ 6	85 „ 75

Note.—Cu–C with a length of 0·3 in. at 6 amperes gave only 47 volts across the arc.

Volt-Ampere Characteristic Curves of Arcs 0·05 in. in Length.

The length of arc, 0·05 in., was chosen as it was the longest which could be steadily maintained for all the arcs under examination. This length is so short that accurate measurements of unstable arcs are extremely difficult, and in such cases it is only fair to consider the curves given as generally approximate. Fig. 5 (Pl. II.) gives the volt-ampere characteristics of the arcs in air in which carbon was one of the electrodes. It will be seen that when the carbon is negative, it makes very little difference what the positive is in regard to the general position of the curve, while if carbon is positive a great variation occurs according to the material of the negative. The arc in which carbon forms both positive and negative partakes of the properties of both of these groups, occupying its individual position under the influence of the carbon negative; but being little influenced by its positive, it keeps the same general shape as the lower group of curves.

Fig. 6 gives this same group of curves for arcs in hydrogen. The metal–carbon and carbon–carbon arcs so nearly coincide that one curve is given to represent them. The real positions of this group may be seen in fig. 10. We now have not a difference in curvature, for the curves are approximately parallel, but merely variations in the distance apart, according to the material of the negative. The position of the curve, therefore, seems to be due to some constant of the material—say, its ability to become ionized in the particular gas.

In air there is superposed the active effect of oxidation. We may infer that the general shapes of the curves,—or, at any rate, variations in shape—are due to the influence of the surrounding gas.

Fig. 7 shows the characteristics of arcs in air when both electrodes are of the same metal. That of aluminium differs considerably from the other two, and in general it was found that where aluminium entered, either as one or both of the electrodes, there was apt to be considerable variation, whereas iron and copper kept pretty close together.

Fig. 8 gives the combinations of the different metals in air. Fig. 9 gives them for arcs in hydrogen. It has already been mentioned that these arcs bear more resemblance to heavy spark discharges than to genuine arcs. But the action seems to follow the same general law, in both cases. Some interesting comparisons may be made from these curves and those of figs. 5 and 6.

In figs. 8 and 9 we find the highest and steepest curves to be Cu–Al, and C–Al occupies a similar position in fig. 5. In figs. 8 and 9, again, the lowest and flattest curves are for Al–Fe. The copper and iron curves bear a general resemblance to each other throughout, but with a certain peculiarity.

We have considered the relative height of the curves to be due, in part, to the effect of the surrounding gas, and in part to a specific property of the material of the negative electrode. There may be other influences, but these two are suggested by a comparison of the copper and iron curves in figs. 5, 6, 8, and 9. The curves both for Cu–Fe and C–Fe are below the curves for Fe–Cu and C–Cu when the arcs are in air. But they are above when the arcs are in hydrogen. With carbon for the negative, copper's place is seen to be above iron in both air and hydrogen.

It was desired to obtain a comparison of the slopes of the curves for C–C in air and metal–C and C–C in hydrogen, in order to show the change in voltage for a given change in current in the several cases. All experiments agreed in the result shown in fig. 10, namely, that the C–C curve in air cuts the other curves in the neighbourhood of 6·5 amperes and 47 volts, and is a much flatter curve. In fact, all the air-curves derived are flatter than those in hydrogen with the exception of that for C–Al. This arc is difficult to measure, but very careful observations led to the conclusion that the curve was, if anything, steeper than as shown in fig. 5.

Curves were also taken of the copper–copper arc in air under three conditions :—

(1) + above, and cooled by water-circulation.
(2) − above, and cooled by water-circulation.
(3) Arc between two rods of 0·375 in. diam., with + above, and no cooling.

The latter curve lies between the others, but is somewhat steeper, as shown in fig. 11.

Voltage Equations of different Arcs.

In determining the equations of different arcs, the form of procedure given by Mrs. Ayrton has been followed. The characteristics have first been sketched in according to the readings obtained. From these curves were drawn up tables of volts, amperes, and watts. The watts were then plotted against amperes. This should give straight lines if the volt-ampere curves are satisfied by equations of the form

$$V_l = a + \frac{b}{A},$$

where a and b are constants, l=length of arc, and A=current in amperes. In all cases, the approximation to a straight line was so close, that no doubt could be entertained that the deviation was due to error of observation.

From these watt-ampere curves new sets of tables were drawn up, giving volts, and from these derived volts and amperes the curves shown in the different figures have been drawn. The derived curves have in all cases been found to satisfy the original points, and may be regarded as more nearly the truth than the original curves. From this we conclude that, provided the electrodes are of solid materials, all arcs of constant length are satisfied by the equation

$$V = a + \frac{b}{A}.$$

It is difficult to obtain long enough arcs to prove the validity of a like assumption with regard to arcs of constant current but variable length, which shall cover the whole range of arcs in air and hydrogen. But no case has been found where the arc of constant current is not represented by the equation

$$V_A = a + bl.$$

Equations of the general form for variable length and current given by Mrs. Ayrton,

$$V = a + bl + \frac{c + dl}{A}$$

have been derived in a number of cases, as follows.

For carbon–carbon in air, both solid, positive above, 0·475 in. diam.; negative 0·4 in. diam.

$$V = 38 + 1{\cdot}8\,l + \frac{19 + 12{\cdot}6\,l}{A},$$

where l is given in millimetres.

For copper–carbon in air, electrodes as described,

$$V = 14{\cdot}6 + 2{\cdot}4\,l + \frac{33{\cdot}6 + 7\,l}{A}.$$

For iron–carbon in air,

$$V = 12{\cdot}8 + 2\,l + \frac{24{\cdot}3 + l}{A}.$$

For iron–iron in air, the positive cooled,

$$V = 18{\cdot}5 + 3{\cdot}97\,l + \frac{18{\cdot}5 + 3{\cdot}97\,l}{A}.$$

For copper–copper in air, the positive cooled,

$$V = 17{\cdot}4 + 3{\cdot}6\,l + \frac{18{\cdot}3 + 10{\cdot}8\,l}{A}.$$

For copper–copper in air, the negative cooled,

$$V = 22{\cdot}2 + 1{\cdot}5\,l + \frac{31{\cdot}1 + 5{\cdot}2\,l}{A}.$$

For copper–copper in air, both solid rods 0·37 in. diam., the positive above,

$$V = 18 + 17{\cdot}5\,l + \frac{52{\cdot}1 + 3{\cdot}43\,l}{A}.$$

For copper–carbon in hydrogen,

$$V = 21{\cdot}5 + 3{\cdot}46\,l + \frac{4{\cdot}3 + 86{\cdot}4\,l}{A}.$$

For aluminium–carbon in hydrogen,

$$V = 8{\cdot}45 + 7{\cdot}1\,l + \frac{84{\cdot}5 + 85\,l}{A}.$$

Alternating-Current Arcs.

The carbon–carbon arc in hydrogen at about 50 volts, and 15 amperes at a frequency of 80, was steady to a length of 0·03 in. On first making contact, the carbons being cold, the arc gave a momentary flash which was repeated at each

contact. By maintaining a bad contact for a few seconds, until the carbons began to glow, and then slowly separating them, the arc could be obtained, and was quite steady. It could not be drawn out longer than 0·03 in. with 100 volts supply.

Copper–carbon in hydrogen gave no arc at 100 volts, 15 amperes and 80 cycles, nor again at 100 volts, 20 amperes and 50 cycles. The arc apparently started at contact, but went out at the first zero of the current after separation of the electrodes.

Carbon–carbon in coal-gas gave an arc of 0·01 in. and 0·02 in., the volts and amperes being about 38 and 14 respectively, and the frequency 80. At times the arc seemed to reach 0·03 in. in length, but it did not appear to draw out as long as the same arc in hydrogen. As in hydrogen, the arc required a certain amount of previous heating of the carbons to start.

Copper–carbon in coal-gas gave at first no arc with 120 volts supply, and current of 15 amperes at 80 cycles. Sparking and heating, however, gradually produced a deposit of carbon on the copper, and when this was sufficiently thick, it was possible to maintain an arc of exceedingly short length.

Copper–carbon in air. The arc would not start until the carbon had been heated to a glow. By this time a deposit of carbon was on the copper. The arc could then be drawn out to 0·05 in., but with some difficulty, and with not less than 15 amperes. At 7 or 8 amperes the arc would hardly form at all.

Copper–copper in air would not start at all. It seems probable that if sufficient heat were concentrated at the ends of the poles, the arc could be maintained.

Interrupted Arcs.

It was desired to compare the lengths of time that various arcs could be interrupted without losing the power of re-starting themselves when the voltage was again applied to their terminals. For this purpose, apparatus was devised in accordance with a suggestion by Dr. Fleming as follows:—

A small electromagnet was fixed so that its armature

would have a free fall when the current was switched off. When the armature was held by the magnet, two springs bore lightly against its sides and, they being in the main arc circuit, the current of the arc passed through the armature. When the armature was allowed to fall, it was guided into another set of spring-contacts in parallel with the first set, thus completing the circuit, which in falling, it had opened.

The length of fall could be measured accurately, and the time was calculated from the formula $t=\frac{\sqrt{d}}{4}$, where d is the fall in feet.

It was especially desired to compare the carbon–carbon arc in air and the copper–carbon arc in hydrogen. With the currents used, the spark on opening the circuit was in all cases so small as to be of inappreciable length, and therefore its effect on the value of t could be neglected. The following table is for solid carbons in air, with arc-length of 0·05 in.

Volts.	Amperes.	Fall in feet.	Time (sec.).	Watts.
45	4·5	0·3	0·137	202·5
49	3	0·146	0·095	147
43	6	0·541	0·184	258
41	7·5	0·833	0·228	307·5
40	9	1·44	0·3	360

Plotting either amperes or watts against time gives nearly a straight line (see fig. 12, Pl. II.).

Both lines, continued downward, cut through the origin. If, instead of curving downward at their upper ends, they are more properly straight throughout, we have

$$A=at, \quad \text{and} \quad W=bt,$$

where A = amperes, W = watts, t = time, and a and b are constants.

Using cored carbons, the arc was found to restart much more easily. In fact it was difficult to make the current so small that the arc would not restart after a time lapse of

0·3 sec. Observations were taken with constant time of interruption to find the effect of variations of arc-length on the current required to restart. The carbons were cored, each 0·47 in. diam.

Volts.	Amperes.	Arc Length (in inches).	Time (in sec.).	Watts.
47	1·2	0·02	0·3	56·4
45	2·5	0·05	0·3	112·5
49	3·7	0·1	0·3	181
55	5	0·15	0·3	275
50	7	0·2	0·3	350
55	8·5	0·25	0·3	467·5

Plotting either amperes or watts against arc-length gives practically a straight line (see fig. 13).

A cored carbon was bored out to a depth of about 1·5 inches, and packed with powdered potassium sulphate. With this as positive and an ordinary cored carbon as negative, a very low-power arc was obtained. The arc was perfectly steady at 20 volts, 1·1 amperes and 0·05 in. length. It was much more difficult to make this arc restart than the preceding. The results agreed fairly well, however, when the amount of energy consumed was the same.

The following observations were taken with this arc.

Volts.	Amperes.	Arc Length (in inches).	Fall (in feet).	Time (in sec.).	Watts.
12·5	5	0·05	0·125	0·089	62·5
12	6	0·05	0·416	0·162	72
12	8	0·05	0·916	0·239	90
11	10	0·05	1·44	0·3	120
12	12	0·1	1·44	0·3	144

Fig. 14 shows the variation of amperes and watts for both fall and time. In the curve of amperes against time there appears to be a lower limit of about 4 amperes under which the arc of 0·05 in. length will not restart. This accords with experiment.

A copper–carbon (solid) arc in air would not restart with an interruption of 0·072 sec., when the readings for the arc were 22 volts, 12·2 amperes, and 0·03 in. arc-length.

A copper–carbon (cored with K_2SO_4) arc in air restarted as follows :—

Volts.	Amperes.	Arc Length (inches).	Fall (feet).	Time (seconds).
12·5	13	0·05	0·083	0·072
12·5	12·5	0·03	0·125	0·088

A carbon (cored with K_2SO_4)–copper arc in air would not restart with an interruption of 0·072 sec., when the readings were 17 volts, 13 amperes, and 0·05 in. length.

A copper–carbon (solid) arc in air could not be made to restart with the smallest obtainable interruption, 0·051 sec., when its readings were 19 volts, 12·5 amperes, and 0·02 in. length. It did restart, however, when the arc-length was reduced to about 0·01 in.

A copper–carbon (solid) arc in coal-gas could not be made to restart at 23 volts, 14 amperes, and 0·01 in. length, in the shortest obtainable time of interruption. Even after running it some time, to obtain a good deposit of carbon on the copper, it would not restart.

A copper–carbon arc in hydrogen could not be made to restart at all. Its readings were as follows :—

Volts.	Amperes.	Arc Length (inches).	Time of Interruption (seconds).
32	12	0·02	0·056
24	13	0·01	0·051
23	14	0·005 (approx.)	0·04

In conclusion, the author wishes to express his indebtedness to Dr. J. A. Fleming, F.R.S., and to Mr. W. C. Clinton, B.Sc., for their kind assistance during the progress of this work.

Discussion.

Mr. W. Duddell said the paper was a most interesting one. He asked if the author had noted the character of the different arcs which he had used. The characteristic curves for arcs in coal-gas or hydrogen were steeper than those in air; and therefore arcs in coal-gas were more suitable for the production of oscillations. He asked if the curves remained steep when the current was quickly changed. He had noticed the phenomenon of an arc between two cored carbons relighting, and said that on high-voltage circuits the length of time was considerable and amounted in some cases to seconds.

Mrs. Ayrton said the paper was a valuable one, but she was sorry the author had not given a careful description of the appearance of the arcs and carbons under different conditions, and also the observations upon which the curves were founded. It was difficult to arrive at any general conclusions from the author's curves. The steep parts depend not only on the composition of the electrodes but on their size and on the length of the arc. The author had only used short arcs which with carbons only gave the flat part of the curve. With metal-carbon you get the steep part. Mrs. Ayrton also expressed her interest in the relighting experiments shown by the author.

Mr. A. Campbell asked if the characteristic curve of a mercury arc was similar to that of the other metal arcs investigated. He pointed out that the use of unscientific units (inches and feet) in giving the results of such interesting researches was most deplorable. In all standard work on the subject the lengths of arc are always given in millimetres, and even in commerce the diameters of carbons are similarly expressed (*e. g.* 12 mms. and not 0·47 inch).

Mr. Alexander Russell stated that Mr. Upson's paper would be extremely useful to all engaged in research in connexion with the electric arc. He agreed with the author that coal-gas was very unsuitable for experiments with the arc. He had recently been experimenting with high-voltage alternating arcs in coal-gas, and in most cases the enclosing vessel was soon filled with black cobwebs. When, however, the high-voltage high-frequency arc between iron-iron electrodes was shunted by a Leyden jar in series with a small coil of wire the discharge between the electrodes became simply a torrent of sparks and no appreciable quantity of soot was formed. The difficulty the author experienced in certain cases in obtaining an arc was due probably to the low voltages used. When high voltages are applied it is easy enough to maintain an arc. More experimental research was wanted on the upper part of the statical characteristic curve of the arc. A good deal of exploration has already been made in this direction in Dr. Simon's Laboratory at Göttingen with the help of a motor generator from which three or four amperes of direct current can be obtained at 5000 volts. The results already obtained were of great importance. The success of the statical characteristic method as an instrument of research was mainly due to the care with which Mrs. Ayrton had specified the method of obtaining

the curve. He hoped that Mr. Upson would make further researches at high voltages.

Dr. FLEMING congratulated Mr. Upson on his paper, and said that it represented a considerable amount of careful work, and even if all the conclusions to be drawn from it were not yet absolutely clear, it would furnish a statement of facts which would assist in future theorising. He said that the interesting part to him at the moment was the difference between the form of the volt-ampere characteristics for the carbon-carbon arc in air and the metal-carbon arc in hydrogen, but on that matter he would defer his remarks until he had shown some other experiments.

Mr. UPSON, in reply, stated that the primary object of the paper was to compare metal arcs with carbon arcs. He had also performed some experiments with charges of voltage and variations in the length of burning of the arc, and he hoped to carry out a series of experiments using high voltages.

II. *Some Observations on the Poulsen Arc as a means of obtaining Continuous Electrical Oscillations.* *By* J. A. FLEMING, *M.A., D.Sc., F.R.S., Professor of Electrical Engineering in University College, London* *.

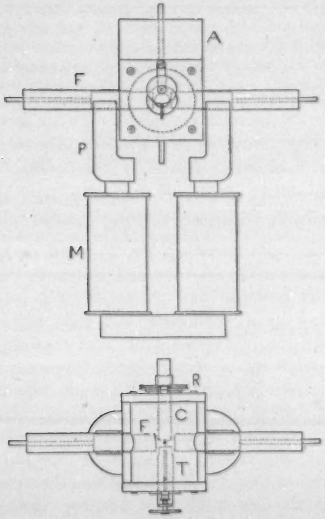
THE interesting discovery of Mr. Poulsen that an electric arc between a carbon and a metal electrode formed in an atmosphere of coal-gas and traversed by a magnetic field, when shunted by a condenser and inductance, as in the Duddell musical arc, furnishes a means of obtaining undamped electrical oscillations, of much higher frequency than with a plain carbon arc in air, has attracted much attention from its technical application in wireless telegraphy. Having made a careful examination of the matter during the last six or seven months, the record of some of my observations in connexion with it may be of interest to other physicists who are studying the arc method of exciting oscillations. The apparatus with which I have worked, constructed on the same lines as that of Mr. Poulsen, consists of a stout brass cylinder 14 cms. in diameter and 19 cms. long, which is soldered across another brass box 7 inches by 18 cms. by 25 cms. high, the cylinder being open at both ends (see fig. 1). The interspace between the cylinder and the containing box serves as a water-jacket, and through it cold water is made

* Read June 14, 1907.

to flow by inlet and exit pipes. The ends of the cylinder are closed gas-tight by marble plates, and these are pierced by holes in which brass sleeve tubes are fitted.

The cylinder has also inlet and exit pipes for coal-gas. In addition, a pair of brass tubes pass through the sides of the box and the cylinder, and in these are fitted soft iron rods 4 cms. in diameter and 20 cms. long. The inner ends are

Fig. 1.



about 3 to 4 cms. apart. In the chamber so constructed an electric arc is formed between the edge of a stout cylindrical carbon rod 2·5 cms. (1 inch) in diameter, which passes through the sleeve in one marble end. This carbon is kept in slow rotation by an electric motor geared to it so that it makes one rotation in about 2 or 3 minutes. In apposition to this carbon is a hollow brass tube closed at both ends, having cold water circulating through it. This passes through

the other marble end, and it is terminated by a copper beak. The electric arc is formed between this beak and the sharp edge of the rotating carbon. The copper electrode can be moved to and fro by an adjusting screw, and is so set that the arc is formed in a magnetic field between the soft iron pole pieces. The magnetic field is created across the arc by setting the brass box upon an electro-magnet and joining its poles to the two iron cylinders by suitable soft iron connectors. With this apparatus it is possible to create an electric arc of suitable length in an atmosphere of coal-gas, the arc being also traversed by a magnetic field. The carbon is the negative and the copper beak the positive pole. The field used did not exceed 1000 C.G.S. units in strength, and generally was about 600 units. This field was so directed as to cause the arc to spring between the upper edge of the carbon and the copper beak, and be forced upwards in an arch.

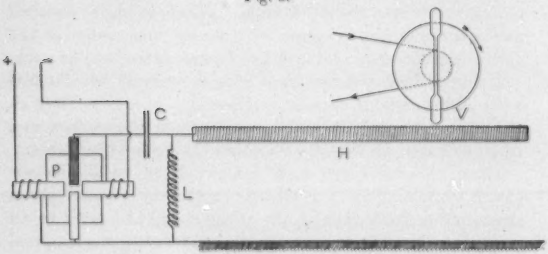
Using a direct current supply at a pressure of 400–500 volts with a suitable rheostat interposed, it is easy to form an arc having a potential difference between the carbon and metal electrode of 300–350 volts and taking 5–10 amperes.

These electrodes are then connected by an oscillatory circuit consisting of a condenser made with metal plates separated by sheet ebonite, the whole being immersed in an insulating oil. The capacity of the condenser generally used by me was 0·00289 microfarad. In series with the condenser was an inductance consisting of a coil of insulated copper wire No. 7/22 wound in square form over a wooden cross so as to make a circuit 60 cms. in the side, having 8 turns. The inductance of this coil for high-frequency currents was 116,200 centimetres. An additional sliding inductance was used consisting of bare copper wire wound in a spiral on a wooden rod $1\frac{1}{4}$ inch or 3·2 cms. in diameter, having 8 turns to the inch. Two such spirals were placed on a support parallel to each other, and cross-connected by a sliding copper saddle, so that by moving the saddle, more or less inductance could be thrown into the circuit without interrupting it. The full inductance of the double spiral was 200,000 centimetres. If then an arc is created between the electrodes and adjusted to be of such a length as to take about 8 amperes with 300–350 volts between the carbons, then powerful

oscillations are set up in the square circuit in series with the condenser. A hot wire ammeter inserted in this circuit indicated a current of about 5 amperes. If an electrostatic voltmeter is joined across the terminals of the condenser, we find a potential difference of 1200 to 1500 volts (R.M.S. value), vastly exceeding the steady potential difference of the arc electrodes.

The existence of a powerful oscillatory current in this square circuit may be shown by holding near to it a circular coil of about 30 turns, 30 cms. in diameter, formed of well-insulated wire, to the ends of which a 50-volt carbon filament glow-lamp is attached. When this round coil is held near

Fig. 2.



and parallel to the square coil, the lamp lights up brilliantly in consequence of the secondary current induced in the lamp coil.

It is, however, seen that the lamp often flickers a good deal, showing that the primary oscillation current is not absolutely constant, and this flickering varies with the adjustment. It is, however, possible to adjust the arc so that the glow-lamp remains fairly steady, and if its incandescent filament is examined in a revolving mirror, its image will be seen spread out into a uniform band of light.

This, however, is not an absolute proof of the uniformity of the oscillatory current. To the above-mentioned condenser circuit, I then attached a long helix of fine silk-covered wire (No. 30 S.W.G.) wound in one layer of closely adjacent turns on an ebonite tube (see fig. 2). The length of this helix was 210 cms., its diameter 4·78 cms., and the

number of turns 5470. This helix was supported horizontally on insulating stands about 45 cms. above the table, a sheet of zinc as long as the helix being placed on the table to act as an earth plate. This earth plate was connected to the metallic electrode of the arc and the helix to the junction point between the condenser and inductance coils in series with it.

The far end of the helix was provided with a number of needle-points formed by a bunch of sewing-needles attached to a wire.

On adjusting the inductance in series with the condenser so that the whole inductance in series across the arc terminals had a value of about 215,000 cms., powerful stationary oscillations were excited in the helix. Under these conditions, vivid blue electric brushes 1 or 2 cms. long appear on the needle-points at the far end of the helix, thus showing that the potential at that end has been increased by resonance to many thousand volts. These brushes are not of the same harmless character as the ordinary electric brush of an electrostatic machine, but are veritable flames or arcs taking place into the air. They are very hot and will melt sealing-wax and set fire to paper, and inflict the most painful burns if incautiously touched. They make a rushing or whistling sound and are of a strong blue colour.

If the inductance in the condenser circuit is varied even a little, the brushes disappear. The above described helix considered simply as an open circuit oscillator, has a natural time period of electrical oscillation of very nearly 5 one-millionths of a second, or a frequency of $0{\cdot}197 \times 10^6$. This value had been exactly determined in previous experiments with it*. As the value of the capacity in the condenser circuit is 0·0029 microfarad, and the total inductance in series with it 215,000 cms., the time period T of the condenser circuit is given by :

$$T = \sqrt{\frac{0{\cdot}0029 \times 215000}{5{\cdot}033 \times 10^6}} = \frac{5}{10^6} \text{ nearly};$$

and this shows that to obtain the brush-discharge effect the condenser circuit has to be exactly tuned to the period of the fundamental electrical oscillation of the helix.

The effect of these oscillations in the wire is to create an intense electric field round the helix extending in all directions for a distance of nearly a metre. Vacuum-tubes of all kinds glow brilliantly when brought near the helix. Especially brilliant is the glow in the form of Neon tube I employ with my cymometer, which, as I showed some three years ago, is extremely sensitive to a high-frequency electric field. The form of tube I use is of the spectrum type, the straight part being a glass tube with a bore of 1 mm., and the enlarged ends cylinders of about 1 cm. diameter and 5 cms. long.

The beautiful discovery made some time ago by Sir James Dewar that Neon and other rare atmospheric gases can be separated from the commoner constituents of air by means of charcoal cooled to very low temperature, renders it possible to fill spectrum vacuum-tubes with nearly pure Neon, and such tubes are of very great utility in high-frequency research. When filled with Neon at the proper pressure they glow with an intensely brilliant orange-red light in a high-frequency field. If one of these tubes is held near the helix when in action, it shows by its glow that there is an electric field increasing in intensity all the way up the spiral to the free end.

But more, the Neon tube shows us that the oscillations are not perfectly continuous. If the tube is waved rapidly to and fro in front of the spiral, by the persistence of vision its image is expanded into a fan-shaped area of light. Examining this, we see it crossed vertically by narrow or broad lines irregularly placed. This indicates that the tube is extinguished at intervals, and this is because the electric field disappears and the oscillations are interrupted. Not only Neon tubes, but all other vacuum-tubes show this effect provided they are narrow tubes. If a wide tube is used, the effect is not seen because the intermittent images of the tube overlap each other. The phenomenon is well shown by rapidly rotating a Neon tube near the spiral. The tube must be fastened by silk threads to a strip of ebonite held on an

* See J. A. Fleming, "On the Propagation of Electric Waves along Spiral Wires," Phil. Mag. ser. 6, vol. viii. p. 434 (October 1904).

ebonite axle, and this rapidly rotated by a turn-table (see fig. 2).

The intermittency is always most marked when the arc is first started and then diminishes, but it is always present more or less, and it shows that the oscillations, even if undamped, are not absolutely continuous, but are cut up into irregular groups. It seems to be caused by the arc changing its point of departure on the carbon electrode. It is increased if the rotation of the carbon is irregular, but I have never been able to abolish it altogether. Another proof of the same fact is derived as follows:—

The helix is removed and another square coil of 12 turns or so of insulated wire wound on a frame 60 cms. inside is placed parallel and at a distance of 2 or 3 metres from the similar coil in the condenser circuit (see fig. 3). This

Fig. 3.

secondary coil is tuned by a condenser to the same frequency as the arc condenser circuit. Across the terminals of the secondary circuit another circuit is joined which consists of the oscillation valve or glow-lamp wireless telegraph detector devised by me four years ago, and already described*. This valve stops all movement of electricity in one direction, and in series with the valve is placed a telephone. On listening to the telephone when the arc is in operation and is quite steady, a crackling sound is heard, which is not sufficiently regular to be called a musical note. If the oscillations in the arc circuit were continuous and absolutely persistent, they

* See J. A. Fleming, "On the Conversion of Electric Oscillations into Continuous Currents by means of a Vacuum Valve," Proc. Roy. Soc. Lond. vol. lxxiv. p. 476 (1905). Also Phil. Mag. May 1906.

would induce similar secondary currents, and my oscillation valve would separate out all the unidirectional constituents, and send through the telephone a perfectly steady continuous current. But this is not the case. The telephone yields a sound which shows that the continuous current through it is interrupted irregularly, and this can only be because the oscillations in the arc circuit are interrupted. If the telephone is not inserted directly in circuit with the valve, but is inserted in the primary circuit of an induction-coil, or air-core transformer the secondary of which is inserted in series with the valve, the sound in the telephone is still more increased.

We have therefore strong indications that the arc method of exciting undamped oscillations gives rise to irregular groups of oscillations which are separated by short intervals of time.

In order to be successful at all in producing high frequency oscillations by the arc method, I find it essential to pay attention to a number of details. One important condition of success is, as Mr. Poulsen has already pointed out, that the carbon electrode should be kept in slow rotation. If the carbon is not rotated, a bulbous deposit of carbon soon grows on the electrode and builds up a beak or pimple of carbon which makes the arc irregular and soon puts it out. The end of the round carbon should be cut off square to begin with, but it soon gets rounded off and a little corrugated or milled, and seems to work better when it has been used for a short time. Hard arc carbons are better than soft for producing oscillations. Secondly, as regards the gas. Pure hydrogen works with difficulty. Coal-gas is better, but the coal-gas cannot be used over and over again, as it is altered in composition by the arc, and this seems to create a difficulty in the use of the Poulsen arc as a transmitter in wireless telegraphy on board ship. The deposit of soot in the chamber is sometimes excessive. The coal-gas must be passed through quickly, but not too quickly, and my experience is that if the sides of the containing chamber are not kept cool, the deposit of soot is greatly increased. The more enriched the coal-gas is the worse becomes this trouble. I find it necessary in any case to clean out the chamber after a few hours of continuous use.

In the next place, the magnetic field required careful adjustment. If it is too strong, the arc cannot be maintained without constant interruption. If too weak, the oscillations are feeble.

The various factors—gas, field, arc-length, rotation of carbon and water cooling—all require exact adjustment to obtain the best result. In laboratory or in lecture experiments, when effects are only required for a few minutes, this is not difficult; but if the apparatus is used as a transmitter in wireless telegraphy, it must be possible to maintain constant for hours together the produ tion of the oscillations, and so far I have not found that this is a very easy thing to do. I have tried many methods for making the arc self-regulating as to length, but without success. The best plan is hand regulation by a skilled assistant. There is a particular length of arc which for given current and voltage produces the oscillations most effectively, and this length has to be maintained constant.

When employed as a wireless telegraph transmitter, the question which presents itself for consideration is the efficiency of the arrangement as an energy transformer. We supply the arc with continuous current power, and we create by it high-frequency persistent oscillations in the antenna. There are several sources of energy di-sipation. In the first place, the arc must be arranged in series with certain regulating or ballast resistance to permit of adjustment. I have found that very good effects are obtained when working off a 400-volt pub ic supply of direct current or from a 500-volt continuous current 5 k.w. dynamo. Then from 50 to 100 volts may be dropped in the regulating resistance when working with an arc of 8–10 amperes.

In the next place, there is a large production of heat in the chamber containing the arc, the greater part of which is removed by the cooling water. We have then a portion of the power converted into high-frequency current in the condenser circuit, and this dissipates itself partly as heat in the inductance and condenser, and only a fraction of this power is radiated by the antenna. We are not really concerned to know the manner in which these heat losses are distributed. We cannot subject the matter to mathematical discussion

unless we make certain assumptions as to the form of the curve representing the oscillations, which form has not yet been determined. I have found by experiment that the power factor of the arc when working is not far from unity, and hence the power given to the arc in the form of continuous current can be very approximately determined by the product of the arc current and the supply voltage. For instance, in one case the potential difference of the arc electrodes was found to be 320 volts and the arc current 8·4 amperes, whilst in two separate measurements with the wattmeter the power taken up was found to be 2632 and 2674 watts. The product of amperes and volts is 2688, and hence the power factor by these measurements is 0·98 or 0·99, or practically unity.

If the current through the arc can be kept sufficiently constant, which is rather difficult, the power radiated may be also measured by the wattmeter. For suppose that we have a direct-coupled antenna joined at some point to the condenser circuit, we may then take wattmeter readings with and without the attached antenna, and the difference will be the power radiated.

In the case of the long helix above mentioned, the power given to the arc when the helix was attached to the condenser circuit was 2674 watts, as measured by the wattmeter, and with the helix removed it was 2632 watts, the arc current remaining constant at 8·4 amperes. Hence in this case it appears that the radiation and power consumption due to the attachment of the helix is 42 watts. The measurements, however, cannot be made very accurately in the case of small power radiation.

In the case of a direct-coupled antenna which is emitting a wave-length corresponding to its fundamental oscillation, the radiation can be approximately calculated from a formula given by Hertz. Hertz shows that in the case of a dumbbell oscillator having an electric moment Q, the energy E radiated per period is given by the expression

$$E = \frac{16\pi^4 Q^2}{3\lambda^3} \text{ ergs per period,}$$

where λ is the wave-length of the radiation.

If C is the capacity of one-half of the oscillator with

respect to the other, and V is the maximum potential difference of the parts during an oscillation, and l the effective length, then $Q=CVl$.

If I is the maximum value of the current in the oscillator during a period, then

$$I = 2\pi nCV \quad \text{and} \quad Q^2 = I^2l^2/4\pi^2n^2,$$

where n is the frequency.

Accordingly,

$$E = \frac{4\pi^2I^2l^2}{3\lambda^3n^2} \text{ ergs per period.}$$

Now $$u = 3\times10^{10} \text{ cms./sec.} = n\lambda$$

is the velocity of radiation.

In the above equation, I is the maximum current in electrostatic measure. If A is the ampere value of the same current, then $\frac{A}{10} = \frac{I}{u}$. If, then, the oscillations are persistent and with frequency n, the energy in ergs radiated per second from the antenna is

$$W = \frac{4\pi^2I^2l^2}{3\lambda^2n} \text{ ergs per second ;}$$

$$= \frac{4\pi^2u}{300}\frac{l^2}{\lambda^2}A^2 \text{ ergs per second ;}$$

$$= 40\pi^2\frac{l^2}{\lambda^2}A^2 \text{ watts.}$$

For any antenna the ratio of wave-length λ emitted to effective length l is a fixed quantity and approximates in practice for single-wire antennæ to 2·5. We have, therefore, this curious result which does not seem to have been previously noticed, that the power radiated in watts from a linear antenna of any length is given by the expression $W=64A^2$, where A is the maximum value of the current flowing into the base of the antenna. If (A) is the current as read on a hot-wire ammeter inserted at that point, then

$$(A) = \frac{A}{\sqrt{2}}, \quad \text{and} \quad W = 128(A)^2 \text{ watts.}$$

Thus, if such a hot-wire ammeter showed an antenna

current of 1 ampere (R.M.S. value), the radiation would be 128 watts, and for the same current is quite independent of the height of antenna or the frequency.

Seeing, then, that open antennæ have such enormous radiative power, we can in practical radiotelegraphic work employ very small antenna currents when using persistent oscillations.

A complete explanation of all the phenomena involved in the Poulsen arc can hardly yet be said to have been reached. It has been shown by experiments made under the Author's direction in the Pender Electrical Laboratory of University College, London, by Mr. W. L. Upson, that the characteristic curve or volt-ampere curve for a carbon-metal arc in hydrogen is a curve falling down with a much steeper slope than the characteristic for a carbon-carbon arc in air.

Also that the time during which the electromotive force can be removed without permanently extinguishing the arc, is very much shorter in the first case than in the second. Both these facts undoubtedly contribute to enable the production of oscillations of a higher frequency to be obtained with a carbon-metal arc in coal-gas, as compared with those which can be reached with a carbon arc in air.

In order that oscillations may be produced when a condenser circuit is shunted across the arc electrodes, the condenser, during its charge and discharge, must cause a variation of current through the arc, and this in turn must cause a variation in the potential difference of the electrodes in such direction as to assist the charge or discharge. Thus when the condenser is robbing the arc of current and so decreasing the arc current, this must automatically raise the potential difference of the electrodes, and when the condenser is discharging it must decrease the potential difference. The greater this P.D. variation for a given current decrement or increment, the greater will be the energy conveyed to the condenser circuit.

In the case of a carbon-carbon arc worked at 10 amperes and say 60 volts arc P.D., the characteristic curve is rather flat. Hence, if we employ a small capacity in the shunt circuit we have very little variation of arc current and arc

P.D., and little or no energy in the condenser-circuit oscillations. We can, however, obtain more energy by using, as Mr. Duddell originally did, a condenser of rather large capacity, say 1 to 5 microfarads. Moreover, the carbon arc has a persistence which renders it irresponsive to very rapid variations of current. Hence with a large current carbon-carbon arc in air, we can only obtain rather slow oscillations.

If, however, we employ a carbon-metal arc, carbon negative and cooled copper, positive electrodes in hydrogen or coal-gas, we then have an arc with a very steep characteristic curve, and one which responds to exceedingly rapid variations of current through it. We can, therefore, employ in the shunt circuit a condenser of small capacity, and yet convey to it a considerable amount of energy owing to the large P.D. variation in the arc caused by a small arc-current variation. Hence we can produce oscillations of high frequency. This theory is confirmed by the observation that in the case of the carbon-carbon arc in air for small currents, and in the case of the carbon-aluminium arc, the characteristic curve is nearly as steep as for large current carbon-metal arcs in hydrogen ; and with these particular arcs, viz., the small current carbon arc in air and carbon-aluminium arc in air, we can obtain oscillations of high frequency.

Much further investigation is, however, necessary to enable us to state precisely the action of the surrounding medium, hydrogen, or better coal-gas, in producing this steep characteristic volt-ampere curve for the carbon-metal arc in these gases.

Discussion.

Mr. W. Duddell expressed his interest in the experiments, especially those in which it was shown that the oscillations were broken up into groups. He pointed out, however, that it was possible to obtain continuous oscillations with a musical arc using high frequencies. The question as to the measurement of the power put into the Poulsen Arc was of great interest, as also was the question as to the amount of power which could be taken from the secondary without stopping the arc. He questioned Dr. Fleming's statement that the power put into the arc was equal to the product "AV," and said that the power should be measured with a wattmeter.

Mr. A. CAMPBELL remarked that Mr. Altberg in Odessa had lately shown that sparks of high oscillation frequency produce actual air vibrations of the same frequency, and by measuring the wave-lengths of these air vibrations by diffraction methods he had been able to determine the high frequency.

Dr. FLEMING, in reply, said he was prepared to accept Mr. Duddell's criticism that the power put into the Poulsen Arc should be measured with the wattmeter. The really important matter, however, was the reason why the metal-carbon arc in coal-gas could produce oscillations of so much higher frequency than the carbon arc in air. This he thought was intimately connected with the slope of the characteristic curve.

III. *A Form of Cosine Flicker Photometer.*

By J. S. DOW, *A.C.G.I., B.Sc.**

[Plate III.]

THE illumination of the white surface employed in any photometer is equal to $\frac{I \cos \theta}{d^2}$, [where I equals the intensity of the source illuminating the surface, d the distance of this source from the surface, and θ the angle between the rays of light striking the surface and a normal to the surface].

Hence, when measuring the intensity of a source of light, we may either vary "d," in which case we utilize the inverse square law, or θ, in which case the cosine law is utilized.

While the inverse square law is almost invariably utilized in photometric measurements, this method is inconvenient in one respect. In order to vary "d" the photometer is usually moved to and fro between the two sources of light to be compared. The observer is therefore obliged to be continually moving his head in order to follow the motion of the photometer, and this is particularly distracting when the eye is applied to a telescope. In order to avoid this necessity, many workers

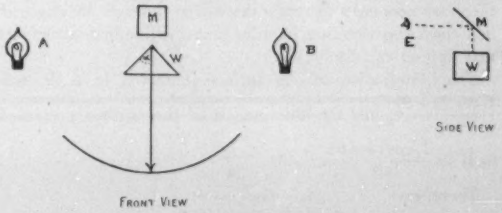
* Read June 28, 1907.

prefer to keep the photometer stationary and to move one of the sources of light. But in the case of gas-lamps and many other sources of light, this method is obviously unsatisfactory, and, even in the case of glow-lamps, is sometimes inconvenient.

The utilization of the cosine law is advantageous in this respect, for the photometer may then be kept stationary and the illumination of the photometrical surfaces adjusted in the photometer itself. The type of instrument about to be described by the author, and shown in fig. 1 (Pl. III.), has this advantage. Indeed, while it is desirable that such a photometer should be mounted on a photometrical bench in the usual way for accurate work, the author has used the apparatus shown in the Plate for many purposes, such as the comparison of the results obtained by the methods of flicker and equality of brightness, without employing a photometrical bench at all. Also it is evident that the convenience of an instrument of this type is independent of the distance between the sources of light, provided the illumination of the photometrical surface is not too low.

The general principle of this instrument is shown in fig. 2.

Fig. 2.



The two sides of the rectangular Ritchie wedge, W, are illuminated by the two sources A, B. Above the wedge a 45° mirror, M, is placed so that the observer's eye at E sees an image of the illuminated surfaces reflected in this mirror. The wedge is arranged to rotate about the line of intersection of the photometrical surfaces as a horizontal axis. To the

observer, therefore, this line appears stationary as the wedge is rotated.

Suppose now that the two sources A, B, are equidistant from the wedge.

Then, if the intensities of the two sources are the same, photometrical balance is obtained when the wedge is placed symmetrically about a vertical axis, as shown. If, however, A, say, is the brighter of the two sources, the wedge must be rotated in a clockwise direction, so that the rays from A strike the surface presented towards A more obliquely than before, while, conversely, the rays from B strike the surface presented less obliquely.

Hence let α represent the inclination of either surface to the vertical, when the wedge is in its symmetrical position.

Let I_1, I_2 represent the intensity of A and B respectively.

The illumination of the surface facing A is then $\frac{I_1 \cos \alpha}{d^2}$.

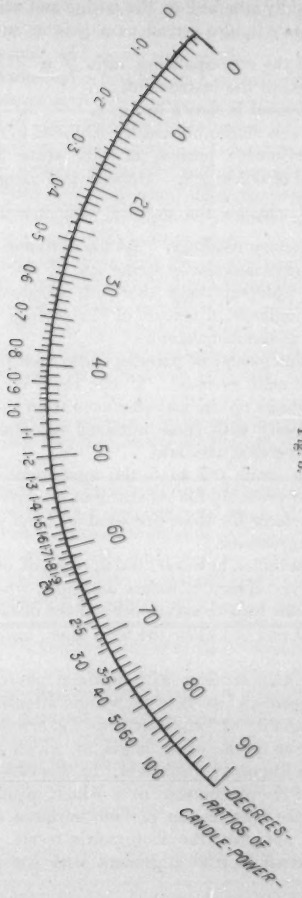
Similarly, the illumination of the surface facing B is $\frac{I_2 \cos \alpha}{d^2}$; hence if $I_1 = I_2$ the surfaces appear equally illuminated and we obtain photometric balance, with the wedge in its symmetrical position.

Suppose now that I_1 is greater that I_2, and that it is therefore necessary to rotate the wedge through an angle θ in a clockwise direction, in order that the illuminated surfaces may appear equally bright.

The illumination of the surface presented to A is now $\frac{I_1 \cos (\alpha + \theta)}{d_2}$, and the illumination of the surface presented to B is $\frac{I_2 \cos (\alpha - \theta)}{d^2}$.

Therefore
$$\frac{I_1}{I_2} = \frac{\cos (\alpha - \theta)}{\cos (\alpha + \theta)}.$$

The ratio $\frac{I_1}{I_2}$ corresponding to each value of θ is therefore known. In this case the angle α was made 45° for, by so doing, the maximum angular range of photometrical reading is obtained.


0
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10
20
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40
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80
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0.7
0.8
0.9
1.0
1.1
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1.4
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1.6
1.7
1.8
1.9
2.0
2.5
3.0
3.5
4.0
5.0
6.0
10.0
-DEGREES-
-RATIOS OF CANDLE-POWER-

Fig. 3.

A lever, rigidly attached to the wedge and utilized by the observer to rotate it, also served as a pointer indicating the value of θ and the corresponding ratio $\frac{I_1}{I_2} = \frac{\cos(45-\theta)}{\cos(45+\theta)}$, on a scale attached to the instrument.

This arrangement is shown in fig. 1.

The scale is also diagrammatically exhibited in fig. 3 (p. 39). The most serviceable portion of this scale lies in the neighbourhood of 0·2 to 5·0. Outside this range the values of $\frac{\cos(45-\theta)}{\cos(45+\theta)}$ change too rapidly with increasing values of θ, for satisfactory readings. At the extreme ends of the scale, too, the rays necessarily strike one of the photometric surfaces very obliquely, and therefore emphasize any inequalities or roughness in texture of the surface and reduce the sensibility of the instrument.

At present fine plaster of paris has been used by the author to secure white matt surfaces. Under these conditions, the theoretical readings on the scale deduced from the cosine law agreed very closely with those obtained experimentally and assuming an inverse square law.

Between the limits 0·2 to 5 the agreement was within 3 per cent., and probably the substitution of perfectly ground photometric surfaces for those prepared by hand would result in still closer agreement.

Beyond these limits, however, the agreement was naturally less satisfactory. Hence it seems desirable, when using an instrument of this type to assign convenient distances to the sources of light and to utilize the most open, central, portion of the scale.

When using a photometer with inclined photometric surfaces such as occur in the Ritchie wedge, attention must be paid to the possibility of "angle-errors," *i.e.* errors introduced by uncertainty as to the exact angle at which the rays of light strike the illuminated surfaces. In this case a considerable rotation of the photometer, as a whole, about a vertical axis will alter the illumination of both surfaces equally and will, therefore, not affect the photometric result. But if the two sources are not in true alignment with the photometer,

or if the photometer is tilted slightly about a horizontal axis, the illumination of the photometric surfaces may be differently influenced thereby, and errors may result.

Suppose, for instance, that we are comparing the intensities of two glow-lamps each half a metre distant from the photometer. Then, if the centre of illumination of one of these sources is raised say, one centimetre, the angle at which the rays from this source strike the surface presented to it is altered by about 0·6 degrees. This corresponds to an alteration of illumination of about one per cent., when the most open, central portion of the scale is used.

It is desirable, therefore, that the sources of light should be distant not less than, say, 1 metre from the photometer. It should then be possible to adjust the alignment of the lamps with sufficient exactitude to avoid appreciable error from this source. Indeed, in making accurate experiments with *any* photometer, it is desirable that the distance of the sources from the photometer should not be less than this value. Otherwise the inverse square law cannot be rigidly applied owing to the fact that the centres of illumination are rarely correctly located *.

A tilt of only one degree of the photometer, as a whole, about a horizontal axis would cause an error of about 4 per cent., even when the most favourable portion of the scale is used. It is, therefore, desirable that no appreciable tilt of this character should be introduced either by inequalities in the level of the bench or undue play in the carriage of the photometer.

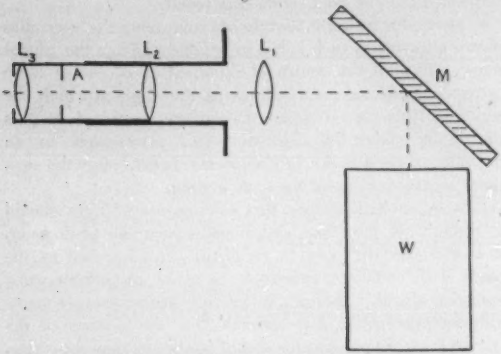
Yet, it may be pointed out that the "double comparison method," now almost invariably used when careful photometric tests are attempted, eliminates all want of photometric symmetry in a stationary photometer and would completely eliminate the above source of error, if the same portion of the scale was used in both experiments.

This instrument can be utilized either on the Equality of Brightness or Flicker principle.

* 'Electrician,' Sept. 14th, 1906, for some notes by the author on this point.

This is accomplished as follows.

Fig. 4.



The image of the wedge, as formed by the mirror M, is outside the focal length of the convex lens L_1, and this together with the lens L_2 forms a real image of the illuminated surfaces in the plane of the aperture at A. The eye of the observer applied to the third lens L_3 sees a magnified image of the aperture and the illuminated surfaces—both simultaneously in focus.

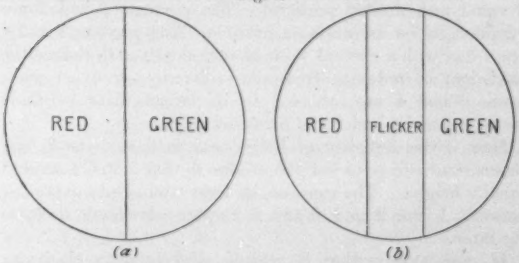
If, now, an observer is comparing, say, a red source with a green one, the field of view appears to him as shown in fig. 5 *a*, when the lens L_1 is stationary. He then adopts as photometric balance the position of the photometer-wedge in which the red and green fields appear equally bright.

The element of flicker may, however, be introduced by a modification of the method due to Rood *. The lens, L_1, is mounted on springs and attached to a cord passing over a pin mounted eccentrically on the pulley of a small electric motor. When the motor is caused to rotate the lens oscillates to and fro and the boundary between the photometrical surfaces appears to oscillate with it. A band of flicker is thus produced, and the field of view assumes the appearance shown in fig. 5.

* Amer. Journal of Science, 1899, p. 194.

The observer may then judge the position of photometric balance by observing the cessation of flicker in the band. Or he may drive the motor so fast that all flicker disappears and the intermediate band merely assumes a colour intermediate between that present in the two adjacent portions of the field of view. He may then use this intermediate colour to assist him in his decision as to the exact point when all three sections of the field of view appear equally bright.

Fig. 5.



The question may now be raised whether, when using this instrument, the results of using the Equality of Brightness and Flicker methods is the same.

The author has found that for widely different colours and for certain portions of the retina this is not the case *. Under the conditions present in this instrument, however, the agreement was very close unless the colour difference was very marked indeed. This is illustrated by the results shown in the following :—

Nature of Lights compared.	Results by Flicker method.	Result by Equality of Brightness method.
White to White	1·031	1·034
White to Apple-Green. (impure)	0·911	0·906
White to deep Green ...	1·21	1·08
White to deep Red	2·23	2·09

* Proc. Phys. Soc. vol. xx.

When comparing lights of the same colour the sensitiveness of the photometer seemed to be very much the same for both methods, but when the comparison of sources of light differing widely in colour was attempted, the Flicker method gave rise to the most consistent results.

It seems possible that the sensitiveness of a photometer of this type might be improved by introducing the element of *contrast* in the manner shown in fig. 6 (Pl. III.)

Each of the photometrical surfaces is shown divided into 3 equal and similar portions. The portions 2 and 5 are intended to be as white as possible. The portions 1 and 6 are tinted with a neutral wash of such depth as to reduce the coefficient of reflexion from the surface by say 4 per cent., while 3 and 4 are intended to be intermediate in shade between 1 and 5 and 2 and 6 respectively.

Now if the Equality of Brightness method is used, the observer attempts to set the wedge so that 3 and 4 appear equally bright. The contrast in light and shade exhibited between 1 and 2 and 6 and 5 respectively should then be the same.

If, now, the wedge is rotated slightly in a clockwise direction, the contrast between 1 and 2 becomes *more* marked but that between 6 and 5 *less* marked than before.

We have thus a *double* change such as occurs in the Lummer-Brodhun contrast photometer and a corresponding gain in sensitiveness. If the Flicker method is used a corresponding effect will exist.

The author has endeavoured to facilitate the comparison of heterochromatic sources of light by applying Crova's method.

As the temperature of an incandescent solid is raised, the luminous radiation in the form of waves of short wave-length increases more rapidly than the luminous radiation of great wave-length.

According to the principle discovered by Crova*, however, the luminosity in the spectrum of such a source in the neighbourhood of $\lambda = 0{\cdot}582\mu$ is a measure of the total luminosity of the source. Crova therefore proposed to view the surfaces

* *Comptes Rendus*, vol. cxix. no. 16, p. 627.

illuminated by two heterochromatic sources through a solution allowing only rays in this portion of the spectrum to pass. This solution was composed of :—

Anhydrous ferric chloride, 22·32 gms.
Crystallized nickel chloride, 27·19 gms.
Distilled water, 100 cc.

It must be noted, however, that the principle on which Crova's method depends, only applies rigidly to an incandescent body yielding a continuous spectrum.

In the case of a luminescent vapour yielding a spectrum in which the luminosity is concentrated in isolated bright lines, the method is inapplicable.

Yet, since the portion of the spectrum which Crova's method proposes to utilize is by far the most valuable portion from an illuminating point of view (*i. e.* the portion which, for a given expenditure of energy, produces the greatest sensation of brightness), the method may prove satisfactory even in many cases of this character.

The chief difficulty which the author has found in applying this method lies in the fact that, if a sufficient depth of solution is employed to restrict the light passing to a very narrow region of the spectrum, and hence to render the colours of the two illuminated surfaces identical, the luminosity of the surfaces becomes, as a rule, inconveniently low. The only case, in fact, in which the author succeeded in obtaining an apparently perfect colour match without undue loss of illumination, was when comparing an arc lamp against an incandescent mantle.

When a *flicker* was utilized, however, it was found that a depth of 3 millimetres of solution, which, while most transparent to yellow rays, actually allowed rays from $\lambda = 0{\cdot}52$ to $\lambda = 0{\cdot}68\mu$ to pass, rendered the comparison of most sources distinctly easier.

The following are a few cases in which this method was applied.

Carbon Glow-lamp compared with Nernst lamp.
Argand Gas-burner „ „ Incandescent Mantle.
White Arc-light „ „ Incandescent Mantle.
White Arc-light „ „ Argand Gas-burner.
Flame Arc-lamp (Excello carbons) compared with Argand Gas-burner.
Flame Arc-lamp (Bremer carbons) „ „ Argand Gas-burner.

In all these cases the means of a set of photometrical results obtained with and without the Crova screen agreed very closely, in the first two cases within 2 per cent.

The unsteady nature of the arc lights rendered very exact comparisons with these lights impossible, but in this case also the difference seemed to be, at any rate, less than 5 per cent.

All these sources of light, however, yield a continuous spectrum. It is true that some isolated bright lines, superimposed over the main spectrum of the arc, serve to increase the efficiency of the flame carbons. Yet the brightest of these lines were congregated in the yellow and orange and would, therefore, not seriously affect the applicability of Crova's method.

Quite otherwise were the results obtained when comparing a mercury-vapour lamp against a carbon-filament glow-lamp. The luminosity of this lamp was concentrated in three isolated bright lines in the yellow, green, and violet respectively, practically no red rays being present.

It was found that the depth of solution utilized in the experiments described above not only failed to very noticeably improve the colour difference between the sources but also seriously affected the photometric results obtained.

The use of yellow screens in order to facilitate the photometric comparison of heterochromatic sources of light is certainly only justifiable in the case of illuminants yielding continuous spectra.

Even in this case it is essential that the yellow screen should exhibit absorbing properties similar to those of Crova's solution, namely minimum absorption in the neighbourhood of $\lambda = 0{\cdot}58\mu$, and a uniformly increasing absorption on either side of this point.

Discussion.

Mr. Patterson complimented the author upon his paper and experiments. With reference to the cosine instrument he asked if the cosine law was trustworthy at very acute incidence of the light. The unequal sensitiveness of different parts of the retina accounted for the fact that it was possible to obtain different balance-points by fixing attention on the different sides of a Ritchie wedge.

Mr. J. S. Dow, in reply, said he had only worked with photometers with wedges of 90°, but he was certain that the cosine law would not hold at acute incidence on account of the large amount of regular reflection.

IV. *On Magnetic Oscillators as Radiators in Wireless Telegraphy.* *By* J. A. Fleming, *D.Sc., F.R.S.**

An oscillator of the open or Hertzian type is commonly called an *electric oscillator* because the effects produced in the external field are to a large extent determined by the potential of the free electric charges which alternately make their appearance on the open ends. If, however, the oscillator consists of a metallic circuit completed by a condenser, the plates of which are very near together, the effects in the external space are mostly or entirely determined by the current in the circuit and little if at all by the condenser-plate charges, because these being of opposite sign and near together neutralize each other's effects in the field. Such a closed or nearly closed circuit is called a *magnetic oscillator*. In the case of an open oscillator or simple doublet Hertz showed that if ϕ is the maximum electric moment, or product of the maximum static charge on each end and the distance between them, and if λ is the wave-length of the emitted radiation, the energy radiated by the oscillations in ergs per period is given by the expression

$$E = \frac{16\pi^4\phi^2}{3\lambda^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This expression can be reduced to a more convenient form for practical measurement as follows :—

Let C be the capacity in microfarads of each sphere or half of the oscillator with respect to the other, and V be their maximum potential difference in volts before discharge, then 9×10^5 C V / 300 = 3000 C V is the maximum value of the charge in electrostatic units.

Hence if l is the length of the oscillator in cms., we have for the electric moment,

$$\phi = 3000 \text{ C.V.} l. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Let the current at the centre of the oscillator have a maximum value A reckoned in amperes, and let a be the

* Read October 25, 1907.

root-mean-square or effective value in amperes as measured by a hot-wire ammeter.

Then, if the oscillations are sinoidal in form, undamped or persistent, and of frequency N, we have

$$A = 2\pi N C V/10^6, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$a = \frac{A}{\sqrt{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence the energy E radiated per period in ergs is given by

$$E = 4\pi^2 10^8 \frac{l^2}{\lambda^2} \frac{A^2}{N}, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and the power W in watts by

$$W = 40\pi^2 \frac{l^2}{\lambda^2} A^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Remembering that $\pi^2 = 9{\cdot}87$ and $N\lambda = 3 \times 10^{10}$, we can write the above formulæ in the convenient form

$$E = 0{\cdot}2632 \frac{l^2}{\lambda} a^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$W = 789{\cdot}6 \frac{l^2}{\lambda^2} a^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The value of the ratio l/λ can be determined experimentally for any oscillator. For a simple linear oscillation it approximates in value to 0·4. Hence for such an oscillator radiating undamped waves, the power radiated in watts is given by the expression $W = 126a^2$. Thus, for instance, if the effective value of the current at the centre of the oscillator is 2 amperes, the radiation will be half a kilowatt. If we then consider the case of a perfectly closed oscillation circuit having an area S and traversed by undamped or persistent oscillations which have a maximum value A and R.M.S. value a reckoned in amperes, it can be shown that such a closed circuit also radiates energy, and we can derive a similar formula for the radiation to that given by Hertz for the open circuit. The author has shown that if M is the maximum magnetic moment of such a closed circuit *,

* See 'The Electrician,' vol. lix. p. 936 (1907). A series of articles on "The Elementary Theory of Electric Oscillators," by J. A. Fleming.

viz., the product of its area S and the maximum value of the alternating current (reckoned in electromagnet measure) in it; and if λ is the wave-length of the radiation emitted by it, and E the radiation in ergs per period, then for the magnetic oscillator we have

$$E = \frac{16\pi^4 M^2}{3\lambda^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

But by definition $M = AS/10$, and if the current is sinoidal $a = A/\sqrt{2}$, hence we have

$$E = 10{\cdot}4\,\frac{S^2 a^2}{\lambda^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Furthermore, if the oscillations are persistent the power in watts W radiated is given by

$$W = 31{,}200\,\frac{S^2 a^2}{\lambda^4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The formulæ (8) and (11) can be written in the form

$$\left.\begin{aligned} W &= 87 \times 10^{-20}\, l^2 a^2 N^2 \quad \text{(open or electric oscillator),} \\ W &= 4 \times 10^{-38}\, S^2 a^2 N^4 \quad \text{(closed or magnetic oscillator).} \end{aligned}\right\} \quad (12)$$

These last two formulæ show us that in the case of the open or electric oscillator the power radiated varies as the square of the current in it, and as the square of the frequency, whereas in the case of the closed or magnetic oscillator it varies as the square of the current but as the fourth power of the frequency. Accordingly, in the case of the magnetic oscillator the radiation is small unless the frequency is very high.

For the purposes of theory, it is most convenient to consider a closed oscillatory circuit as made up of a series of Hertzian or dumbbell oscillators placed in series with their electric poles of opposite sign overlapping. We can then easily determine the nature of the electric and magnetic field produced by it by the summation of those due to the elementary oscillators. As the writer has given lately in another place a discussion of the problem and furnished the expressions for the electric and magnetic forces at any point in the field of such a closed oscillator and the derivation of the formula (10) for the radiation, it is unnecessary to go

over the ground again here. (See 'The Electrician,' vol. 59, p. 936 *et seq.*)

The object of the following experiments was to obtain some information as to the actual distance-effects of closed oscillators of certain dimensions, and to ascertain whether true radiation was at all influential in producing the inductive action of one closed oscillatory circuit on another at moderate distances, or if the whole of the action could be accounted for by ordinary electromagnetic or Faradaic induction.

The following circuits were employed for this purpose :—

Two pair of square circuits were constructed by winding a few turns of highly insulated copper wire round a wooden cross.

One pair had sides 8 feet in length, each frame carrying two coils of 5 turns of 7/21 stranded insulated copper wire. These will be referred to as the "large square coils."

One pair had sides 2 feet in length, each frame carrying one coil of 8 turns of the same sized wire. These are referred to as the "small square coils."

The experiments consisted in setting up continuous or undamped oscillations in one of these coils by means of a Poulsen arc and then measuring the current created in the corresponding coil placed at a distance by means of the Author's oscillation-valve or glow-lamp detector. The two coils were placed at various distances and in various respective positions. The coils themselves were erected or placed in the large quadrangle of University College, London. As the arc apparatus had to be placed indoors in the Electrical Laboratory it was necessary to connect the coil used as a transmitter or primary coil with the oscillation-producing apparatus by means of a long pair of insulated parallel copper (7/21) wires 14 cms. apart and 27½ feet or 836 cms. long.

The first step was to measure the inductance of these wires for high frequency currents. Anderson's bridge method was employed with a commutator as described by the Author*.

* See J. A. Fleming, "A Note on the Measurement of Small Inductances and Capacities," Phil. Mag. May 1904, p. 586.

See also J. A. Fleming and W. C. Clinton, "On the Measurement of Small Capacities and Inductances," Phil. Mag. May 1903, p. 493.

The inductance of the parallel wires or tails above mentioned was found to be 50,910 cms., and that of one large coil of 5 turns on the 8-foot square together with the tail (L) was 310,740 cms. Hence the inductance of the coil itself of 5 turns was 259,830 cms. The inductance of the two coils each of 5 turns in parallel was 198,350 cms., and in series 779,300 cms.

The inductance of each of the small or 2-foot square coils was in the same manner found to be 116,200 cms.

In series with these coils was placed a condenser having a capacity (C) of 0·0026 mfd. and the coil and condenser shunted across a 400-volt Poulsen arc.

The arc current was generally about 8 amperes, the potential difference of the arc terminals (V_0) 260 volts (continuous). The potential difference (V_1) (R.M.S. value) of the condenser terminals was 1580 volts as measured by an electrostatic-voltmeter and the R.M.S. current (a) in the square coil was 4·22 amperes as measured by a hot-wire ammeter. The frequency N of the oscillations is given by the formula $N=(5{\cdot}033\times10^6)/\sqrt{CL}$. Hence if $C=0{\cdot}0026$ mfd. and $L=310740$ cms., we have $N=177{,}100$. If the R.M.S. current (a) in the coils were strictly sinoidal in wave-form, the value should be calculable from the formula,

$$a=\frac{2\pi}{10^2}CVN$$

where $V=\sqrt{V_1^2-V_0^2}=\sqrt{(1580)^2-(260)^2}=1560$ volts. Hence we should have

$$a=\frac{44}{7}\times\frac{26}{10^{10}}\times1560\times177100=4{\cdot}52 \text{ amperes.}$$

The actual measured value was 4·22 or 7·3 per cent. less, which may be accounted for by the known fact that the actual wave-form of the current in the coil is not by any means truly sinoidal.

The wave-length λ corresponding to a frequency of 177100 is nearly 169,000 cms., or rather more than 1 mile in length.

The total area S of the large coils using the 5-turn circuit is $5\times(243{\cdot}2)^2$ cms.$=295{,}731$ sq. cms, and the value

of S^2a^2 is therefore $(295731)^2 \times (4{\cdot}22)^2 = 156 \times 10^{10}$. Also $(177100)^2 = 554523 \times 10^{10}$. For a closed square radiator the energy in ergs radiated per period is given as already mentioned by the expression

$$E = 10{\cdot}4 \frac{S^2a^2}{\lambda^2}.$$

Hence we have in this case for the power W radiated in ergs per second,

$$W = \frac{10{\cdot}4 \times 156 \times 177000}{554523} = 522 \text{ (nearly)}$$

or about 52 microwatts.

This is the energy sent out per second through the surface of a sphere described round the oscillator the radius of which is large compared with the wave-length of the oscillator in question.

It is evident therefore that the true radiative power of a closed circuit of the above dimensions is extremely small compared with that of an open oscillator of the same order of linear dimensions.

Hence when a closed circuit traversed by a high frequency current acts to produce secondary currents in another similar circuit at less than a wave-length distance from it, the chief part of the effect cannot be due to true radiation or detached energy but is the result of ordinary electromagnetic induction, or the creation of secondary currents produced by the ebb and flow or pulsation of the lines of magnetic force permanently connected with the primary circuit*.

Nevertheless experiment shows that such circuits may act appreciably upon one another at very considerable distances, when the primary circuit of one is traversed by a high frequency current, and the circuits are in resonance with each other.

The old form of wireless telegraphy based upon such mutual induction between closed coils, as in the arrangement

* Moreover, just as in the case of the Hertzian oscillator true radiation or detachment of energy only takes place about $\frac{1}{4}$ or $\frac{1}{2}$ a wave-length from the oscillator, so in the case of the closed or magnetic oscillator true radiation effects can only be produced at a similar distance from it.

suggested by Trowbridge and Stevenson, or the parallel wire method of Preece, Rathenau and others, employed relatively low frequency currents because the receiver used was the telephone, and the frequency was therefore limited by the range of audition. Even when the circuits were put in resonance, as by Lodge, the difficulties connected with the disturbance of commercial telephony restricted the possibilities of its use apart from other considerations.

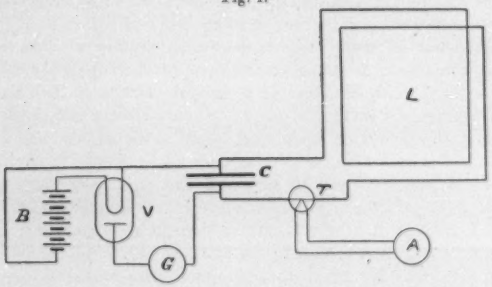
One object of the present experiments was to obtain data for an opinion whether such inductive wireless telegraphy could be improved by the use of electric oscillations of high-frequency in completely non-earthed circuits so that no disturbance of telephones or telegraphs could possibly arise combined with the use of a suitable oscillation detector. With this end in view, a closed primary-circuit had created in it continuous or undamped electric oscillations, and its inductive and radiative effect upon a similar closed receiving circuit was measured by means of the Oscillation Valve or Glow-Lamp Detector first described by the Author in 1904.

It consists of a 12-volt carbon filament glow-lamp with a metal cylinder surrounding the filament sealed into the bulb. The cylinder is carried on a platinum wire sealed through the glass. The filament is incandesced by a 6-cell secondary battery. If the terminals of a condenser inserted in an oscillatory circuit are connected, one to the cylinder and one to the negative terminal of the lamp filament, and if an ordinary movable coil galvanometer is inserted in this circuit, then when oscillations occur in the condenser the glow-lamp acts as an electrical valve and the galvanometer is traversed by a continuous current which bears a definite relation to the potential difference of the condenser terminals and therefore to the current into it*.

* This glow-lamp detector or oscillation valve was first described by me in a British Patent Specification, No. 24,850 of 1904, and in a Paper to the Royal Society (see Proc. Roy. Soc., vol. lxxiv. p. 476) in February 1905. Also in a United States Patent Specification, No. 803,684 of 1905, and in a Paper to the Physical Society of London (see Phil. Mag. May 1906), as well as in a book on the Principles of Electric Wave Telegraphy, pubished in May 1906. These publications apparently failed to make it known in the United States, for in October 1906 Dr. Lee de Forest described to the American Institute of Electrical

To calibrate the valve the following arrangement was employed. An oscillatory circuit was arranged consisting of one of the small square coils L (see fig. 1) and a condenser or Leyden jar C in series with it. In this circuit at one point was inserted a fine constantan wire having a thermo-junction (T) of iron and bismuth or bismuth and tellurium in contact with it. The arrangement actually employed was either the thermo-junction used with the Author's Cymometers or else

Fig. 1.



the thermoelectric receiver described some time ago* (see fig. 1).

The galvanometer A associated with the junction was a 5-ohm single-pivot Paul galvanometer. The arrangement was calibrated by first passing measured continuous currents through the constantan wire and observing the corresponding scale-readings of the Paul galvanometer. The curve plotted out proved to be very approximately a parabola, the galvanometer deflexion being very nearly proportional

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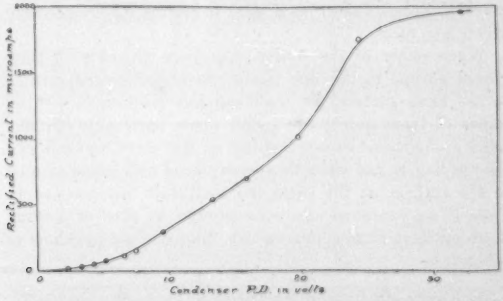
Engineers precisely the same device, viz., a carbon filament glow-lamp having a plate sealed into the bulb as a detector for electric oscillations, as his own invention and re-christened this two-year old appliance an *audion*. No re-christening, however, affects the fact that the appliance had been in use by me long previously and especially described as a receiver for wireless telegraphy.

* See "Radiation from Bent Antennæ," by J. A. Fleming, Phil. Mag. December 1906, p. 592.

to the square of the heating current sent through the constantan wire. When electric oscillations were set up in the circuit they heated the constantan wire and caused a deflexion of the Paul galvanometer from which by the previous calibration their mean-squre value could be determined. In addition the oscillation valve V joined in as shown in fig. 1 passed a continuous current through the galvanometer G in its circuit, and readings were taken of this current in microamperes corresponding to various measured values of the oscillatory current in the square circuit which passed through the constantan wire.

The object of this calibration was to be able to use the valve subsequently as a means of measuring the oscillatory

Fig. 2.

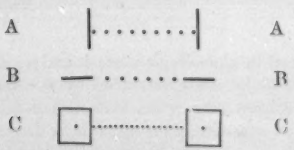


current induced in the circuit when it had a value less than that which could be easily measured by the thermojunction, for when associated with a suitable mirror-galvanometer it provides the means of measuring a high frequency current of a few microamperes. Moreover it does not introduce any extra resistance into the circuit, as would have been the case if the constantan wire and themojunction had been permanently inserted into the circuit. Observations showed that the galvanometer current did not increase proportionately to the condenser plate potential difference, but tended to a "saturation" value as shown by the curve in fig. 2.

The large (8-feet) square coils were then set up in the quadrangle of University College, one being used as a transmitter and the other as a receiver circuit.

The transmitter coil, in series with a suitable condenser, was connected by the long parallel wires or tails with a Poulsen arc in the adjacent electrical laboratory, and persistent oscillations excited in it. The similar receiver circuit at a distance had a condenser of equal capacity in series with it and a variable inductance for tuning. The oscillation valve had its terminals connected to the receiver circuit condenser-plates and wires taken from it to a mirror galvanometer placed in a distant room. The transmitter coil had then persistent oscillations set up in it by the arc, and the transmitter and receiver coils were placed at various distances and in various relative positions, and the current created in the received circuit was in each case measured by the galvanometer and valve.

When either of the square coils were placed with their planes normal to the line joining their centres and vertical to the earth surface, we shall call this position A. When either of them were placed with plane parallel to the line joining their centres and vertical to the earth we shall call the position B, and when they were placed with plane parallel to the surface of the earth we shall call this position C. Thus if we represent the coils as seen in plan or looking down on them from above we can denote these positions as follows: —



The dotted lines represent the line joining their centres. The following experiments were then made :—

Exp. I. *Effect of varying the position of the coils.*—The two 8-feet coils of 5 turns each were set up with centres 60 feet apart. The transmitter current was 5·4 amperes. The

capacity in the transmitter circuit 0·0026 mfd. The receiver was tuned to a frequency of 164,000 in syntony with the transmitter and undamped oscillations were set up in the latter by the arc. The electric arc was adjusted to keep a constant high-frequency current in the transmitter circuit of 5·4 amperes. The arc current itself was 8 amperes and arc P.D. from 260 to 320 volts.

The transmitter was put successively in positions A, B, and C, and in each case associated with receiver in positions A, B, and C, and the receiver or secondary current in milli-amperes being measured. The results are tabulated in Table I.

TABLE I.

Transmitter Position.	Receiver Position.	Secondary Current in milliamperes.
A	A	24·16
"	B	10·77
"	C	20·98
B	A	5·25
"	B	20·45
"	C	18·10
C	A	6·56
"	B	22·58
"	C	37·00

When the coils were in position C their planes were 6 inches above the ground.

The figures in Table I. show that when both coils were in position C the receiver current exceeded that in any other position, and that although the receiver current was small with the coils in the positions of zero mutual induction, viz. (A.B.), (B.A.), or (C.A.), it was not small in the position (A.C.).

In the case of two small coils placed at a distance large compared with their dimensions the magnetic induction for a steady field should by theory be twice as great in the position (A.A.) as in the position (B.B.), and should be the

same in position (C.C.) as in position (B.B). Also zero in the 6 positions (A.B.), (B.A.), (C.B.), (B.C.), (A.C.), (C.A.).

But when using the high-frequency primary current the secondary current is still large in the cases (A.C.),(B.C.), and (C.B.), viz., when either the receiver or transmitter coil has its plane parallel to the earth's surface.

Subsequent results showed that the height of the coils above the ground affected this result considerably. Nevertheless the exalted effect in position (C.C.) is interesting and convenient because it is easy to construct large circuits out of doors in positions (C.C.) and very much less easy without the aid of tall masts to make them in the positions (A.A.) or (B.B.).

Exp. II. *Effect of varying distance. Coils vertical.*—The same pair of coils were set up on positions (A.A.) and their distances varied. The receiver circuit was tuned to a frequency of 162,000, and the primary current was 4·7 and 5·4 amperes in two sets of observations respectively.

Table II.—Coils in position (A.A.).

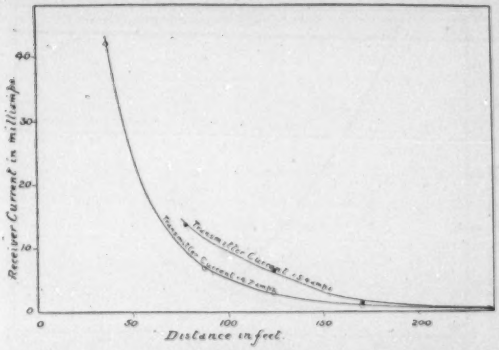
Distance of Centres in feet.	Primary Current in amperes.	Secondary Current in milliamperes.
78·0	5·4	13·7
123·5	,,	6·46
170·5	,,	1·17
237·0	,,	0·12
36·2	4·7	41·9
88·0	,,	6·99
123·2	,,	3·05
169·7	,,	1·01
237·0	,,	0·11

The results are plotted in the curve in fig. 3.

The curve shows that the inductive effect decreases at first very rapidly with the distance, less rapidly than the

inverse cube but rather more than the inverse square of the distance.

Fig. 3.



Exp. III. *Effect of varying distance. Coils horizontal.*—The same coils were set in positions (C.C.) with centres 6 inches above the ground. The receiver circuit was tuned to a frequency of 162,000, and the primary current was 5·4 amperes.

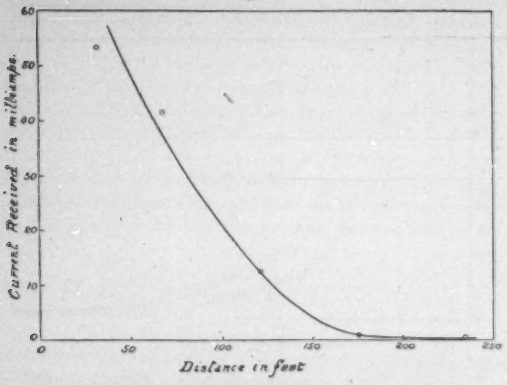
Table III.—Coils in position (C.C.).

Distance of Centres in feet.	Primary Current in amperes.	Secondary Current in milliamperes.
33	5·4	53·1
68	"	41·4
121	"	12·4
175	"	0·57
236	"	0·28

Here again the current falls off at a rate between the inverse cube and inverse square, but the secondary current has a much greater absolute value for the same distance than

with the coils in positions (A.A.). The results are set out in the curve in fig. 4.

Fig. 4.



Exp. IV. *Effect of varying distance. Small coils horizontal and 6 inches above the ground.*—A similar set of measurements were made with the small coils 2 feet in side and having 8 turns on each coil. The primary current was 5·4 amperes. The receiver circuit was tuned to a frequency of 214,400 in syntony with the transmitter current.

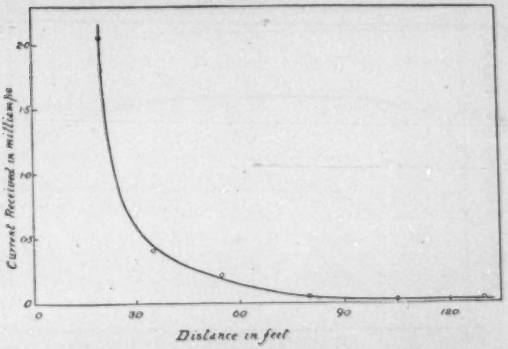
The coils were placed at various distances and the receiver currents measured.

Table IV.

Distance of Centres in feet.	Primary Current in amperes.	Secondary Current in milliamperes.
20	5·4	2·045
35	"	0·402
55	"	0·223
80	"	0·045
105	"	0·027
130	"	0·034

The same law of variation holds good in this case also as shown by the curve in fig. 5.

Fig. 5.



Exp. V. *Effect of varying height above the ground. Coils in position* (A.A.) *vertical.*—The large coils were employed and raised step by step to various heights above the ground. The primary current was 5·4 amperes. The receiver was tuned to a frequency of 162,000. The distance of the coils was 68 feet. The coils had their planes vertical and parallel to each other except in the last case when they were placed horizontal or in positions (C.C.) for comparison.

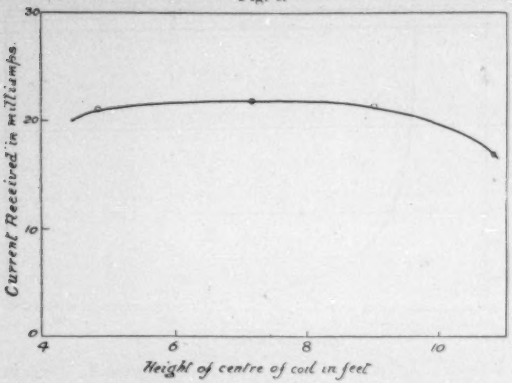
Table V.

Height of centre of Coil above ground, in feet and inches.	Receiver Current in milliamperes.	Coil Position.
ft. ins.		
4 10	20·98	Coils vertical (A.A)
7 2	21·76	" " "
9 0	21·50	" " "
10 10	16·84	" " "
0 6	31·08	Coils horizontal (C.C.)

Here again is shown the improved effect of placing the

coils with their planes parallel to the earth's surface. These observations are plotted in the curve shown in fig. 6.

Fig. 6.



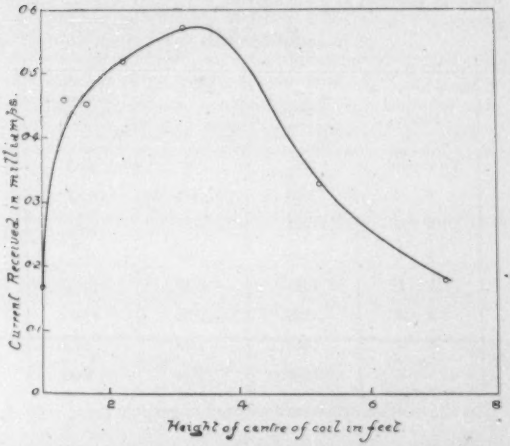
Exp. VI. The same measurement was made with the small coils 2 feet square of 8 turns, placed at a distance of 60 feet. The primary current was kept at 5·4 amperes. The receiver was tuned to a frequency of 214,000 and the secondary current was measured for various heights of the coils above the ground, the coils being vertical and in positions (A.A.).

Table VI.

Height of centre of Coil above the ground, in feet and inches.	Secondary Current in milliamperes.
ft. ins.	
1 0	0·166
1 4	0·460
1 8	0·452
2 3	0·515
3 2	0·570
5 3	0·327
7 2	0·176

The results of the last two experiments is to show that for constant primary current the receiver current is a maximum for a certain height of the coils above the earth's surface. The observations are delineated in fig. 7.

Fig. 7.



It is obvious, however, that all the measurements in which the coils are placed horizontal and at varying heights above the ground need caution in interpreting them. When the coil is placed with its plane parallel to the earth, currents must be induced in the soil, and the effective inductance of the coil and therefore the frequency of the current must be altered. The nearer the ground the less is the effective inductance of the coil, but on the other hand the capacity is increased. Hence, even if the primary high-pressure current is kept constant, there will be variations of frequency which affect the inductive action of the primary on the secondary. That such is the case was shown by the results of a further set of measurements with the large coils, the power taken up in the arc and its power-factor being observed at the same time.

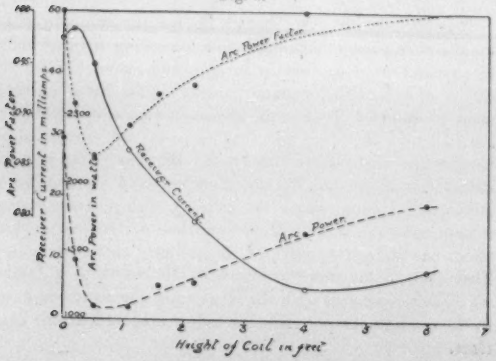
Exp. VII.—Large square coils placed parallel to the earth. Distance 68 feet. Receiver tuned to a frequency of 162,000 approximately in syntony with the transmitter. The coils were raised to various heights above the ground and the secondory current measured. The transmitter current was kept constant at 5·4 amperes.

Table VII.

Height of Coils above the ground, in feet and inches.	Power taken up in Arc in watts.	Power factor of Arc.	Secondary Current in milliamperes.
ft. ins			
On ground	2328	1·0	45·8
0 2	1424	0·91	47·4
0 6	1096	0·85	41·2
1 1	1104	0·89	27·7
1 7	1248	0·92	22·5
2 2	1272	0·93	16·0
4 0	1640	1·00	5·18
6 0	1856	0·98	8·03

The above observations are reduced to graphic form in fig. 8.

Fig. 8.



They show how much the receiver current varies with the height of the coils above the ground, the maximum value being reached when the coils are about 2 inches above the surface. Also they show that when the coils are laid flat on the ground there is a great increase in the power taken up by the arc.

Raising the coils from 6 inches to 6 feet above the ground decreased the secondary currents to one-fifth.

Exp. VIII. *Comparison of coils of different areas.*—The two large or 8-feet coils were compared with the two small or 2-feet coils. In both cases the distance between primary and secondary was 79 feet, and the coils were parallel to the earth and at 6 inches above the surface. In both cases the primary current was 5·4 amperes.

The small coils had 8 turns and were tuned to a frequency of 214,400.

The large coils had 5 turns and were tuned to a frequency of 162,000.

Table VIII.

Coils used.	Number of turns.	Frequency.	Secondary Current in milliamperes.
Small. 2 feet in side ...	8	214,400	0·082
Large. 8 feet in side ...	5	162,000	27·4

The large coil therefore produces 340 times the effect of the small coil.

Exp. IX. *Comparison of the same coils with different numbers of turns.*—The two large coils were wound with two circuits each of 5 turns and they could be joined in series so as to make a coil of 10 turns. The coils were placed parallel to the earth and at a distance of 6 inches above it. The primary current used in both cases was 5·4 amperes. The inductance of the primary circuit was 380,000 cms., and when 10 turns were used it was 1,000,000 cms. The power

taken up in the arc therefore was not the same in the two cases.

TABLE IX.

Number of turns on Coil.	Power taken up in Arc in watts.	Frequency of Oscillations.	Secondary Current in milliamperes.
5	2495	162,000	27·4
10	1537	99,600	21·4

The increase in the number of turns by increasing the inductance lowers the frequency of the oscillations, and this lowers the power taken up in the arc, and therefore the secondary current is reduced roughly in proportion to the decrease in frequency. In all the above experiments the Poulsen arc was used as a generator of undamped oscillations. It was considered desirable, however, to compare the effects obtained in this case with those when using a spark method of exciting intermittent oscillations.

The large square coils were placed vertical and facing each other in positions (A.A.) and at 68 feet apart. They were tuned to a frequency of 162,000. Oscillations were then excited by an arc and a spark and the primary and secondary currents measured and volt amperes given to the generator in each case, as follows:—

TABLE X.

Generator.	Volts.	Amperes.	Volt amperes.	Primary Current in amperes.	Secondary Current in milliamperes.
Spark	12·2	2·6	31·7	2·15	0·56
Arc	260	8·0	2080	5·4	21·0

Roughly speaking, therefore, 70 times more power was being expended on the arc than on the induction-coil used to create the spark, but the secondary current only increased some 38 times.

This seems to indicate that the intermittent spark method of exciting the oscillations is not inferior but rather superior in efficiency to the arc method, for the particular purpose here considered.

The conclusions to be drawn from this preliminary set of experiments are that the inductive effects observed between the closed oscillatory circuits used are in the main due to electromagnetic induction and not to true radiation. Secondly, that the most advantageous position of the coils is with the plane of the coils parallel to the earth's surface and at a small distance above it. Thirdly, that the increase in the area of the coils is especially advantageous, and that therefore the coils used should consist of single turns of wire enclosing as large an area as possible. Fourthly, that the inductance and capacity of the circuit must be kept low to keep up the frequency. Lastly, the spark method of exciting the oscillations can be advantageously employed.

The best method, therefore, for constructing and using such closed circuit antennæ for high-frequency closed circuit wireless telegraphy which seems to be indicated by the foregoing experiments is as follows:—

In some large field set up a number of short wooden posts having telegraphic insulators attached to them. Round these lay a number of wires, which may be bare copper wires, so as to form a closed circuit of one turn of a number of parallel wires, the circuit being say a foot or two from the ground, or if cows, horses, or sheep are kept in the field it may be better to keep it up 7 or 8 feet from the ground. Complete this circuit by a condenser and spark-gap and make provision by a high-tension transformer for exciting oscillations in the circuit.

In a similar distant and syntonised circuit include any of the receiving devices now employed in radio-telegraphy, and signals can be transmitted and received without high antennæ and without interruption to commercial telephonic circuits. Such closed horizontal receiving circuits will not pick up signals from the vertical antenna transmitters commonly used.

The method suggested is in fact the old form of magnetic induction space-telegraphy, but conducted with electric

oscillations in place of the low-frequency currents obtained by an ordinary alternator or interrupted battery-current. On the other hand, direct experiment with low-frequency alternating currents of 5·4 amperes passed through the small coils used in these experiments, showed that no inductive effects detectable at more than a few yards at most were apparent when using an ordinary telephone as a receiver. Even if the whole of the inductive effect is due to simple electromagnetic induction, the use of high-frequency currents is a distinct advantage.

The experiments here described are merely a preliminary to large scale experiments in actual closed circuit telegraphy the writer hopes to be able to try later on.

One drawback to the magnetic induction form of telegraphy is the rapid rate at which the effect falls off with the distance. In the case of true radiation at long distances the forces vary inversely as the distance, but a more rapid rate of decay, something between the inverse cube and inverse square, holds good for the inductive effect at least at short distances. Hence the use of magnetic oscillators as transmitters is never likely for this reason alone to rival the electric or open oscillator, but there may be circumstances under which it is possible to use them with advantage. In conclusion the author desires to mention that the actual measurements recorded in this paper were taken by his assistant, Mr. G. B. Dyke, B.Sc., with the kind help of Mr. K. W. McMillan, and to these gentlemen is due an acknowledgement of their share in the work, in making these observations with much intelligence and care.

Discussion.

Mr. A. Russell thanked the author for his clear exposition of the theory of electric oscillators and for the valuable formulæ he gave for magnetic oscillators. The experiments described were exceedingly interesting and instructive. It was, however, difficult to interpret some of the results obtained. He asked whether the author had been able to find a position for the secondary coil in which no current was induced in it. Possibly buried pipes or metallic conductors, iron girders in adjacent buildings or iron railings in the neighbourhood, might have had some influence in modifying the inductive effects between the coils.

Mr. Taylor said that the Post Office had tried arrangements similar

to those described for magnetic space telegraphy, and had found that the closed circuit method was inferior to that in which earth-connected vertical coils were used. The ratio between the received currents when using an earth-connected and a closed circuit was often as great as eighty to one, and he thought the improvements in the closed-circuit method would have to be very great to make it equivalent to the earthed-circuit method.

V. *On the Use of Variable Mutual Inductances.* By Albert Campbell, *B.A.** (From the National Physical Laboratory.)

[Plate IV.]

Contents.

1. Introductory.
2. Mutual versus Self Inductance.
3 & 4. Construction and Adjustment of Variable Mutual Inductances.
5, 6, 7 & 8. Uses of the same for measuring unknown Mutual Inductance, Self Inductance, and Capacity.
9. Appendix. The Scale of the Excentric-Coil Mutual Inductance.

1. *Introductory.*

The accurate measurement of small self inductances and capacities has of late years assumed an increased importance owing to its direct practical application in connexion with wireless telegraphy; and much important and interesting work has been done on the subject especially as regards the calculation and measurement of self inductance. For some time past I have found that suitably designed mutual inductances are of the greatest assistance in such measurements, and I incline to think that by their use some of the sources of error inherent in other methods are avoided. The chief method (B) here described was designed for the measurement of the small self inductances (1 to 200 microhenries) used in wave telegraphy, and it has proved most useful in such work. Before I proceed to describe the designs and the methods of use which I have found valuable, let us consider and compare mutual and self inductances in relation to their capabilities in practical use.

* Read October 25, 1907.

2. *Mutual versus Self Inductance.*

For a number of reasons mutual inductances are more easily dealt with than self inductances. Those of invariable value, when properly designed, have the following advantages:—

(*a*) The absolute values can be calculated with much more certainty from the geometrical dimensions, since the formulas for mutual inductance are of high theoretical accuracy while those for self inductance are much less satisfactory *.

(*b*) Unless the conductors are highly stranded, the current distribution varies with frequency and in general the self inductance will also vary. By keeping the two circuits at a relatively large distance from one another the mutual inductance is practically free from this effect.

(*c*) The effects of distributed capacity are probably less in mutual than in self inductances. In all cases the distributed capacity of one of the two coils can be made very small by sufficiently decreasing the number of turns in it (or opening them apart) while increasing the number in the other coil to keep the M constant.

When the mutual inductance is of the variable type, it can always be designed so that its value can be varied continuously from *negative to positive through zero*. This is a very great advantage, for with variable self inductance standards the incapability of reaching a zero value is a distinct drawback.

3. *Construction of Variable Mutual Inductances.*

For short range standards the construction used by Lord Rayleigh may be employed †. In this the primary coil is inside the secondary and can be rotated round a common diameter as in the Ayrton-Perry variable self inductances, and the whole positive range is comprised within 90° of turning, which gives a scale not nearly open enough for high accuracy. I have therefore designed a different and

* For example see Rosa, *Bull. Bur. Stands.* vol. ii. p. 161 (1906); and Strasser, *Ann. der Phys.* vol. xviii. p. 763 (1905).

† Phil. Mag. vol. xxii. pp. 469–500 (1886); Collected Papers, vol. ii. p. 155.

more elaborate arrangement which gives a much more extended scale. I proceed to describe the first model, which has been in use for a considerable time with most satisfactory results.

The general arrangement of the apparatus is shown diagrammatically in figs. 1 and 2, which are plan and side

Fig. 1.

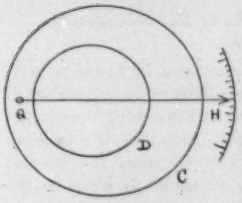
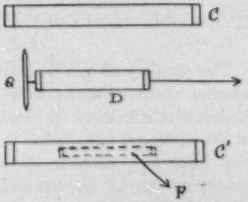


Fig. 2.



view respectively. The primary * circuit consists of two equal coaxial coils C and C′, which are connected in series, their windings being in the same direction of turning. The secondary consists of the coils D and F in series. Of these coils, D is movable, being mounted on an excentric axis Q so as to be free to turn in a plane parallel to those of C and C′ and midway between them. Rigidly connected with the movable coil is a pointer H which moves over a circular scale of about 180° in extent and graduated to read directly. The coil F is subdivided into ten sections, which are in series, each of them being 0·1 millihenry, and their junctions are brought to a set of separate terminals or studs with a turning head. The range of the moving coil extends from −0·002 to +0·11 millihenry. This gives a continuous range from 0 up to 1 millihenry, readable near zero to 0·02 microhenry, to 1 part in 500 at 0·1 millihenry, and to 1 in 5000 at 1 millihenry. The subdivision of the coil F is easily carried out by the following artifice. The coil is wound with uniformly stranded wire of ten insulated strands, all the strands are connected in series, and the whole adjusted

* It is only for convenience of description that the circuits are here distinguished as *primary* and *secondary*. They are really quite interchangeable.

to give 1 millihenry *. If the stranding has been properly done, it will be found that no one of the sections differs from its neighbours by more than 1 part in 1000, and each is 0·1 millihenry. The placing of the movable secondary coil midway between the planes of the two primary ones ensures that small axial displacements shall have very little effect on the mutual inductance.

4. *Adjustment and Calibration of Variable Mutual Inductances.*

The equality of any pair of sections can be tested by connecting them in series with their windings in opposition in circuit with a ballistic galvanometer and reversing the current in the primary. It should be noticed here that, if a primary coil have any number of secondary circuits, the mutual inductance to all the secondaries in series is equal to the algebraic sum of their separate mutual inductances (+ or − according to the direction of the winding). Owing to this very important property we can build up and step down in the values as easily as if we were dealing with resistances, and there is the further simplification that we can subtract as well as add the values. The marking of the scale and the setting of the coil F are done by comparison with a fixed standard mutual inductance such as I have described elsewhere †. The comparison may be made by Maxwell's method ‡, using a sensitive ballistic galvanometer or a vibration galvanometer as detector. When a vibration galvanometer is used as in fig. 3, it should be remembered that, for a balance, *two* conditions must be satisfied, viz.,

$$M_1/M_2 = R_1/R_2$$

and

$$L_1/L_2 = R_1/R_2,$$

where R_1 and R_2 include the resistances of the secondary coils. In order that the second condition may hold, it is

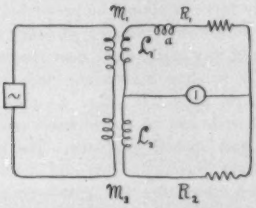
* Mutual Inductances with stranded wires were used many years ago by Prof. Brillouin (*Thèses presentées à la Faculté des Sciences de Paris*, 1882).

† Phys. Soc. May 1907; Phil. Mag. [6] vol. xiv. p. 494, Oct. 1907. Also see Proc. Roy. Soc. A. vol. lxxix. p. 428, June 5, 1907.

‡ Maxwell, Elect. and Mag. 2nd edition, ii. § 755.

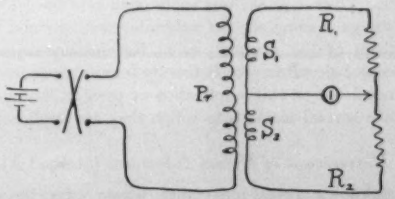
necessary to introduce into one of the secondary branches a coil a whose self inductance can be continuously varied; by alternate adjustments of R_1/R_2 and the self inductance of this coil, a balance is easily obtained. The fact that R_1 and R_2 are partly of copper coils is apt to introduce some inaccuracy. The copper resistance, however, can usually be largely swamped without losing too much of the sensitivity.

Fig. 3.



In connexion with this method I may mention that it is perfectly applicable to the case of comparing the mutual inductances between one primary circuit and two separate secondaries. This case is shown in fig. 4.

Fig. 4.



By this means a thorough intercomparison (or adjustment) can be made between the various sections of a subdivided mutual inductance. By further sets of secondary coils for higher multiples (10, 100, and so on) any desired range may be obtained. By stranding (or otherwise subdividing) the primary coils as well as the secondary ones a multiple-range

inductance is the result. In all cases the subdivision can be effected by other means than stranding the wires, which is merely used to avoid the separate adjustment of the sections. The main objection to the subdivision by stranding is that it increases the distributed capacity of the coils. With coils of many turns this might be a serious drawback, but with the relatively few turns required for 1 millihenry the effect can only be very small and I have not yet noticed it in practice. It should be kept in mind that, as in all standard inductances, eddy currents should be prevented by using only highly stranded wire for the winding of every section and by avoiding the use of any metal parts near the coils.

By varying the relative dimensions and positions of the two fixed primary coils CC′ (fig. 2) and the movable secondary D, the scale can be considerably modified, and can be designed to suit special purposes. The most generally useful type of scale becomes more and more open as the readings approach zero, for this gives good accuracy of reading even at very low values. Fortunately in the present instance this type of scale can readily be attained. Further discussion of this point will be found in an Appendix.

5. *Employment of Variable Mutual Inductances.*

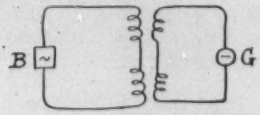
Variable inductances of this type may be used for several purposes. Their most obvious application is to the calibration of ballistic galvanometers for magnetic testing; and I have found them of the greatest value for quickly adjusting a galvanometer circuit to read B directly for any given iron ring. I need not dwell on this application at present, but proceed to describe several methods to which they are applicable.

6. *Measurement of Mutual Inductance* (Method A).

Any unknown mutual inductance, whose value lies within the range of the variable standard, can be at once determined by connecting the primaries of the unknown and the variable in series to B (fig. 5), a source of alternating or intermittent current, while the secondaries, with their windings in opposition, are connected in series to a vibration galvanometer G. The variable M is then adjusted to bring the deflexion to

zero, and the reading gives directly the value of the unknown M. This is an extremely simple method as it involves no knowledge of any resistances. A ballistic galvanometer and a commutated current may be used. This

Fig. 5.

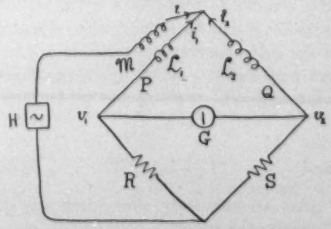


method does not apply to mutual inductances higher than the maximum value of the variable standard; for these another method will be described below (see § 8).

7. *Measurement of Self Inductance* (Method B).

In fig. 6 let a variable mutual inductance M whose *primary* includes the subdivided coil be connected into a Wheatstone's network, as shown, along with a self inductance L_2. Let the resistances of the arms be P, Q, R, and S respectively,

Fig. 6.



the self inductance of the arm P being L_1 (the secondary coils of M) and that of Q being L_2. Let H be a source of periodic current and G a vibration galvanometer tuned to resonance with it, so that we may take the wave form of the currents to be a sine curve. Let the instantaneous potentials of the three upper corners be v_1, 0, and v_2 respectively, and

the instantaneous values of the currents into the upper corner be i_1, i_2, and i as marked. Let $p=2\pi n$ where n is the frequency, and for convenience of writing let $p\sqrt{-1}$ be denoted by α, so that $\alpha^2=-p^2$. The mutual inductance M may be made positive or negative according to the way in which the coils are connected; and in all that follows we might write $\pm$M for M throughout. When the galvanometer shows a balance, $v_1=v_2$, and the instantaneous value of the current through G is zero.

Also
$$i=-i_1-i_2.$$

Accordingly we may write
$$(P+L_1\alpha)i_1-M\alpha i=(Q+L_2\alpha)i_2\,;$$
therefore
$$[P+(L_1+M)\alpha]i_1=[Q+(L_2-M)\alpha]i_2$$
also
$$Ri_1=Si_2.$$
Hence
$$S[P+(L_1+M)\alpha]=R[Q+(L_2-M)\alpha].$$

Equating the real and imaginary parts each to zero we have
$$SP=QR, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
and
$$S(L_1+M)=R(L_2-M). \quad . \quad . \quad . \quad . \quad . \quad . \quad *(2)$$

Exactly the same equations hold when the positions of the source and the galvanometer are interchanged.

The most useful case is when the noninductive arms are made equal, *i. e.* S=R; then (1) and (2) become
$$P=Q,$$
and
$$L_2-L_1=2M. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

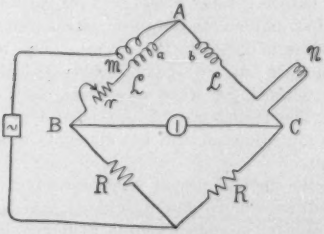
This case gives an extremely convenient way of measuring small self inductances, which is done as follows.

The arrangement is shown in fig. 7. The noninductive arms are equal (R, R). In the arm AB there is the secondary coil a of self inductance L in series with a practically non-

* (Nov. 22, 1907.) I find that a case slightly more complicated than that of Equation (2) has been worked out and applied in an ingenious manner by Grätz (Wied. *Ann.* vol. 50. p. 766, 1893).

inductive rheostat r. In the arm AC is placed a "balancing" coil b also of self inductance L and of resistance equal to or slightly greater than that of b. By adjusting r the bridge will balance when M=0. The small self inductance N to be measured is now inserted in series with coil b in arm AC, and a balance obtained by altering r and M. Then, by (3), N=2M. Thus N is found directly from the reading of M, and the range of values that can be measured runs from 0 up to twice the highest reading of the variable mutual inductance.

Fig. 7.



[For values of N above this range the more general case (equations (1) and (2)) may be used.] The L's of the coils a and b should be adjusted to equality once for all by putting M at zero and setting one of the coils. An exact setting is convenient, but not necessary, for if L_a and L_b differ slightly, they can be balanced (without N) by a small reading M_0. If M be the reading for balance when N is inserted, then $N=2(M-M_0)$*.

Even if a and b are well matched, it is always well to begin by reading their difference if any.

It will be noticed that the method is really a differential one; when N is introduced into the arm AC no alteration has

* (Oct. 1st, 1907.) If, instead of the coil N, a condenser K with series (absorption) resistance q be added, then

$$1/p^2K=2M,$$

and q = change in r.

to be made in the other arms except to increase the resistance of AB by an amount equal to the resistance of the coil N. But although it has *all the advantages of differential measurement, the reading can be made to give* N *directly* without having to take a difference at all. This is due to the use of the inductive balancing coil b *.

The method has the advantage that it does not require the knowledge of the absolute value of any resistance. The non-inductive bridge arms must be equal; to check the equality they can be interchanged. For the noninductive adjustable resistance r I usually employ a special rheostat consisting of two slightly flattened thin wires running parallel to one another at a few millimetres distance, with a sliding contact piece across them to complete the circuit. The inductance of such a rheostat can be approximately calculated, and may thus be allowed for when measuring very small self inductances. The inductance of the part added to compensate for the introduction of N has merely to be subtracted from the result.

In practice the method proves very convenient; with the variable mutual inductance described above, self inductances of any value from 0·1 microhenry up to 2000 microhenries can be measured directly without the bridge being altered in any way except in the rheostat r. In a later model shown in the accompanying Plate IV. the whole scale of the movable coil corresponds to 20 microhenries, and at this value it can be read to 1 or 2 in 1000—at 200 microhenries to 1 or 2 in 10,000. All the resistances of the coils are very low, and the sensitivity can be considerably increased by using M. Wien's method of connecting the vibration galvanometer to the bridge by means of a transformer of suitable ratio (n_1/n_2 small). The method will also give the difference between two unknown self inductances introduced into AB and AC.

The addition of a condenser to Method (B) gives an interesting corroborative method (Method C). If (fig. 8) the condenser K is in shunt with S,

* If coil b be made noninductive we revert to Maxwell's method of comparing the M of a pair of coils with the L of one of them. Equation (2) then reduces to $L/M = -(1+R/S)$. Maxwell, 'Elect. & Mag.' 2nd edit. vol. ii. § 756.

it may be shown that, for a balance,

$$SP-RQ=p^2RSK(L_2-M), \quad . \quad . \quad . \quad . \quad (4)$$

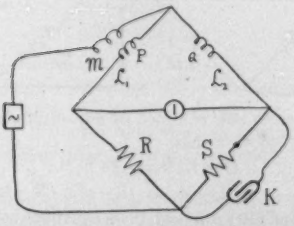
and $$S(L_1-M)-R(L_2+M)=QRSK. \quad . \quad . \quad . \quad (5)$$

When R=S and $L_1=L_2=L$,

$$P-Q=p^2RK(L-M), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and $$2M=QRK. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Fig. 8.



Equation (7) gives a check of K against M without having to measure the frequency. The adjustment is dependent on p, however, and therefore a steady frequency is necessary, which is not required in Carey Foster's Method.

From (6) and (7) we get

$$Q(P-Q)=2p^2M(L-M). \quad . \quad . \quad . \quad . \quad (8)$$

8. *Measurement of Mutual Inductance greater than Standard* (Method D).

When the unknown M is beyond the range of the variable standard the following method is applicable.

Let the primaries of the variable standard and the unknown M be connected up as in fig. 9, adding to the latter if necessary a coil c of sufficient self inductance to make N considerably greater than L. Let a balance be obtained (by Maxwell's Method) and hence

$$PS=RQ, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and $$NS=LR. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Now, without altering any other branches, let the secondary coils be introduced (in opposition) into the galvanometer circuit as shown in fig. 10.

Fig. 9.

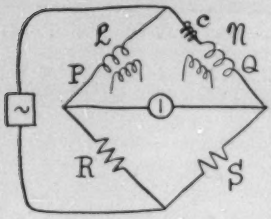
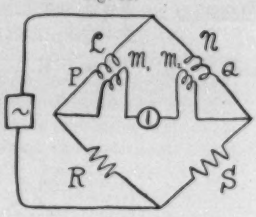


Fig. 10.



If a balance be now obtained by adjusting the variable M, then it is easy to show that

$$(P+R)M_2=(Q+S)M_1, \quad . \quad . \quad . \quad . \quad (11)$$

and $$PS-p^2M_2(L-M_1)=QR-p^2M_1(N-M_2). \quad . \quad (12)$$

Since (9) and (10) still hold, these equations both reduce to

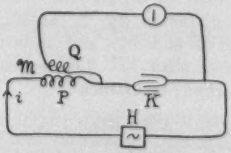
$$RM_2=SM_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

which gives M_2 immediately in terms of the reading M_1 and the resistances R and S (neither of which include copper coils). The self inductance of the added coil c need not be known, but if R and S have not a slide wire between them, c ought to be slightly adjustable, which is easily arranged by having part of it a small coil of a few turns that can be shifted about over the remainder.

9. *Measurement of Capacities* (Method E).

As in fig. 11, let the primary P of the variable standard

Fig. 11.



be connected in series with a condenser K to a periodic

source H, while the secondary Q is connected to a vibration galvanometer as shown. Let M be adjusted until the galvanometer gives no deflexion, in which case we have

$$\pm \mathrm{M}\alpha i = \frac{1}{\mathrm{K}\alpha} i, \quad \ldots \ldots \quad (14)$$

or $$\pm p^2 \mathrm{MK} = -1.$$

The secondary must be connected in such a direction that the lower sign is taken, and thus

$$p^2 \mathrm{MK} = 1, \quad \ldots \ldots \quad (15)$$

where M is in henries and K in farads.

It is interesting to notice that this formula is similar in form to that for resonance with a condenser and self inductance, viz. $p^2 \mathrm{LK} = 1$. With self inductance we usually tune to get a maximum current through the galvanometer; with mutual inductance, on the other hand, we tune for zero current. If M is in henries and K in microfarads, equation (15) gives

$$n \doteqdot \frac{159{\cdot}2}{\sqrt{\mathrm{MK}}}, \quad \ldots \ldots \quad (16)$$

and if M and K are known, the frequency can be determined.

If the frequency is not fairly high, the product MK becomes inconveniently high, *e. g.* M = 1 henry, K = 10 mfds. gives $n \doteqdot 50 \sim$ per sec. In contrast, $n \doteqdot 10{,}000$ for 256 microhenries and 1 mfd.

The following is a practical example of the method:—The source was a microphone hummer for $2000 \sim$ per sec. and the detecting instrument a tuned telephone. A balance was obtained for K = 10·06 mfds. and M = $0{\cdot}626_6$ millihenry, whence the frequency was found to be $2004 \sim$ per sec. With another condenser (a carefully calibrated subdivided one), capacities of 1·000, 0·900, 0·800, 0·701, 0·601, and 0·551 gave

$n = 1999$, 2005, 2009, 2001, 2004, and $2001 \sim$ per sec.

respectively; the variable M was not, however, so accurately readable in this case.

In this method it is very important that the insulation

resistance S of the condenser shall be high. If the leakage is not negligible, equation (14) becomes

$$\pm \mathrm{M}ai = \frac{\mathrm{S}}{1+\mathrm{SK}\alpha} i$$

$$= \frac{\mathrm{S}(1-\mathrm{SK}\alpha)}{1+p^2\mathrm{S}^2\mathrm{K}^2} i. \quad . \quad . \quad . \quad . \quad (17)$$

Putting the real part equal to zero we have

$$1/(1+p^2\mathrm{S}^2\mathrm{K}^2)=0,$$

showing that an exact balance can only occur when $p^2\mathrm{S}^2\mathrm{K}^2=\infty$. In practice the insulation resistance of a good mica condenser is quite high enough to give a good balance; even a good paraffined paper condenser will work fairly well, but an ordinary one gives only a minimum deflexion on the galvanometer *.

The application of this null method to the high frequencies used in wave telegraphy is made difficult by reason of the coexistence of two frequencies in the waves used.

Appendix.

Scale of Excentric-Coil Variable Mutual Inductance.

The problem of finding the mutual inductance between two circles (thin circular coils) not in the same plane and whose axes are parallel has not yet been solved, as far as I know. For any particular case the result may be calculated by quadratures as follows.

In fig. 12 let the two circles be H and G, of radii a and A respectively, b being the distance between their planes, and q the distance between their axes. Let the circle H′ be the projection of H on the plane of G, and let BCD be an elementary strip concentric with H′ of radial width ∂r, where $\mathrm{O'B}=r$. Let $\mathrm{CO'B}=\theta$. Then the M between circle H and strip BCD

$$= \frac{\theta}{\pi} \cdot (\mathrm{M \text{ of circle H and annulus TBD}})$$

$$= \frac{\theta}{\pi} \cdot \frac{\partial}{\partial r}\left[-4\pi n_1 n_2 \sqrt{ar}\left\{\left(k-\frac{2}{k}\right)\mathrm{F}_1 + \frac{2}{k}\mathrm{E}_1\right\} 10^{-9}\right] \partial r,$$

* This may, however, be due more to absorption than leakage.

or

$$\partial M = \frac{\theta}{\pi} \cdot \frac{2\pi n_1 n_2 10^{-9}}{\sqrt{(a+r)^2+b^2}} \left\{ 2a(F_1 - E_1) - (r-a)\left(\frac{k}{k'}\right)^2 E_1 \right\} \partial r,$$

where . . . (18)

$$k = 2\sqrt{ar}/\sqrt{(a+r)^2+b^2} = \sin\gamma,$$

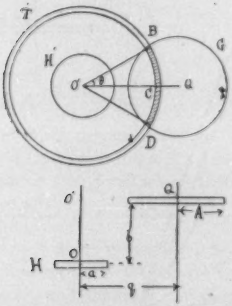
$$k' = \cos\gamma,$$

and F_1 and E_1 are complete elliptic integrals to modulus k.

Also

$$\theta = \cos^{-1}\frac{r^2+q-A}{2rq}.$$

Fig. 12.



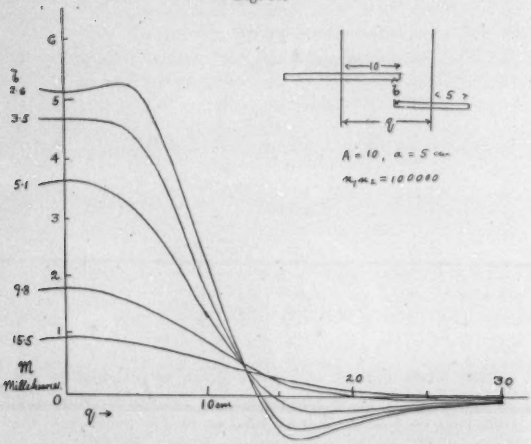
Thus when a, A, b, and q are given, by calculating ∂M by (18) for a series of equally spaced successive values of r, from $r = q \sim A$ to $q + A$, and adding in the usual way, the value of M can be approximately found. By this process, which is rather laborious, the following approximate values were found, when $A = 20$, $a = 10$, $b = 5$ cm and $n_1\ n_2 = 100{,}000$:—

q. cm.	M. millihenries.
0	9·3
5	9·7
10	9·9
15	6·7

It is interesting to notice that the maximum value of M

does *not* occur when the axes of the coils are coincident ($q=0$). To avoid the labour of the calculation by quadratures, some rough experiments were made with coils of finite section (about $0{\cdot}6 \times 0{\cdot}6$ sq. cm.). From one set of these the series of curves shown in fig. 13 was obtained. The coils were of mean radii $A=10$, $a=5$ cm. respectively, n_1 n_2 being 100,000, and q and b were given various values. It will be seen that for $b=2{\cdot}6$ the maximum M does not occur

Fig. 13.

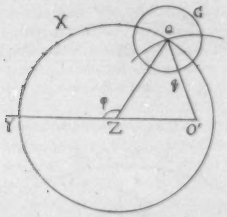


at the symmetrical position (just as in the case calculated, above). For equal coils, however, the coaxial position gives the maximum M. The curves also show that the value of q for which M vanishes and changes sign becomes greater and greater as the mean planes of the coils are set at greater distance from one another. The ratio of A/a has been made 2 in this example, as that is a suitable proportion for use in a practical instrument. Having now the curves connecting M with q, the distance between the axes of the two coils, it is easy to investigate the variety of angular scales obtainable

by turning one of the coils round an axis parallel to its own axis.

Let a definite value of b be taken, say $b=5$, which is convenient in practice, and in fig. 14 let the smaller coil be

Fig. 14.



movable about an axis perpendicular to the plane of the paper at Z, and let the radius $ZQ=\rho$. Let the distance ZO' of the axis Z from O′ (the point where the axis of the other coil cuts the plane of the paper) be h. Let the angular scale reading $YZQ=\phi$.

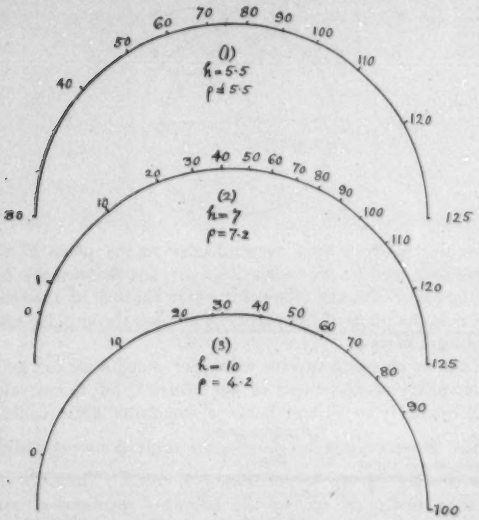
Then by choosing various values for ρ and h we can get a wide variety of movement of the centre Q (of the movable coil) relatively to O′, and hence a variety of scale calibrations. Since $\cos\phi=\frac{q^2-h^2-\rho^2}{2h\rho}$, the angle ϕ corresponding to a reading M can be got from q, h, and ρ. The scale can be most easily set off by the following geometrical construction. Draw a circle with centre Z and radius ρ, and make ZO' equal to h. Then find on the ($b=5$) curve in fig. 13 the q corresponding to a particular value of M. With this value of q as radius, a circle with centre O′ will cut the circle X at the point of the scale required. By taking successive values of M the direct-reading scale can be set off round the circumference of the circle X.

As an example, in fig. 15 are given three scales for $h=5{\cdot}5$, $\rho=5{\cdot}5$; $h=7$, $\rho=7{\cdot}2$; and $h=10$, $\rho=4{\cdot}2$ respectively, when $A=10$, $a=5$, $b=5$, and $n_1 n_2 \doteqdot 33{,}000$.

Although these scales give very different ranges, the

general characteristic of all three is an openness at both ends; no proportions that I have tried tend much to make the scale uniform throughout. The openness at the lower values is a distinct advantage, however, when dealing with

Fig. 15.



small inductances; but care must be taken, when marking that part of the scale, that the subdivisions shall be accurately marked and not put in merely by interpolation.

Scale (2) in fig. 15 is the best of the three and is nearly that which I have used in practice.

In conclusion I desire to express my best thanks to Dr. Glazebrook, for his kind encouragement and valued advice.

Sept 21, 1907.

Discussion.

Dr. Fleming asked the author how his method compared in accuracy with the Carey Foster method, using a commutator to add up the impulses. He had frequently measured mutual inductance by determining the self-inductances of the primary and secondary joined in series, with the windings first helping each other and then opposing each other. He asked if the author's methods were applicable when dealing with the high frequencies used in wireless telegraphy.

Mr. A. Russell agreed with the author in thinking that mutual inductance was sometimes easier to calculate and easier to measure than self-inductance. The variable mutual inductance standard shown was very compact, and would be a great acquisition to a physical laboratory. In calculating self-inductance the variable current density over the cross-section of the wire sometimes brought in serious difficulties. The measurement of self-inductance in these cases might possibly be made by Mr. Campbell's method, but the limitations introduced into his formulæ by the assumption that the back E.M.F. was proportional to the rate at which the total current was changing would have to be considered.

Mr. Campbell, in reply to Dr. Fleming, said the methods of measuring mutual inductance described in the paper were more convenient and direct reading than that of Carey Foster, although they were not so absolute. They were of higher accuracy than the difference method, in which the primary and secondary were put in series and their combined self-inductance measured both with the windings helping and opposing. In reply to Dr. Fleming and Mr. Russell, the methods of measurement described were not intended, in general, for use with high frequencies of the order of those used in wave telegraphy, where resonance methods appear to be the only ones available. Mr. Campbell mentioned that the correction for the small inductance of the rheostat in method (B) could be got rid of by making the rheostat of constant inductance, this being attained by running a manganin and a copper wire side by side with a travelling contact across them, the current entering by the copper at one end and leaving by the manganin at the other.

VI. *A Freehand Graphic way of determining Stream Lines and Equipotentials.* *By* L. F. RICHARDSON *.

[Plate V.]

SCHEME OF PAPER.

I. On the need for new methods.
II. The first idea of freehand solution and confirmation of its accuracy.
III. The conditions which the solution of $\nabla^2 V = 0$ must satisfy in order that it may be determinable by a single graph.
  (*a*) When the guiding lines are normal to a family of surfaces. Possible types—test cases.
  (*b*) Thin shells.
  (*c*) Screw symmetry—example.
IV. Points and lines of equilibrium.
V. Equations other than Laplace's—variable conductivity.
VI. Boundary conditions.
VII. Miscellaneous notes on draughtsmanship.
VIII. Estimation of errors.

I. *The Need for New Methods.*

THE Laplacian differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

has received an extraordinary amount of attention during the last century owing to the great number of physical quantities, the space distribution of which can be determined from its integrals. The analytical integrals hitherto obtained by such means as Fourier series, Bessel functions, spherical and other harmonics make it possible to determine the distribution when the boundary conditions bear relation to certain simple types of surface, such as parallelipipeds, cylinders, spheres, ellipsoids, anchor rings, &c.

Now for physical research this is well enough. It is usually possible to arrange the instruments so that the parts involved are of these simple forms. The wires may be wound in circular rings of small cross-section, as in Helmholtz's galvanometer. The pieces of substance for the measurement

* Read November 8, 1907.

of specific properties may be shaped into square bars, as in Forbes's experiments on the flow of heat. Or, as in Kelvin's attracted disk electrometer, parallel plates may be made practically infinite by his device of the guard-ring.

But for the purposes of the engineer this is of very limited application. If he is to handle partial differential equations freely, they must be applicable to bodies of most various shapes, such, for example, as the toothed core-plates of dynamos, the water surrounding ship shells and screw propellers, the space between turbine blades, and a host of other forms, too irregular to be readily described.

Further than this, the method of solution must be easier to become skilled in than are the usual methods with harmonic functions. Few have time to spend in learning their mysteries. And the results must be easy to verify—much easier than is the case with a complicated piece of algebra. Moreover, the time required to arrive at the desired result by analytical methods cannot be foreseen with any certainty. It may come out in a morning, it may be unfinished at the end of a month. It is no wonder that the practical engineer is shy of anything so risky.

Harmonic functions have, however, one very strong point in comparison to the methods put forward in this paper, and that is their accuracy. Once we have determined V as an infinite series of harmonic functions, it is usually not much more labour to obtain an accuracy of 1 in a million than of 1 in ten.

Now it is true that in the determination of absolute electric standards measurements are made to 1 in 100,000 or to an even greater refinement. But for most chemical and physical work 1 in 1000 is more like the limit attained. And in any new branch of research, two, five, or even ten per cent. are very welcome. The root of the matter is that the greatest stimulus of scientific discovery are its practical applications. And here, in the design of machinery for example, cost rules everything, and this can seldom be foreseen as near as 1 per cent.

To sum up. The existing methods of solving Laplace's equation are susceptible of great accuracy, but they are slow and uncertain in time and, most serious of all, they can only

be applied to very special boundary conditions. There is obviously a demand for a method of solving that group of partial differential equations — of which we may regard Laplace's as the simplest type—which shall, if necessary, sacrifice accuracy above 1 per cent., to rapidity, freedom from the danger of large blunders, and applicability to more various forms of boundary surface.

II. *The First Idea of Freehand Solution.*

The real simplicity of the space distributions of electric and magnetic phenomena,—so much disguised in the algebraic integrals of the differential equations, but rescued from confusion and clearly set forth by the vector analysts, Heaviside, Walker and others,—leads one to hope for equally simple methods of calculating their numerical values with reference to any boundary whatever.

The beautiful figures of stream and equipotential surfaces published by Maxwell, Lamb and others as the result of harmonic analysis, and by Hele Shaw as the result of experiment, suggest that by imitating their characteristic properties freehand we may, in some small part, attain the result desired.

Maxwell in § 92 of his 'Elementary Treatise on Electricity and Magnetism' speaks of tentative methods of altering known solutions of the Laplacian equation by drawing diagrams on paper and selecting the least improbable. The object of the present thesis is to point out that this method can do far more than merely alter known results, and that it may be so far from tentative as to yield an accuracy of one per cent of the range.

This method of treating potentials, although still far from combining all desirable qualities, and suffering from the restriction to certain types of symmetry, yet from its great freedom within those types may, it is hoped, supply to a certain extent the demand we have indicated.

On turning to Maxwell's figures and picking out those in which V is independent of z so that we have

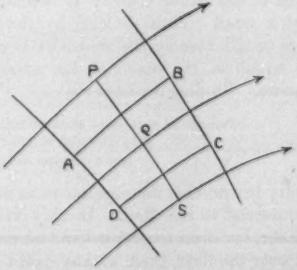
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0,$$

it will be seen that while the curves are of the most various shapes yet the chequerwork of all the diagrams possesses these two properties in common :—(1) the corners are orthogonal, (2) when the chequers are small enough the ratio of their length to breadth is the same in all parts of the field.

The proof of this follows most conveniently from the consideration of the motion of a liquid when the lines of flow lie in parallel planes and the motion is the same at all points of any normal to these planes. Draw three adjacent stream lines defining two adjacent tubes of flow.

Take two points A and B on the mid line of one tube, and from A and B draw normals to the direction of flow cutting the mid line of the other tube in D and C respectively.

Fig. 1.



Halfway between AD and BC draw a line PQS normal to the direction of flow so that PQ is the width of one tube and QS of the other. Now if the fluid is incompressible and we have drawn the tubes so that the flow in each is the same, then the respective velocities are to one another inversely as PQ and QS. Let the velocities be $\frac{k}{PQ}$ and $\frac{k}{QS}$. Next let us take the line integral of the velocity round the small rectangle ABCD. The sides AD and CB are normal to the flow and so contribute nothing. The sides AB and CD

contribute

$$AB \times \frac{k}{PQ} - \frac{DC \times k}{QS} = k\left\{\frac{AB}{PQ} - \frac{DC}{QS}\right\}.$$

Now $\frac{AB}{PQ}$ is the ratio of the length along the flow to the breadth across the flow of the small chequer which has A, Q, B, P, at the mid points of its four sides. It will be convenient to have a special name for this quantity, and I propose to call it the "chequer ratio" with the understanding that length along the flow is always in the numerator, and that the chequer is so small that its size no longer causes an appreciable deviation from the accuracy obtained by using infinitesimals. Then we have :—

$$\text{Line integral of the velocity around ABCD} = k \times \left\{\begin{array}{l}\text{Difference between successive chequer ratios in a direction perpendicularly across flow.}\end{array}\right.$$

Now the curl of the vector velocity is defined as the line integral round a small circuit divided by the area of that circuit—that is in this case by the area ABCD which will in the limit be equal to the mean of the areas of the two adjacent chequers. So that we have :—

$$\text{curl of the velocity} = k\,\frac{\text{difference of successive chequer ratios in a direction perpendicular to velocity}}{\text{mean chequer area}}.$$

If the velocity has no curl the chequer ratio must not vary along any line normal to the flow. It may vary from one normal to another, but if on the other hand we prefer to make it constant all over the field, then at any point the distance between successive normals will be inversely as the flow, so that these normals will be contours drawn at equal intervals of a velocity potential.

To return : since the fluid is incompressible the condition for the existence of a stream function is satisfied, and since the stream-lines are drawn so that the flow between each successive pair is the same, it follows that these stream-lines are the contours drawn at equal intervals of a stream function ψ. Now it is proved in works on Hydrodynamics that $\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}$ is equal to the curl of the vector velocity.

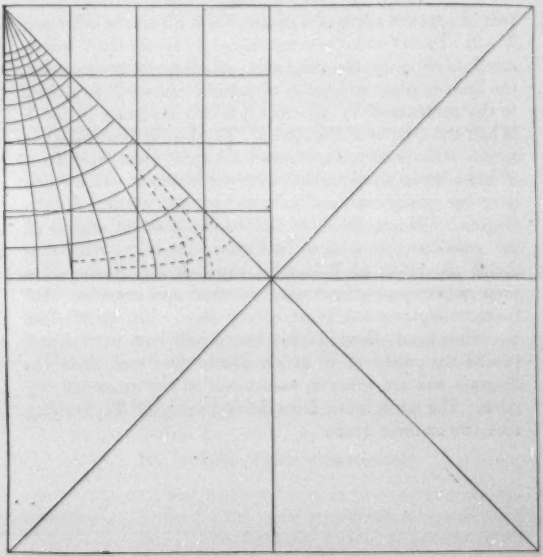
Therefore :—

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=k\,\frac{\text{difference of successive chequer ratios in a direction perpendicular to the contours of }\psi}{\text{mean chequer area}}.$$

And since ψ may be any one-valued function of position on the plane, it is seen that all hydrokinetical considerations have been eliminated from the above equation, which is purely a proposition in differential geometry. The only implication being that contours are drawn at equal intervals of ψ whatever be its physical meaning.

To draw chequers freehand so as to satisfy a difference relation of this sort between the chequer ratios is likely to be toilsome, and we will here consider only the case when $\nabla^2\psi=0$.

Fig. 2.



Supposing then that a chequerwork has been obtained in which the chequer ratio is everywhere the same and in which the given boundary conditions are satisfied, then by the uniqueness of the solution of $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$ this chequerwork gives us what we want.

It remains to be shown what accuracy may be expected from the freehand method. This is of course largely a personal matter. I exhibit my own handiwork. Others will doubtless obtain greater precision. Throughout this paper I have chosen test cases in which the analytical verification should not be too difficult. Hence the diagrams look rather stiff and formal and do not in any way do justice to the freedom of this graphic method.

Example of the solution of $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$.—Along one pair of opposite edges of a square $V = 1$, along the other pair $V = 0$. Find V at all points inside. By symmetry V will be equal to $\frac{1}{2}$ along the diagonals. And again by symmetry, the lines joining mid-points of opposite sides will be normal to the contours of V. So that it is only necessary to find V in half one-quarter of the square. Further, by symmetry, at a corner of the square the contours drawn at equal intervals of V must make equal angles with one another. One starts then by ruling out an accurate square, putting in the diagonals, joining the mid-points of its sides and setting off the equal angles with a protractor. It is convenient to divide the range of V into ten equal parts. Having thus prepared the paper, lines were sketched and amended until further improvement became very slow. The pencil-lines were then firmly fixed in ink. Coordinate lines were drawn in and the values of V at six points were read from the diagram and are given in parenthesis in the accompanying table. The whole work from the beginning of the drawing took two or three hours.

				·5
			·5	(·47) ·466
1·0		·5	(·40) ·396	(·365) ·364
1·0	·5	(·307) ·300	(·23) ·223	(·20) ·202
·5	0	0	0	0

Not until this had been done did I look up the correct values which had been computed from the analytical solution

$$V = \frac{4}{\Pi} \sum_{m \text{ odd}} (-1)^{\frac{m-1}{2}} \frac{1}{m} \operatorname{sech} \frac{m\Pi}{2} \cos mx \,.\, \cosh mz.$$

These are given in the table beside the numbers read from the graph. From these we find the errors +·007, +·007, −·002, +·004, +·001, +·004. Treating these as all of the same sign, their mean is ·0042.

The error of a graph may well be compared with the total range of V within which the determination was made freehand. In this case the range was 0·5 so that the mean error was 0·84 per cent. of the range, a degree of accuracy which would be sufficient for many purposes.

Having shown that the freehand method is a practical one in a plane, it will be well next to enquire to what types of symmetry it may be extended.

III. *Possible Types of Symmetry.*

Inasmuch as a single chequerwork is to determine the distribution in the whole of the space considered, we are confined to two coordinates. The freehand method at present offers

nothing to compete with the analytic forms in which three coordinates appear, such as:

$$V=\sum_{m,n}(A_m \sin mx+B_m \cos mx)(C_n \sin ny+D_n \cos ny)(Fe^{-\sqrt{m^2+n^2}.z}+Ge^{+\sqrt{m^2+n^2}.z}),$$

$$V=\sum_{m,k} e^{\pm kz}(A_m \cos m\phi+B_m \sin m\phi)J_m(kr),$$

(Whittaker, 'Modern Analysis,' p. 318)

$$V=\sum_{n,m} C_m r^m(A_n \cos n\phi+B_n \sin n\phi)(\sin\theta)^n.\frac{d^n P_m\mu}{d\mu^n},$$

(Byerly, 'Fourier's Series,' p. 196)

or others like them.

The expressibility of V in terms of two coordinates implies that V is constant along a certain family of lines in space, namely, the intersections of the surfaces over which the said two coordinates are respectively constant. Any particular type of symmetry is most conveniently distinguished by specifying the family of lines along each of which V must be constant.

As it will frequently be necessary to refer to these lines and to distinguish them from the normals to the surfaces V=constant, I propose to call them the "guiding lines."

It is indeed conceivable that by adding together several space distributions in each of which V is constant along a different family of lines, we might attempt the solution of problems which it may be impossible to treat by two co-ordinates directly, such for example as the motion of a perfect fluid past a three-bladed screw-propeller, or the electrostatic field due to a ring of electrons. It may even be possible to treat the most general distributions by means of sections of the potential surfaces drawn on the leaves of a book of tracing-paper. But these extensions must be left to those who desire the results. This paper deals only with two coordinates.

Case (a). The guiding lines are everywhere orthogonal to a family of surfaces.—Let these be the surfaces over each of which $\gamma=F_3(x, y, z)$ is constant.

Then choosing a particular surface, say $\gamma=\gamma_0$, we wish to draw thereon a chequerwork of orthogonal lines, and we wish this chequerwork, by the motion of each point of it along the guiding line at that point, to sweep out two families of surfaces

in space, in such a way that one family may be equipotentials and the other stream-surfaces. This requires that these two families, which we may denote by

$$\alpha = F_1(x, y, z) = \text{const.}, \qquad \beta = F_2(x, y, z) = \text{const.},$$

should be everywhere orthogonal. Therefore the surfaces α, β, γ are mutual orthogonal, and consequently the surfaces γ must satisfy the condition necessary in any member of a triply orthogonal system (Salmon, 'Geometry of Three Dimensions,' 4th ed. §§ 476 to 486).

But more than this. For we wish to be unrestricted as to the direction of the orthogonal traces of α and β drawn upon the surfaces γ. Therefore, since three mutually orthogonal surfaces necessarily intersect in their lines of curvature (*loc. cit.* § 304), it follows that at every point of the surfaces γ there are lines of curvature in every direction. The only form which possesses this property is the sphere or its limit the plane. Therefore the surfaces γ are either spheres or planes. This is necessary. We have not proved that it is sufficient. As frequent reference will be made to the theorems proved in Lamé's *Leçons sur les coordonnées curvilignes*, it will be convenient to employ expressions such as (Lamé, § xi. 15) to indicate equation 15 of § xi. of this treatise. The relation of our notation to Lamé's is that his $\rho\ \rho_1\ \rho_2$ are replaced by $\alpha\ \beta\ \gamma$ and that $\alpha\ \beta\ \gamma$ are used as subscripts respectively instead of absence of subscript, 1 and 2. Otherwise the notations are the same.

In particular, if F is any function of position, we will denote by H_F the quantity

$$\frac{1}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}}$$

which is the reciprocal of the space-rate of F along the normal to the surface F = constant.

Consider the lamina bounded by the two spheres γ and $\gamma + \delta\gamma$. The thickness of the lamina is $H_\gamma \delta\gamma$. If $\alpha = V =$ the potential, then the surfaces $\beta =$ constant are surfaces of flow. Denoting in like manner the distances between two adjacent surfaces of these families by $H_\alpha \delta\alpha$, $H_\beta \delta\beta$, we see that $H_\alpha \delta\alpha$

and $H_\beta \delta\beta$ are the length and breadth of a chequer traced on the surface γ.

A tube of flow is bounded by the four surfaces β, $\beta+\delta\beta$, γ, $\gamma+\delta\gamma$. And its cross section is therefore $H_\beta \,.\, H_\gamma \,.\, \delta\beta \,.\, \delta\gamma$.

Now if the flux has no divergence, then along a tube of flow magnitude of flux multiplied by cross-section = constant. But the magnitude of flux is equal to the negative space-rate of the potential α along the line of flow, and this is $-\frac{1}{H_\alpha}$. Therefore along a line of flow $\frac{H_\beta \,.\, H_\gamma}{H_\alpha} \,.\, \delta\beta \,.\, \delta\gamma$ must be constant in order that the vector space-rate of the scalar α—the Hamiltonian $\nabla\alpha$—shall have no divergence. Or equivalently the condition that

$$\nabla^2\alpha = 0 \text{ is } \frac{\partial}{\partial\alpha}\left(\frac{H_\beta \,.\, H_\gamma}{H_\alpha}\right) = 0.$$

Now we may by freehand trial and amendment so arrange the orthogonal lines on the surface γ_0 that the above relation shall hold true on the surface γ_0; but we must further enquire what conditions the spheres γ must satisfy in order that $\frac{\partial}{\partial\alpha}\left(\frac{H_\beta \,.\, H_\gamma}{H_\alpha}\right) = 0$ shall be true for all values of γ when it is true for one γ_0; and this moreover when α and β are otherwise undetermined.

At this stage the fact that the surfaces γ are spheres makes a remarkable simplification. For supposing for a moment that they did not possess this property and that r_γ^α and r_γ^β were their principal radii of curvature at any point, then by (Lamé, § xxx. 24)

$$\frac{1}{r_\gamma^\alpha} = \frac{h_\gamma}{h_\alpha} \cdot \frac{\partial h_\alpha}{\partial\gamma}, \qquad \frac{1}{r_\gamma^\beta} = \frac{h_\gamma}{h_\beta} \cdot \frac{\partial h_\beta}{\partial\gamma},$$

where $h_\alpha = \frac{1}{H_\alpha}$, and similarly for β and γ.

Equating the two curvatures,

$$h_\beta \cdot \frac{\partial h_\alpha}{\partial\gamma} = h_\alpha \cdot \frac{\partial h_\beta}{\partial\gamma}.$$

Now

$$\frac{\partial}{\partial\gamma}\left(\frac{H_\beta}{H_\alpha}\right) = \frac{\partial}{\partial\gamma}\left(\frac{h_\alpha}{h_\beta}\right) = \frac{-h_\alpha \frac{\partial h_\beta}{\partial\gamma} + h_\beta \frac{\partial h_\alpha}{\partial\gamma}}{h_\beta^2}$$

and therefore vanishes; so that the chequer ratio $\frac{H_\alpha}{H_\beta}$ is a function of α and β only.

As has already been stated, to make $\nabla^2\alpha=0$ on the surface γ_0 we must have

$$\frac{\partial}{\partial\alpha}\left(\frac{H_\beta H_{\gamma_0}}{H_\alpha}\right)=0,$$

which may conveniently be arranged by making

$$\frac{H_\alpha}{H_\beta}=k_0 H_{\gamma_0}$$

where k_0 is a constant, so that the chequer ratio is given as a function of position on γ_0. This is more than sufficient in that it makes

$$\frac{\partial}{\partial\beta}\left(\frac{H_\beta H_{\gamma_0}}{H_\alpha}\right)$$

vanish as well, but the loss of generality involved is found not to matter, while the simplicity gained is a great convenience. Next, because $\frac{H_\alpha}{H_\beta}$ is independent of γ it follows that on any other surface γ_1 we still have

$$\frac{H_\alpha}{H_\beta}=k_0 H_{\gamma_0}.$$

But if $\nabla^2\alpha=0$ is to be satisfied on this second surface we must there have

$$\frac{H_\alpha}{H_\beta}=k_1 H_{\gamma_1}$$

where k_1 is a second constant. Therefore regarding γ_0 as fixed and $\gamma_1=\gamma$ as movable we have $H_\gamma=H_{\gamma_0}\times$ a function of γ only.

But H_{γ_0} is a function of α and β only.

This relation is equivalently expressed by the two equations

$$\frac{\partial^2}{\partial\gamma\partial\alpha}(\log H_\gamma)=0, \qquad \frac{\partial^2}{\partial\gamma\partial\beta}(\log H_\gamma)=0,$$

whence by (Lamé, § xxx. 24)

$$\frac{\partial}{\partial\gamma}\left(\frac{1}{h_\alpha r_\alpha^\gamma}\right)=0, \qquad \frac{\partial}{\partial\gamma}\left(\frac{1}{h_\beta r_\beta^\gamma}\right)=0.$$

But if the surfaces γ are planes, then by the equations

(Lamé, § xxx. 24) already quoted

$$\frac{\partial}{\partial\gamma}\left(\frac{1}{h_\alpha}\right)=0=\frac{\partial}{\partial\gamma}\left(\frac{1}{h_\beta}\right)$$

and consequently $\frac{1}{r_\alpha^\gamma}$ and $\frac{1}{r_\beta^\gamma}$ are independent of γ. But it is shown by Lamé (§ xxxviii.) that the curvature of the arc of intersection of the surfaces α and β is equal to

$$\sqrt{\left(\frac{1}{r_\alpha^\gamma}\right)^2+\left(\frac{1}{r_\beta^\gamma}\right)^2}.$$

So that if the radius of curvature of this arc be p then p is independent of γ. But p is equal to the length of the normal from the point considered onto the line of ultimate intersection of two consecutive planes of the family γ which pass one on either side of the point considered. As the plane moves this length must remain constant. And as this is to be true for every point in space, it is easy to see that if the surfaces γ are planes they must intersect in a common axis. We have in this case symmetry about an axis. Or if the axis be at an infinite distance, the planes are parallel, and we have V independent of one of the Cartesian coordinates x, y, z. But if, quite generally, the surfaces γ are spheres we have only $\frac{h_\alpha}{h_\beta}$ independent of γ and therefore $\frac{r_\alpha^\gamma}{r_\beta^\gamma}$ independent of γ. If the centres of the spheres γ lie in a straight line, then since the orthogonal traces of the surfaces α and β on a sphere γ may turn round anyhow, we may choose for β the planes intersecting in the line of the centres of the spheres γ. Then $r_\beta^\gamma=0$, and consequently r_α^γ is independent of γ so that the traces of $\alpha=$const. on the planes β are circles. This is the system of toroidal coordinates which has been treated by Professor Hicks in Phil. Trans. 1881, Part II. Now the above reasoning would lead us to expect in these a type of symmetry which can be dealt with by two coordinates—other than symmetry about an axis. But on referring to Hicks's formulæ it is easy to show that this is not possible, for if V be made independent of either of those two of his coordinates which determine position in a plane passing through the axis, then the other of these two will not divide out of the equation $\nabla^2 V=0$, so that all three coordinates

must still be present in the integrals. Clearly then, our deductions, though necessary, are not sufficient. I have little doubt that the omission lies in this: that to leave us unrestricted as to the direction of the orthogonal traces of α and β upon the surfaces γ, it is not sufficient that the surfaces γ should be spheres. For the curves normal to γ which we have called the "guiding lines" must be such that they form one set of lines of curvature of any surface whatever passing through them. To satisfy this condition it seems likely that except when the radius of the spheres γ is infinite, the guiding lines will have to be straight and the spheres concentric. This is the symmetry when V is independent of the radius in spherical coordinates, but may vary anyhow with the latitude and longitude.

The only case remaining uninvestigated is that in which the surfaces γ are spheres with centres which do not lie on a straight line.

By this application of Lamé's formulæ, aided by those due to Hicks, we have discovered no new type of symmetry which allows two coordinates to be used instead of three. We have proved that within the stated limits the well-known types are the only possible ones. A summary of these may be useful.

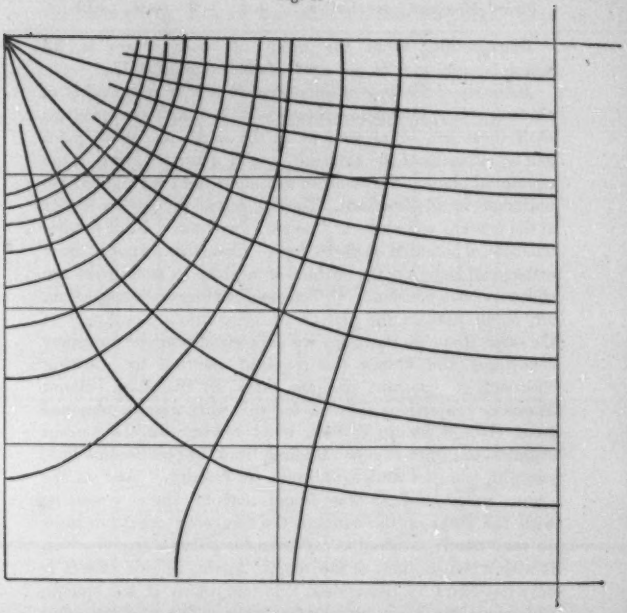
Summary of Types of Symmetry when the guiding lines are orthogonal to a family of surfaces.

Guiding lines.	Chequer ratio.	If $\nabla^2 V$ is made equal to $f(V, \alpha, \beta)$ over one surface γ_0 its value on the others will be	Analytical methods.
Parallel straights.	Constant.	$\nabla^2 V = f(V, \alpha, \beta)$.	Conjugate functions.
Circles with their centres on a common axis and their planes normal thereto.	Proportional to distance from axis.	$\nabla^2 V = f(V, \alpha, \beta)$.	Zonal harmonics of the cylindrical, spherical, spheroidal, and toroidal systems.
Radii from a common point.	Constant.	$\nabla^2 V = \frac{f(V, \alpha, \beta)}{r^2}$ where r is the distance from the radiant point.	

Example of Symmetry about an Axis.—Byerly in his 'Fourier's Series and Spherical Harmonics,' p. 230, sets the following problem:—"A cylinder of radius one metre and altitude one metre has its upper surface kept at temperature 100°, and its base and convex surface at the temperature 15°, until the stationary temperature is set up. Find the temperature at points on the axis 25 cm., 50 cm., and 75 cm. from the base. and also at a point 25 cm. from the base and 50 cm. from the axis." To solve this the first thing necessary is to prepare a chart bearing chequers of the appropriate shape for each distance from the axis. The graph of any solution of $\nabla^2 V = 0$ symmetrical about an axis would serve this purpose. For example several of the figures out of Maxwell's 'Electricity and Magnetism' would do. But I preferred to prepare a standard chart by ruling equidistant parallel equipotentials normal to the axis of revolution, and then stream-lines parallel to the axis at distances from it proportional to the square roots of the natural numbers 0, 1, 2, 3, 4, 5, &c. The cross section of the cylindrical shell enclosed between successive stream-lines is then the same for every pair, and the chequer ratio proportional to the distance from the axis. This having been done in red ink, a sheet of tracing-paper was pinned over it, the section of our given cylinder was drawn in black and equipotentials and lines of flow were drawn in pencil. These were then rubbed out and amended with the aim of making the pencil chequers everywhere very similar to the red rectangles underneath. When improvement became slow, the blurred lines were made firm and definite with ink and the chequers considered individually and marked as to whether they were too square or too thin. The lines were then drawn on a clean sheet of tracing-paper, the chequers again examined individually, and finally the lines fixed in ink (see fig. 3). Coordinate lines were then ruled and the values of V at their intersections were read from the graph. This process, from the ruling in of the given contour to the determination of V in numbers, took me four hours. The analytical method would perhaps have been more rapid *in this case*; but for an irregular shaped contour with an irregular boundary distribution the freehand solution would still take about the same

time, while analytical methods may be almost indefinitely tedious.

Fig. 3.



The results are tabulated below :—

Distance from base of cylinder	25	50	75	25 cms.
„ axis „	0	0	0	50 „
Potential read from the graph as decimal of unit range.	·175	·38	·64	·14
Last multiplied by the actual range (100°−15°)	14·9	32·3	54·4	11·9
Last +15° = actual temperature ..	29·9	47·3	69·4	26·9

Not until these numbers had been written down did I look

at the correct values found by Byerly from the Bessel function series, namely :—

	29·6	47·6	71·2	25·8
Errors of freehand determination..	+·3	−·3	−·8	+1·1

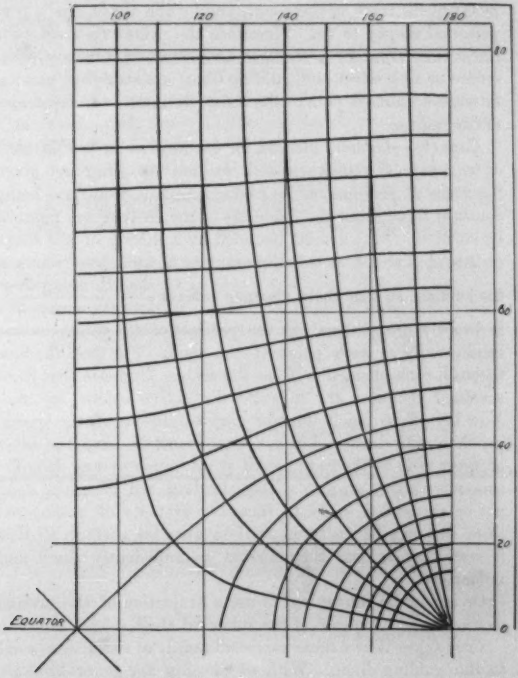
Disregarding signs the mean of these errors is ·63 which is equal to 0·74 per cent. of the range of 85°.

Example of the type of symmetry when V *is independent of the radius in spherical coordinates.*—"On a uniform spherical shell there are equal sources at the north and south poles and equal sinks at the extremities of a diameter lying in the equatorial plane. The sources and sinks send out and receive uniformly in all directions. The flux has no divergence except at the sources and sinks and no curl anywhere. Find the distribution of potential on the surface." To do this we might draw orthogonal lines on the surface of a globe so as to make the chequers ratio constant. Or because, in Mercator's projection, any small part on the globe transforms into a small part of the same shape on the map, we may transform the boundary conditions and obtain the required solution by drawing chequers of constant chequer ratio on the map. Blank Mercator projections suitable for this work may be obtained from George Philip & Son, Fleet Street. In the present example the lines of flow radiating from the pole become lines straight, parallel and equidistant at infinity. And as the graph progressed it was found that by their symmetry with the sinks on the equator, the foregoing condition must be very nearly satisfied at 10° from the poles, a region which is within the confines of the map. Again, in this case it is only necessary to determine V in one octant of the sphere, and symmetry helps us in other ways. The accompanying graph (fig. 4) is the best of four or five separate attempts. The time taken to make these was collectively four hours. Special attention was given to the equipotential curve which passes mid-way between the two equatorial sinks, and as the result of the aforesaid trials it was found to pass through a point 44° due north of the sink on the equator. This suggested that the true value should be 45°, and on looking at a sphere this is seen to follow from symmetry although it was not obvious on the map. Thus again we have a confirmation of

the passable accuracy of the graphic method—the error here is 1 degree in 90 or 1·1 per cent. of the range.

So far we have only treated the problem as relating to a spherical shell. But we may next suppose the sphere solid

Fig. 4.



and V to be independent of the radius. We will then have a solution of Laplace's equation in space. Since the chequer ratio is constant, the magnitude of the flux is inversely proportional to the linear dimensions of the chequer (on the

sphere not on the map) and is consequently proportional to $\frac{1}{r}$ along any guiding line, r being the radius. But if we draw any small cone enclosing the polar axis—which is now a line source—we see that the outflow between two spheres r and $r+\delta r$ is proportional to $\delta r \times$ (magnitude of flux) $\times$ (perimeter of the trace of the cone), and by the above this is proportional simply to δr. Therefore the polar axis must be a line source of uniform strength and similarly for any other source or sink when the guiding lines are straights passing through a common point—the strength must be independent of the radius.

Case (*b*).—General method for conduction in a *thin shell of any shape*, the thickness and conductivity being any given functions of position on its surface, and all conditions being constant throughout the thickness of the shell at any point of its surface. Take a solid bounded by a surface of the shape of the shell and draw small rectangles at numerous points of the surface, so that their chequer ratio $= \frac{\text{length along flow}}{\text{breadth across flow}}$ is directly proportional to the product of the thickness and conductivity at each point of the shell. For then the flow through each chequer will be the same. Suppose that these standard chequers are in some distinctive colour, say red. Now lay off in black the boundary conditions of the special problem and draw a black chequerwork to have the same chequer ratio as the red at each point, much as was done for symmetry about an axis. The standard red chequers need not be connected so as to form two systems of orthogonal lines but may be scattered anyhow over the surface, all that is necessary is that they should be sufficiently small and numerous.

Or it may be convenient to use a projection of the surface as was done in the case of the spherical shell above.

Case (*c*).—When there exists no family of surfaces normal to the guiding lines. Without pausing for generalities we will proceed at once to:—

Screw symmetry about an axis.—Let us discuss this with the aid of cylindrical coordinates r, ϕ, z. At a point P on

the axis OZ let a perpendicular be drawn extending to infinity. This perpendicular, which is to project only on one side of the axis, is imagined to revolve round the axis and slide along the same with proportional velocities. In one rotation round the axis let it move l along the axis. Then the line sweeps out a surface, at all points of which the expression $z - \frac{l}{2\pi}\phi$ is constant. Let us put $z - \frac{l}{2\pi}\phi = \omega$.

Then as ω varies we pass from one of these screw surfaces to another formed by shifting the first parallel to z. The range of the coordinate ω is from 0 to l. The intersections of ω=const. with the cylinders r=const. are a family of screw-threads.

Let $d\eta$ be an element of distance measured along any screw guiding line, so that $\frac{d\eta}{dz} = \frac{\text{length of turn of screw}}{l} =$ a function of r only. And let us make V a linear function of distance along each screw-thread so that $\frac{\partial V}{\partial \eta} =$ a function of r only.

Then as $d\eta$, $d\omega$, and dr are in perpendicular directions they are independent and

$$\frac{\partial V}{\partial z} = \frac{\partial V}{\partial \omega}\cdot\frac{\partial \omega}{\partial z} + \frac{\partial V}{\partial \eta}\cdot\frac{\partial \eta}{\partial z} + \frac{\partial V}{\partial r}\cdot\frac{\partial r}{\partial z},$$

which reduces to $\frac{\partial V}{\partial z} = \frac{\partial V}{\partial \omega} +$ a function of r only.

Therefore

$$\frac{\partial^2 V}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial \omega}\right) = \left(\frac{\partial \omega}{\partial z}\cdot\frac{\partial}{\partial \omega} + \frac{\partial \eta}{\partial z}\cdot\frac{\partial}{\partial \eta}\right)\frac{\partial V}{\partial \omega}$$

$$= \frac{\partial^2 V}{\partial \omega^2} + \frac{\partial \eta}{\partial z}\cdot\frac{\partial}{\partial \omega}\left(\frac{\partial V}{\partial \eta}\right) = \frac{\partial^2 V}{\partial \omega^2}.$$

Again,

$$\frac{\partial V}{\partial \phi} = \frac{\partial V}{\partial \omega}\cdot\frac{\partial \omega}{\partial \phi} + \frac{\partial V}{\partial \eta}\cdot\frac{\partial \eta}{\partial \phi} = \frac{l}{2\pi}\cdot\frac{\partial V}{\partial \omega} + \text{a function of } r \text{ only.}$$

Since

$$\frac{d\eta}{d\phi} = \frac{\text{length of turn of screw}}{2\pi} = \text{a function of } r \text{ only.}$$

And so

$$\frac{\partial^2 V}{\partial \phi^2} = \frac{l}{2\pi}\cdot\frac{\partial}{\partial \phi}\left(\frac{\partial V}{\partial \omega}\right) = \frac{l}{2\pi}\left(\frac{\partial \omega}{\partial \phi}\cdot\frac{\partial}{\partial \omega} + \frac{\partial \eta}{\partial \phi}\cdot\frac{\partial}{\partial \eta}\right)\frac{\partial V}{\partial \omega}$$
$$= + \frac{l^2}{4\pi^2}\cdot\frac{\partial^2 V}{\partial \omega^2}.$$

Now substitute these values of $\frac{\partial^2 V}{\partial z^2}$ and $\frac{\partial^2 V}{\partial \phi^2}$ in the expression for $\nabla^2 V$ in cylindrical coordinates, and we have

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \left(1 + \frac{l^2}{4\pi^2 r^2}\right)\frac{\partial^2 V}{\partial \omega^2}$$

which contains only two coordinates r and ω. So that if we make $\nabla^2 V = f(V, r, \omega)$ over any surface the same will be true throughout the whole region filled by the screw-threads passing through the surface, provided that $\frac{\partial V}{\partial \eta}$ is such as to make $\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r}$ constant along every guiding screw. One way of satisfying this is to make V increase by the same amount per turn of the screw, along every screw-thread and so that $\frac{\partial^2 V}{\partial r^2}$ and $\frac{\partial V}{\partial r}$ are both constant along every guiding screw.

In the following pages, except where specially indicated, we will consider only the case $\frac{\partial V}{\partial \eta} = 0$.

As this result does not appear to be given in the text-books, it may be well to confirm it by a slightly different line of reasoning, as follows. The tangent of the angle between the tangent to any screw-threads and a plane normal to the axis of symmetry is $\frac{l}{2\pi r}$.

Therefore the first space rate of any function of position V along the tangent to a screw-thread is

$$\frac{1}{\sqrt{l^2 + 4\pi^2 r^2}}\left\{2\pi r \frac{\partial}{r\partial \phi} + l\frac{\partial}{\partial z}\right\} V.$$

If $\frac{\partial V}{\partial \eta}$ = a function of r only, along every screw-thread then we have a function of r only $+ 2\pi\frac{\partial V}{\partial \phi} = -l\frac{\partial V}{\partial z}$

throughout the whole region. Therefore this last equation will still remain true after differentiation by ϕ or by z, thus

$$-\frac{2\pi}{l}\frac{\partial^2 V}{\partial \phi^2} = \frac{\partial^2 V}{\partial \phi \partial z},$$

$$\frac{\partial^2 V}{\partial \phi \partial z} = -\frac{l}{2\pi}\frac{\partial^2 V}{\partial z^2}.$$

Equating the two values of $\frac{\partial^2 V}{\partial \phi \partial z}$ thus obtained we have

$$\frac{\partial^2 V}{\partial \phi^2} = \frac{l^2}{4\pi^2}\frac{\partial^2 V}{\partial z^2},$$

which on substitution in the expression for $\nabla^2 V$ gives

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \left(1 + \frac{l^2}{4\pi^2 r^2}\right)\frac{\partial^2 V}{\partial z^2}.$$

But it is now to be observed that if the distribution of V on any plane passing through the axis of symmetry is known, then V is determined everywhere. And on such a fixed plane the contours of z are identical with those of ω. So that we may replace z by ω in the last equation, and the previous result is confirmed.

We have shown that if we make $\nabla^2 V = f(V, r, \omega)$ over any surface intersecting all the screw-threads, the same will be true throughout the whole region, with the stated provisions as to the value of $\frac{\partial V}{\partial \eta}$. The geometrical meaning of this result is that if we draw any infinitesimal rectangle normal to one of the screw-threads and draw screw-threads through each of its four corners, then the infinitesimal tube thus formed will be everywhere rectangular in normal cross section, and more than this, the rectangle will have the same ratio of length to breadth and will be of the same size at all points along the tube. For if we consider one pair of opposite faces of the tube as equipotentials and the other pair as lines of flow, then these properties are seen to follow from the fact that $\nabla^2 V$ is constant along a screw-thread when V is constant along the same. And indeed these properties are immediately obvious from the appearance of the system.

Consequently, if we take any family of surfaces α passing through the guiding screw-threads, there will always be an orthogonal family of surfaces β, also passing through the screw-threads. If the surfaces α are the contours drawn at equal intervals of the potential V the surfaces β are stream-surfaces. And $\frac{H_\alpha}{H_\beta}$ may be named the "Chequer Ratio" consistently with what has gone before. If $\nabla^2\alpha$ is to vanish we must have

$$\frac{\partial}{\partial\alpha}\left(\frac{H_\alpha}{H_\beta \times \begin{array}{c}\text{(length of portions of successive screw-threads intercepted}\\ \text{between two stream-lines lying on the same stream-surface)}\end{array}}\right)=0.$$

Since the screw system is uniform the length of the portions of successive screw-threads intercepted between two stream-lines lying on the same surface β can be proportional to nothing else than the length of one turn of the screw-thread at the radius considered. For the two stream-lines in question must by symmetry make equal angles with planes normal to the axis of the screw, at each pair of points lying on the same screw-thread. So that the projection of the distance between the said pair of points onto the axis of the screw will be always the same fraction of l as the points move from one screw-thread to another.

Now the length of an arc ds of a screw-thread being

$$\sqrt{dz^2+r^2d\theta^2}=d\theta\sqrt{\frac{l^2}{4\pi^2}+r^2},$$

the length of one turn is

$$\int_{\theta=0}^{\theta=2\pi} ds=2\pi\sqrt{\frac{l^2}{4\pi^2}+r^2}=l\sqrt{1+\frac{4\pi^2r^2}{l^2}}.$$

Therefore we may satisfy the equation $\nabla^2\alpha=0$ by making $\frac{H_\alpha}{H_\beta}$ proportional to $\sqrt{1+\frac{4\pi^2r^2}{l^2}}$ as the radius varies. A table giving the values of $\sqrt{1+\frac{4\pi^2r^2}{l^2}}$ for a number of values of $\frac{r}{l}$ is annexed.

$\frac{r}{l}$.	$\sqrt{1+\frac{4\pi^2r^2}{l^2}}$ l times this equals arc of one turn of thread.	For standard chart. H_α.	H_β.	$\frac{l}{2\pi r}$ equals tangent of angle between helix and circle.
0	1·0000	·5507	1·8157	∞
·05	1·0482	·5639	1·7735	3·1831
·10	1·1811	·5985	1·6707	1·5916
·15	1·3741	·6456	1·5490	1·0610
·20	1·6060	·6979	1·4328	·7958
·30	2·1338	·8045	1·2430	·53052
·40	2·7049	·9058	1·1040	·39789
·50	3·2969	1·0000	1·0000	·31831
·60	3·9003	1·0877	·9194	·26526
·70	4·5105	1·1697	·8550	·22736
·80	5·1251	1·2468	·8021	·19894
·90	5·7426	1·3198	·7577	·17684
1·00	6·3623	1·3892	·7199	·15916
1·5	9·478	1·6955	·5898	·10610
2·0	12·606	1·9554	·5114	·07958
2·5	15·740	2·1850	·4577	·06366
3·0	18·876	2·3928	·4179	·05305
3·5	22·014	2·5840	·3870	·04547
∞	∞	∞	0	∞

It is hoped that these values are correct to less than half a unit in the last place.

As there is no surface normal to the screw-threads, it is not possible to draw standard rectangles of the appropriate chequer ratio for each distance from the axis. But as the whole distribution of V is determined when the section of it by a plane passing through the axis of the screw is known, we may draw on this plane the sections of tubes formed by the surfaces α and β in such a way that $\frac{H_\alpha}{H_\beta}$ is proportional to

$$\sqrt{1+\frac{4\pi^2r^2}{l^2}}.$$

The sections of these tubes will in general not be rectangles; in fact, the angles and ratio of sides of the chequers formed by the traces of α and β on the plane ϕ = constant will both now depend on the orientation of the chequer as well as on its distance from the axis. It will therefore be necessary

to make a chart of standard chequers in various orientations at a number of distances from the axis. Plate V. is such a chart. The rectangles in the right-hand margin represent normal cross sections of the tubes formed by the surfaces α and β. In a line with each of these are five sections of a tube of the same size and shape by the plane of the paper, when the angle between one face of the tube and the normal to the axis of the screw is successively 0°, 22½°, 45°, 67½°, 90°. In order to be clearly visible these parallelograms are drawn quite large. What each really represents is the shape of an infinitesimal chequer situated at the central point of the large one. Practically the difference will not be important.

Now this standard diagram can be covered by a sheet of tracing-paper, and two intersecting families of lines drawn on the tracing-paper in such a way that the parallelograms formed by them are everywhere similar to the chequers underneath, which have the same distance from the axis and the same orientation on the paper. Then if this tracing-paper plane rotate round the axis and slide along it so as to follow the guiding lines, the equipotential lines on the paper will sweep out the contours at equal intervals of V in space in such a way that $\nabla^2 V = 0$ and the other family of lines will sweep out stream-surfaces.

A quantity which it is frequently necessary to determine is the magnitude of the flux

$$= \sqrt{\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2} = \frac{1}{H_\alpha}.$$

Since
$$\frac{H_\alpha}{H_\beta} = A\sqrt{1 + \frac{4\pi^2 r^2}{l^2}},$$

where A is an absolute constant, we must have

$$\frac{1}{H_\alpha} = A\frac{\sqrt{1 + \frac{4\pi^2 r^2}{l^2}}}{H_\beta}.$$

So that β is a stream function analogous to the forms in use when the guiding lines are parallel straights or circles with their centres on, and their planes normal to, a common axis.

In types previously studied, when the graph was drawn on a surface normal to the guiding lines, H_α and H_β were proportional to the length and breadth of a chequer and could be measured directly. But here we must first compare the linear dimensions of a freehand chequer with those of the standard oblique section of the tube bounded by two stream-surfaces and two equipotentials, and then refer to the normal section of the same tube in the right-hand margin of the chart.

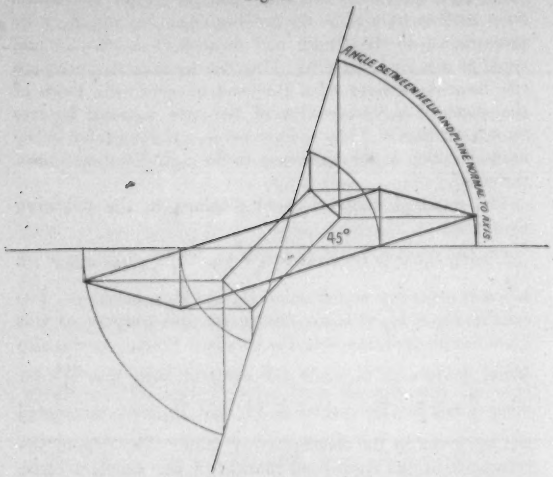
The standard chequers were obtained in the following manner:—

$\frac{H_\alpha}{H_\beta}$ being equal to constant $\times \sqrt{1+\frac{4\pi^2 r^2}{l^2}}$, some other relation is necessary to determine H_α and H_β separately. The relation $H_\alpha \times H_\beta = 1$ was chosen for this purpose, as this gives a neat appearance to the standard chart. It was also found convenient to make the constant such that $\frac{H_\beta}{H_\alpha}=1$ when $\frac{r}{l}=0{\cdot}5$. The values of H_α and H_β were calculated and are given in the accompanying table. The sides of the rectangles in the right-hand margin of the standard chart were drawn proportional to $2H_\alpha$ and $2H_\beta$.

To obtain the slant section, the tangent of the angle between the tangent a guiding-line and the plane normal to the axis of the screw, was first calculated. It is equal to $\frac{l}{2\pi r}$, and is given in the table under that head. The rectangles were then projected with ruler and compasses in a manner which is perhaps sufficiently indicated by fig. 5, which shows the construction when $\frac{r}{l}=0{\cdot}05$ and the angle between a radius from the axis of symmetry and the tangent plane to the surface $\alpha=$constant meeting at the point considered is 45°.

Of the innumerable solutions of $\nabla^2 V=0$ possessing screw symmetry of the sort described, which may be obtained by the aid of this standard chart, perhaps the simplest is the field due to a helical line source, such for example as the distribution of temperature in a mass of electrically insulating

Fig. 5.

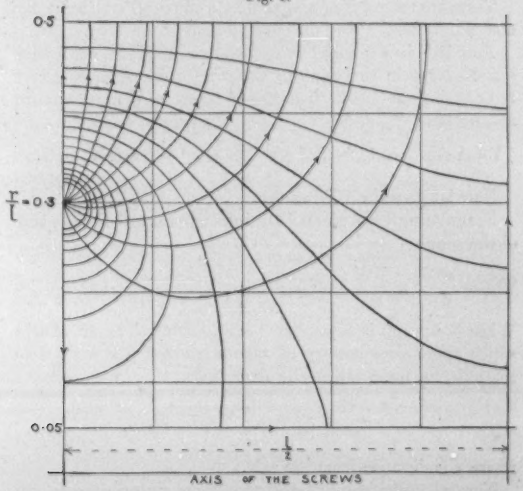


material which encloses a helical copper wire carrying an electric current. To avoid the introduction of a difficulty not characteristic of screw symmetry, I have assumed a core of non-conducting material in the form of a circular cylinder surrounding the axis. This relieves us of the necessity of considering the axial line of equilibrium, which would otherwise have to be treated by an extension of the method in Section IV. The external surface of the medium is also taken as a circular cylinder and is assumed to be at constant temperature. Consistently with our boundary conditions we may suppose that $\frac{dV}{d\eta}=0$. Now symmetry will help us in several ways, for since the chequer ratio on the standard chart is the same whether any particular half-turn of the screw passes over or under the chart, one sees on beginning to make the drawing, that the two surfaces $\omega=$ const., which pass respectively through the electric current and half-way between two adjacent turns of the current, must be surfaces

of flow. Again, very close to the electric current the flow of heat will be nearly the same as that due to a straight current tangential to the helix, that is to say, the lines of flow will be normals to the helix and the isothermals will approximate to circular cylinders concentric about the tangent.

The particular dimensions chosen were $\frac{r}{l}=0{\cdot}05$ for the core, 0·3 for the source, and 0·5 for the outer cylinder.

Fig. 6.



Owing to orientation of the chequer affecting its shape this graph took twice or thrice as long to adjust as did the others in this paper. Its errors are discussed in Section VIII. hereafter.

The magnetic field due to the helical current may doubtless be determined in a very similar manner. Here $\frac{\partial V}{\partial \eta}$ will be a constant other than zero and the cyclical properties of the field will add a further complication.

IV.

We have hitherto passed over without mention the peculiarities relating to points of equilibrium—these are *points at which the first space-rate of the potential vanishes in all directions*. In the neighbourhood of these the chequers become unusually large, and if any chequer goes right up to an equilibrium point it will not have the shape characteristic of its neighbours, but will take a peculiar form of its own.

There are several diagrams of this in Maxwell's 'Electricity and Magnetism.' See, for example, vol. ii. fig. xvii.

Now if V be expressed in terms of rectangular coordinates u and v lying in the plane of the graph with their origin at the equilibrium point, then linear terms in V must vanish, and we have

$$V = Au^2 + Buv + Cv^2 + Eu^3 + Fu^2v + Guv^2 + \text{terms of higher degree.}$$

Now let us make $\nabla^2 V$ vanish.

For guiding lines parallel straight and normal to the plane of the graph

$$\nabla^2 V = \frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial v^2} = 2(A+C) + u(6E+2G) + v(2F+6H).$$

When the graph is on a plane passing through an axis about which there is symmetry of revolution and u is normal to this axis, we must add to the above value of $\nabla^2 V$ the term

$$\frac{1}{r}\frac{\partial V}{\partial u} = 2\frac{u}{r}A + \frac{v}{r}B + 3\frac{u^2}{r}E,$$

where r is distance from the axis.

Now when the point considered is not on or close to the axis, it will be possible to put in so many chequers that the first two chequers in any direction from the equilibrium point require for their measurement so small a range of u and v that the fractions $\frac{u}{r}$, $\frac{v}{r}$, &c., will be small, and therefore the additional terms which come in for symmetry about an axis may be neglected, and we have the same form for $\nabla^2 V$ in both cases.

Further, since Mercator's projection does not alter the shape of any small pieces, $\nabla^2 V$ will have the same form in the neighbourhood of an equilibrium point on the Mercator's plan of the distribution on a sphere.

This being so, the general form of V in all three cases is

$$V = a(u^2 - v^2) + b\,.\,uv + g(u^3 - 3uv^2) + h(v^3 - 3u^2v) + \text{higher terms},$$

where a, b, g, and h are arbitrary constants. When the ratios of a and b to the succeeding coefficients do not vanish, then the first two terms are all that we need consider. Now it may easily be shown that by a proper rotation of the axes of reference, so that $u\,v$ tranform to $u_1 v_1$, the sum of these two terms may be transformed into either of them separately We need therefore only consider one, say bu_1v_1. The contours of this function are hyperbolas and are orthogonal to those of $a(u_1^2 - v_1^2)$. The ratio $\frac{a}{b}$ is determined by the chequer ratio in the neighbourhood of the equilibrium point.

A graph of this function for the special case of unit chequer ratio is given in Webster's 'Dynamics,' p. 525, and shows that two equipotentials meet at right angles at the equilibrium point, and that two stream-lines also pass through the same point and bisect the angles between the equipotentials. The eight curved chequers which meet in the point each have consequently three corners of 90° and one of 45°. A graph of this function may be used as a "standard equilibrium point" to keep the eye informed of the necessary proportions of the first and second ring of chequers surrounding the point.

If, however, the coefficients a and b vanish, while g and h do not, then the terms of the 3rd degree become all important.

By rotating the axes the sum of the two terms of the 3rd degree may be reduced to either separately. A rough graph of the contours of these functions is given by Fiske in Merriman & Woodward's 'Higher Mathematics,' p. 248. Here three equipotentials intersect in the equilibrium point. And three stream-lines bisect the angles of 60° which are formed in this way.

Now when a graph has to be drawn and is found to contain an equilibrium point, the general arrangement of the potential will give us the clue as to whether two, three, or more equipotentials intersect in the equilibrium point. And this being known, we have only to draw in the standard type at the proper dimensions and chequer ratio.

When the graph is drawn on a plane passing through an axis about which there is screw symmetry of the sort described in Section III *c*, then the appearance is different, for we have to add to the value of $\nabla^2 V$ for circular symmetry about an axis the term

$$\frac{l^2}{4\pi^2 r^2} \cdot \frac{\partial^2 V}{\partial v^2} = \frac{l^2}{4\pi^2 r^2}(2C + 2uG + 6vH).$$

And therefore writing

$$1 + \frac{l^2}{4\pi^2 r^2} = R^2$$

we have

$$V = a(R^2u^2 - v^2) + b \,.\, uv + g(R^2u^3 - 3uv^2) + h(v^3 - 3R^2u^2v).$$

A simpler way of looking at the matter is to consider a tiny plane element normal to the guiding screw which forms the line of equilibrium. The normals to the surfaces $\omega = \text{const.}$ lie in this plane. If dS_ω distance along such a normal, then

$$d\omega = dS_\omega \sqrt{\left(\frac{\partial \omega}{\partial z}\right)^2 + \left(\frac{1}{r}\frac{\partial \omega}{\partial \phi}\right)^2} = dS_\omega \sqrt{1 + \frac{l^2}{4\pi^2 r^2}}.$$

Substituting this in the expression of $\nabla^2 V$ in terms of r and ω we have

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial S_\omega^2},$$

just as if S_ω was z in circular symmetry about an axis. From this we see that the appearance of the equilibrium point on a small plane element normal to the guiding screw will be exactly similar to the forms already dealt with. Its appearance on a plane which passes through the axis of the screw may be sketched without much difficulty by comparing the chequers in the right-hand margin of the standard chart with their projections as drawn in the middle of the chart.

V. *Equations other than Laplace's.*

It has been shown above that in order to solve the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \text{any given function of V, } x, y,$$

a relation between differences of chequer ratios has to be satisfied. And the same will be found to be true for the other forms of the equation $\nabla^2 V =$ a function (of V and of position) which can be treated by two coordinates. A difference relation of the sort referred to would involve the comparison of each chequer with a standard set having graded chequer ratios, followed by the calculation of $\nabla^2 V$ by arithmetic. And although it would doubtless be possible to carry out the necessary operations, yet it would almost certainly be quicker and more accurate to use arithmetical finite differences altogether, writing in the numerical values of V at a set of points on the paper and adjusting these numbers until the finite difference equation is satisfied,—in a manner which may be described in a future paper. In view of this I will not attempt to elaborate freehand methods for $\nabla^2 V =$ a given function of V and of position.

There are, however, certain common space distributions which may be treated graphically with simplicity although they do not satisfy $\nabla^2 V = 0$.

Firstly, *when the conductivity is a continuous function of position*, and the direction of the flux is normal to the contours of a potential, and the magnitude of the flux is the maximum space-rate of the potential multiplied by the conductivity, and the flux has no divergence. For example: the flow of heat and electricity in isotropic but non-homogeneous bodies, or the soakage of water in a saturated subsoil the upper layers of which are more porous than those below. Let K be the conductivity and suppose that it is constant along each guiding line but varies from one such line to another. Then, when the lines have a family of surfaces normal to them we must have

$$\frac{\partial}{\partial \alpha}\left(\frac{H_\beta \cdot H_\gamma \cdot K}{H_\alpha}\right) = 0$$

in order that the flux shall not diverge. This is very easily assured by preparing the paper with standard chequers having their chequer ratio $\frac{H_\alpha}{H_\beta}$ proportional to $H_\gamma \,.\, K$. In fact, we have an example of this in Section III *a* above; for circular symmetry about an axis may be regarded for this purpose as flow between parallel planes in a medium having conductivity directly proportional to the distance from the axis. And reciprocally.

Similarly in the case of screw symmetry, standard chequers are to be prepared having

$$\frac{H_\alpha}{H_\beta} \text{ proportional to } K\sqrt{1+\frac{4\pi^2 r^2}{l^2}}.$$

Two other cases can probably be treated freehand, namely, the *flow of heat in bodies where the conductivity varies with the temperature*, and, of great practical importance, *the distribution of magnetic induction in soft iron, taking into account the variation of the permeability with the force*. But these again will be left to those who need the results.

VI. *Note on Boundary Conditions.*

It may be convenient to the reader if we bring together certain well-known facts concerning boundary conditions.

Let us regard V simply as a function of position, not necessarily satisfying $\nabla^2 V = 0$ or any other equation; and, as always, let contours be drawn at small intervals of V each equal to k. Then the first space-rate of V in any direction at a point is inversely as the intercept cut off from a line in that direction by two contours of V one on each side of the point, and is directly as K. Suppose, further, that the whole distribution of V can be represented by a single graph.

1. If we have to make V such that the magnitude and direction of its maximum first space-rate, the Hamiltonian vector ∇V, satisfies given values over a boundary of a given shape. Then it is easy to set off the ends of the contours of V with a ruler and scale, for their directions are known and also the distance apart of successive pairs.

2. If we are not given ∇V over the boundary but only the first space-rate of V in a given direction. Then there are an indefinite number of ways in which the contours of V may cut the boundary ; and as it will not generally be possible to say which of these is consistent with the internal conditions, they must be drawn and modified freehand as the approximation to the internal conditions proceeds. This is usually not difficult.

3. To *make* V *continuous* at any surface cutting the distribution, all that is necessary is that the ends of the contours of V approaching from the two sides should meet one another at this surface. Whether they meet at an angle or not does not matter.

4. To *make the first space-rates of* V *in every direction continuous* at any surface where V is continuous, not only must the contours of V meet one another, but they must pass smoothly into one another without making an angle. For if they made an angle and a straight line were drawn tangent to one branch of the contours at the angle, then the ratio of successive intercepts of this line by the contours of V would not become unity when the contours were drawn at indefinitely small intervals of V, so that the second space-rate along this straight would be indefinite at the angle.

5. Suppose next that a non-divergent vector is normal to the surfaces V = const., and that the magnitude of the vector is equal to the space-rate of V along the said normal, multiplied by a scalar function of position ; which according to the particular application will be the conductivity, permeability or some other specific constant. Then we may require the conditions which must hold at a *boundary where the specific constant has a discontinuity* while V is continuous—as, for example, where magnetic flux passes from air into mild steel. These conditions, which I take from Prof. J. J. Thomson's 'Elements of Electricity and Magnetism,' may be stated thus :—If K_1, K_2 are the aforesaid specific constants on the two sides of the boundary, and θ_1 and θ_2 are the corresponding angles which the direction of the vector makes

with the bounding surface, then

$$\frac{1}{K_1}\tan\theta_1 = \frac{1}{K_2}\tan\theta_2.$$

Now if the graph be drawn on a surface which is normal to the guiding lines, the direction of the vector lies in the graph, and θ_1 and θ_2 are the actual angles which one sees. The same is true of the Mercator's map of a spherical surface distribution, since the angles are unchanged by projection. But with our method for screw symmetry, the angles θ_1 and θ_2 do not immediately appear, and comparison must be made with the angles of the slant sections of the rectangular tubes given on the standard chart.

VII. *Miscellaneous Notes on Draughtsmanship.*

(*a*) Since with the exception of given boundaries and lines deduced from symmetry no part of the field can be said to be correct until the whole field is correct, it is advisable to begin by covering the whole field with intersecting lines, however erroneous they may be, and then to carry out amendments over wide areas at one time.

(*b*) In the final stages of a drawing intended improvements often overshoot the mark or cause unforeseen disturbances in the surrounding chequers. It seems well, therefore, to lay aside the indiarubber after a certain accuracy has been reached, and, placing a sheet of tracing-paper over the rough diagram, to draw the intended improvements upon this. And so with all later stages. The tracing-paper diagrams are then compared with one another and the best selected.

(*c*) The graphic addition of two scalar functions of position is conveniently performed in the way described by Maxwell ('Elementary Treatise on Electricity') by laying the contours drawn on a sheet of tracing-paper at equal intervals of the one over those of the other, covering the two with a clean sheet of tracing-paper, and drawing the diagonals of the chequers formed by the intersecting contours.

VIII. *Estimation of Errors.*

To one reading an account of this freehand method without having worked an example, it might seem as if there were no way of setting a limit to the errors of any particular graph. This, if it were true, would be a serious fault. But, happily, it is not so; for it is commonly necessary to make several drawings and then select the best of them: so that by the time the draughtsman has reached a drawing which he can scarcely improve upon, he has before him deviations from it in divers directions. The difference, then, between the selected graph and the second best graphs is a measure of the errors of the latter and an outside limit to the errors of the former. The actual errors of the selected graph will be less than this limit, and may be estimated by comparing the errors in the shape of the individual chequers in the best and second best graphs, and taking a fraction, thus:—

$$\frac{\text{individual chequer error in best graph}}{\text{same in second best graph}} \left(\begin{matrix} \text{difference between best and} \\ \text{second best graphs} \end{matrix} \right).$$

This is the true measure of the errors of the best graph. It depends, of course, on a general mental estimate or appreciation, and is consequently not susceptible of exact definition. But this does not much matter, for if the value of an error be known within two times either way it is usually sufficient.

The difference between the best and second best graphs is less dependent on a mental estimate, and consequently sets a firmer limit to the possible error.

Taking, for example, the graph of the field round a helical line source given in section III. *c*, and laying over it the tracing of the unpublished second-best graph, one sees that the difference in position of the lines in the two graphs nowhere exceeds $\frac{1}{3}$ the linear dimension of the chequer, at the point and in the direction considered. Now I should estimate that the error of the shape of individual chequers in the published graph averaged $\frac{1}{2}$ of the same quantity in the other; so that $\frac{1}{4}$ of the linear dimensions of the chequer may be taken as the error of position of the lines in the published graph. Now the graph exhibits ten tubes of flow; so that $\frac{1}{4}$ of one

tube is $2\frac{1}{2}$ per cent of the range. This is in the worst parts of the field. Elsewhere the error will be less, but it may still be expected to exceed the errors found when the graph is drawn on a surface normal to the guiding lines, because in the case of screw symmetry we have the added difficulty that the shape of the chequers depend upon its orientation.

Discussion.

Mr. A. Russell said that electricians had often to draw lines of force freehand. As a rule they had little more than their intuition to guide them, as the values of many of the variables were only known roughly and the bounding-curves were not simple geometrical figures. They would therefore welcome the paper, as it would enable them to improve their diagrams. The problem of the flow in two-dimensional space could in many cases be satisfactorily solved by experiment—for instance, by Prof. Hele Shaw's method. The author had shown, however, that the graphical method could be applied usefully to certain classes of problems in three dimensions which it would be extremely difficult to attack by any other method. The speaker had found difficulty in determining the "chequer ratio" in electrical problems where the extent of the field was unlimited.

Mr. A. Campbell asked if the author's method would be applicable to a helix of finite length carrying a current.

The Chairman remarked that in the method described much depended on the eye, judgment, and patience.

The Author, in reply to Mr. Campbell, said his method would not apply to the case of a helix of infinite length.

Fig. 4.

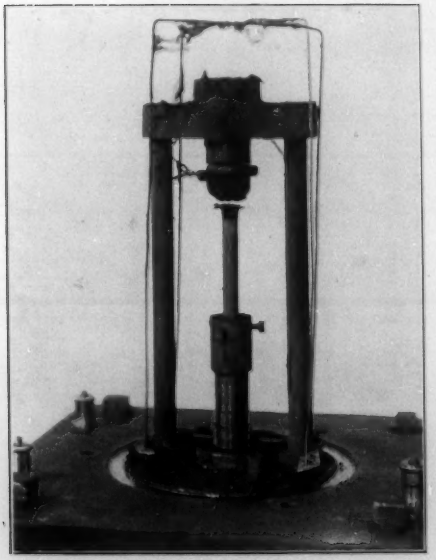


Fig. 5.

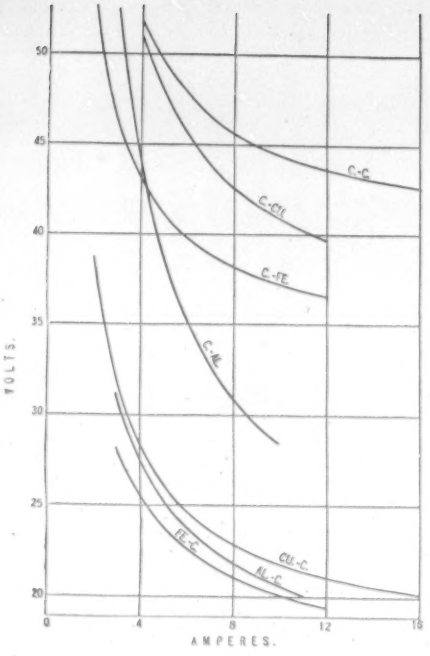


Fig. 6.

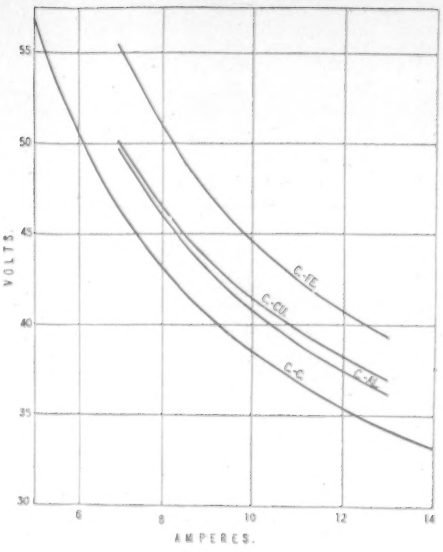


Fig. 7.

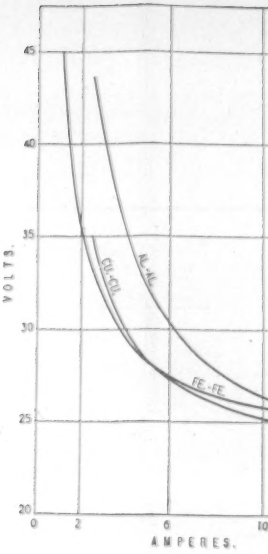


Fig. 10.

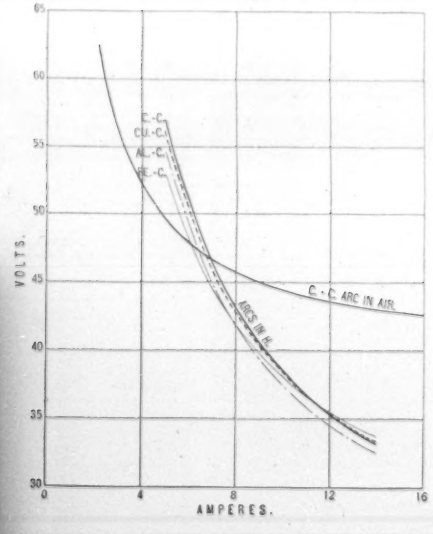
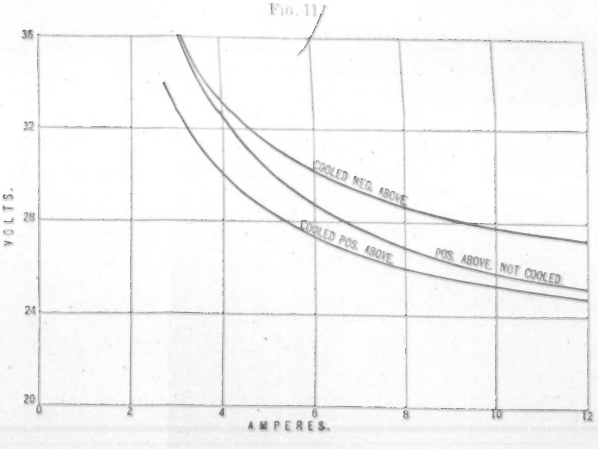
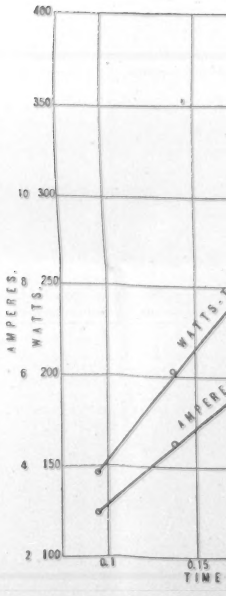


Fig. 11.





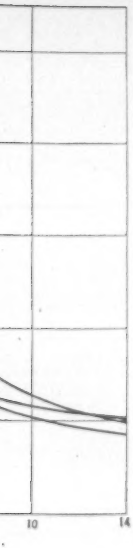


FIG. 8.

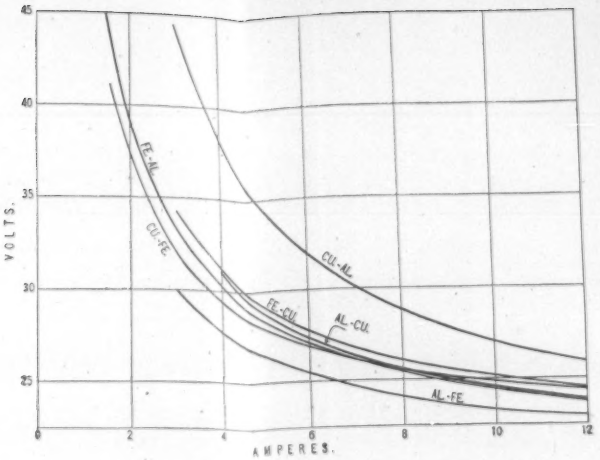


FIG. 9.

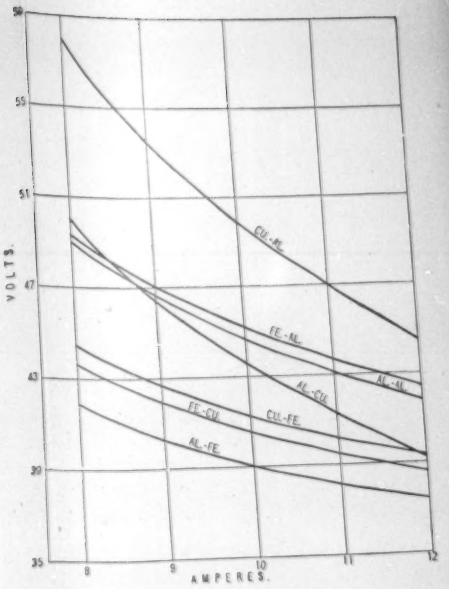


FIG. 12.

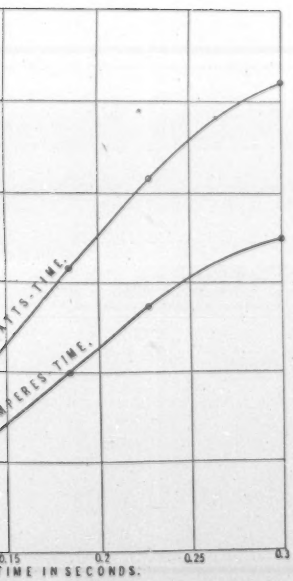


FIG. 13.

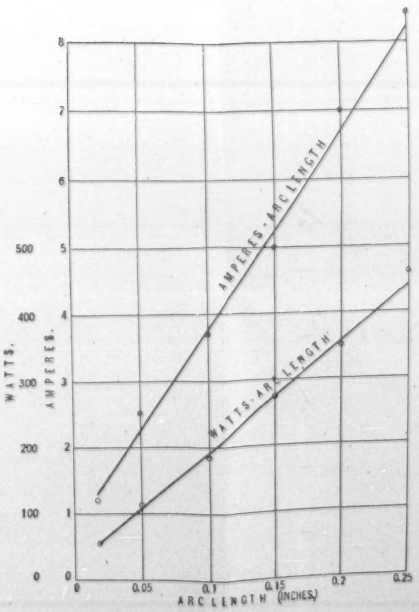


FIG. 14.

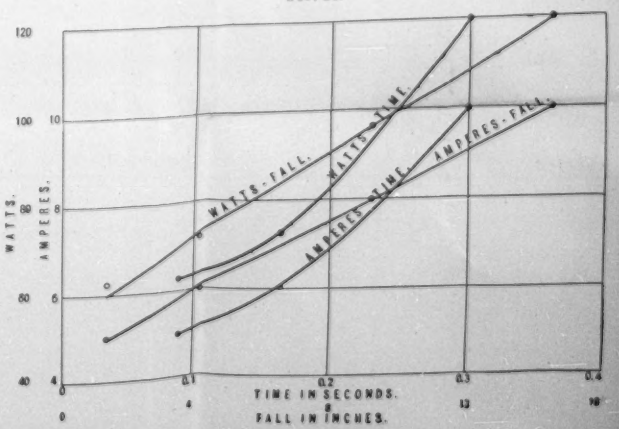


FIG. 1.

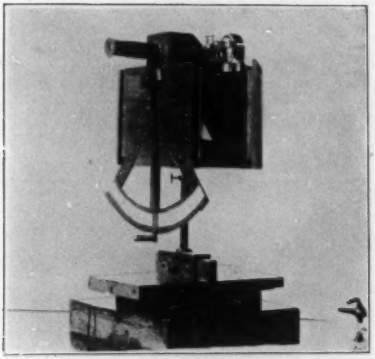
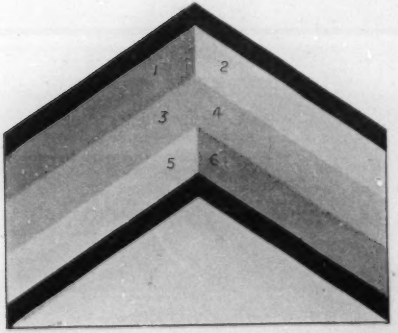
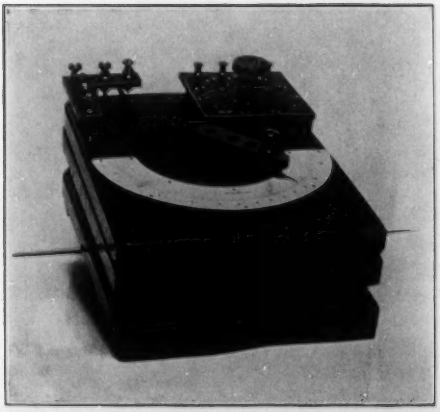


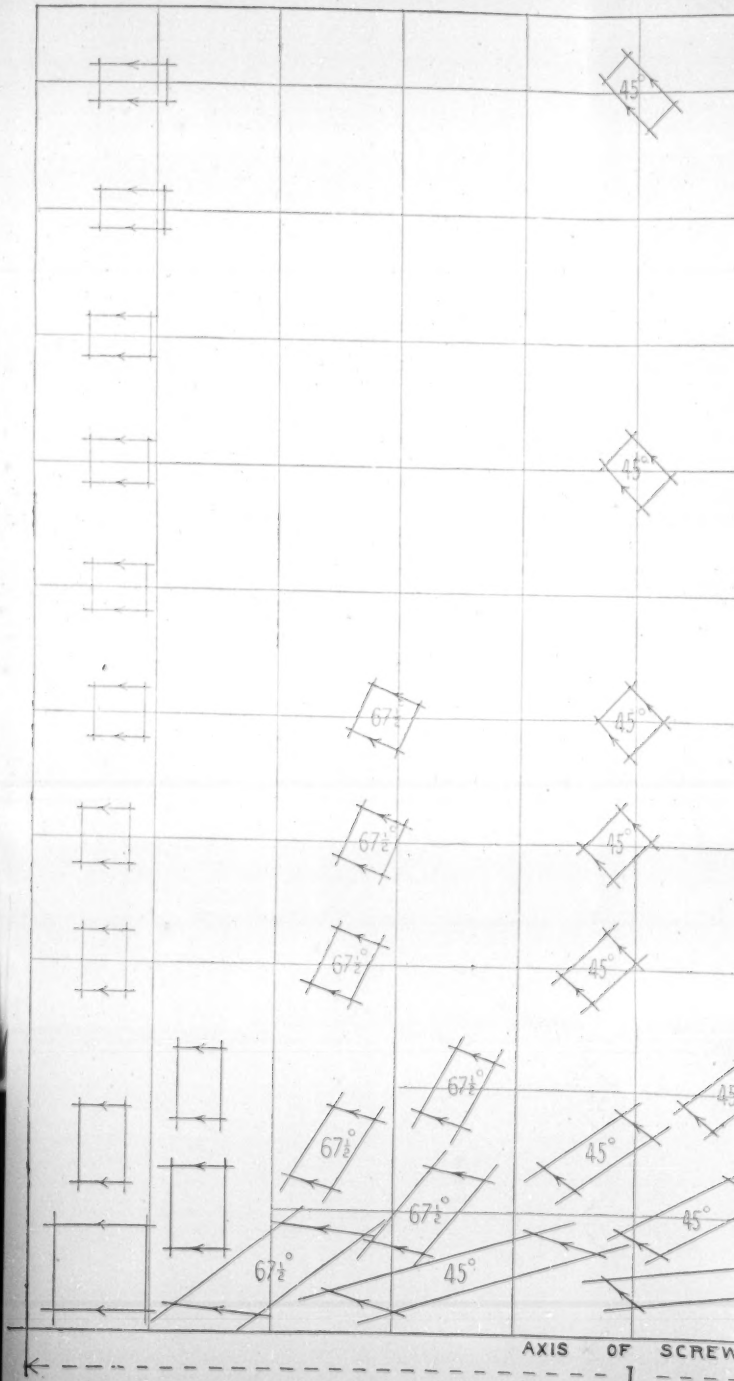
FIG. 6.

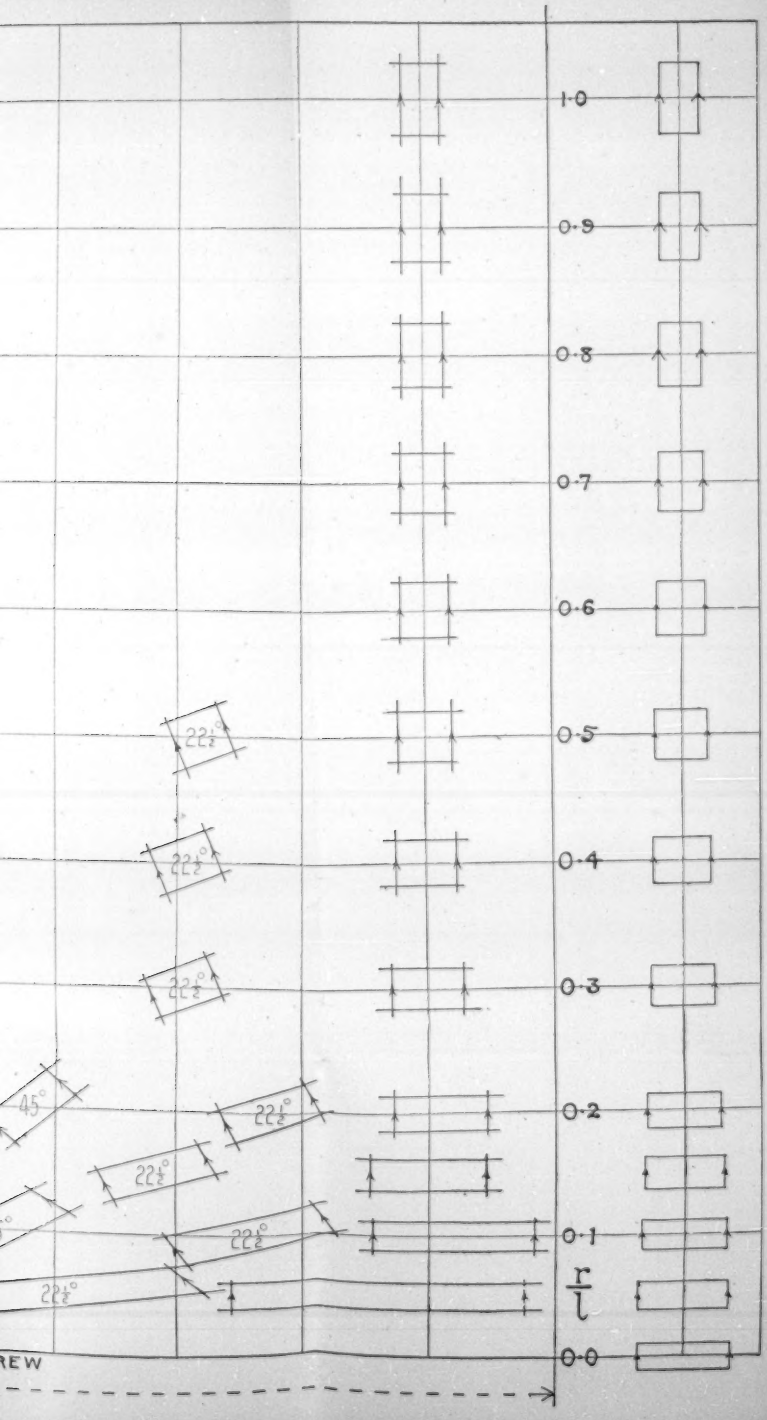




CAMPBELL'S VARIABLE MUTUAL INDUCTANCE.

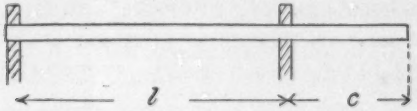
STANDARD CHART FOR




1·0
0·9
0·8
0·7
0·6
0·5
0·4
0·3
0·2
0·1
0·0
r/l
22½°
45°

VII. *On the Lateral Vibration of Bars supported at Two Points with One End overhanging.* *By* JOHN MORROW, *M.Sc., D.Eng.; Lecturer in Engineering, Armstrong College (University of Durham)* *.

WHEN a bar is carried on two supports and has one end overhanging by an amount c, as shown in the figure, its



natural period of vibration is to be determined from the equation

$$(\cosh ml \sin ml - \sinh ml \cos ml)(\cosh mc \sin mc - \sinh mc \cos mc) - 2 \sinh ml \sin ml (1 + \cosh mc \cos mc) = 0,$$

in which

$m = (2\pi N)^{\frac{1}{2}} (\rho\omega/EI)^{\frac{1}{4}}$;
E = Young's Modulus for the material;
I = the moment of inertia of the section about an axis perpendicular to the plane of bending;
ω = sectional area of bar (assumed uniform);
ρ = density of material;
N = frequency of natural vibrations.

For given ratios c/l we can obtain values of ml from the above equation. This has been done incidentally by Professor Dunkerley in connexion with his work on the whirling of shafts †, but the calculations were not sufficiently extended to give very accurate results. By comparison with the table given below, it will be seen that the approximate formula used by Dunkerley cannot be relied on to give more than the first two significant figures.

Having recently required more accurate solutions, I have

* Read November 8, 1907.
† Phil. Trans. A, 1894, p. 279.

found it necessary to make a more elaborate calculation, and as the results have been obtained to six figures it appears desirable to place them on record.

If we write a for c/l and θ for ml, and expand each term of the equation in ascending powers of θ and $a\theta$, we get, as far as the twenty-first powers,

$$4\theta^2-(\cdot0\dot4+\cdot\dot4\,a^3+\cdot\dot3\,a^4)\theta^6+(\cdot000035273369+\cdot0\dot02116\dot4\,a^3+\cdot0\dot037\dot2\,a^4+\cdot0\dot02116\dot4\,a^7+\cdot00\dot079365\dot2\,a^8)\theta^{10}-(\cdot58730+106\cdot8890\,a^3+293\cdot945\,a^4+1007\cdot811\,a^7+881\cdot834\,a^8+106\cdot889\,a^{11}+26\cdot722\,a^{12})10^{-8}\theta^{14}+(\cdot033+13\cdot2\,a^3+49\,a^4+508\cdot8\,a^7+700\,a^8+509\,a^{11}+297\,a^{12}+13\cdot2\,a^{15}+2\cdot4\,a^{16})10^{-11}\theta^{18}=0.$$

Assuming values of a and calculating θ we obtain the numbers tabulated below:—

Ratio c/l.	Value of θ.
Unity	1·50592
Three-quarters ..	1·90170
One-half	2·51895
One-third	2·94042
One-quarter	3·05881
One-fifth	3·09975
One-sixth	3·11752
One-seventh	3·12647
One-eighth	3·13148
One-ninth	3·13449
One-tenth	3·13641
Zero	3·14159

Dr. Chree has given the following approximate formula for use when c/l is small *:—

$$ml=\pi(1-\tfrac{1}{6}\pi^2 c^3/l^3).$$

It is stated to be satisfactory so long as $(c/l)^3\pi^3/6\coth\pi$ is small compared with unity. As a matter of fact we find

* Phil. Mag. [6] vol. vii. p. 517, May 1904.

that, when c/l is one-sixth, Chree's formula is correct to within 0·005 per cent., and the error is less than one per cent. even when c/l is one-half.

The number of vibrations per second is given by the formula

$$N = \frac{\theta^2}{2\pi}\sqrt{\frac{EI}{\rho\omega l^4}}.$$

September 1907.

VIII. *On the Contact Potential Differences determined by means of Null Solutions.* *By* S. W. J. Smith, *M.A.*, *and* H. Moss, *B.Sc.*, *Royal College of Science, London* *.

§ 1. Introduction.
§ 4. Palmaer's "null solutions."
§ 7. A method of search for other null solutions.
§ 9. Null solution of KCN.
§ 12. Effect of Na_2S upon the p.d. between Hg and KCl.
§ 16. Null solution of KCl.
§ 17. Effect of oxygen upon Paschen's relation.
§ 18. Inferences from results obtained with null solutions of KCN and of KCl.
§ 19. Null solution of KI.
§ 20. Null solution of KOH.
§ 21. Summary of conclusions.

§ 1. *Introduction.*—The processes which occur during the polarization of electrodes are of considerable theoretical and practical importance. Being largely surface effects, they can be followed, in the case of mercury, by observation of the changes of the surface-tension which result from polarization. Mercury electrodes have been the subject of much study in this way, but while the interpretation of some of the results seems clear, there are others which have been the subject of frequent discussion. Chief amongst these is the significance of the maximum surface-tension between mercury and the electrolyte which occurs at some particular degree of polarization in almost every solution that has been examined. The maximum was for long supposed to indicate

* Read February 28, 1908.

zero potential-difference between the mercury and the solution; but the adequacy of this hypothesis, as a general interpretation of the significance of the maximum, is now very doubtful.

§ 2. Evidence was given by one of us (Phil. Trans. A. 1899, pp. 47–87), in discussing the validity of the Lippmann-Helmholtz theory of electrocapillarity, that the maximum surface-tension between mercury and certain electrolytes is not reached when the potential-difference is zero. It was shown that, for equally concentrated solutions of the particular electrolytes KCl and KI, the potential-differences at the respective maxima differ to such an extent that if one is assumed to be zero the other must be nearly a quarter of a volt (*l. c.* pp. 70, 71). The assumption of zero potential-difference in either case is arbitrary. It is possible that the potential-difference at the maximum surface-tension is zero in some electrolytes; but the evidence from the electrocapillary curves (*l. c.* pp. 68 & 82), while not conclusive, favours the view that it is not so in any of the moderately dilute solutions of KCl and KI (giving "depressed" maxima) referred to above.

Palmaer has, however, published experiments recently from which it would seem that in the case of $n/10$ KCl, the potential-difference at the maximum surface-tension is, if not zero, at least very small. This result, if conclusive, would be of great theoretical value, and we have therefore performed the experiments described below to test the validity of Palmaer's deductions.

§ 3. Palmaer published in 1903 (*Zeitsch. f. Elektrochemie*, ix. pp. 754–757) a summary of certain experiments with drop electrodes, by means of which he sought to show that the true contact potential-difference between mercury and $n/10$ KCl solution is about 0·57 volt, and he has quite recently (*Zeitsch. f. physik. Chemie*, lix. pp. 129–191, 1907) given a full account of these experiments and of others, from which his final conclusion is that the potential-difference in question is $0{\cdot}5732 \pm {\cdot}0003$ volt at 18° C.

One of us found that the maximum surface-tension between

mercury and $n/10$ KCl solution was produced by a polarizing electromotive force of 0·568 ± ·01 volt (*l. c.* p. 71). It would thus appear that, after all, in the case of $n/10$ KCl, the potential-difference at the maximum surface-tension is practically zero, for it is represented by the difference between the two numbers just given. We shall attempt to show however that, although of much interest in connexion with electrocapillarity, Palmaer's deduction from his experiments is not necessarily true, and that his results leave the knowledge of the contact potential-difference between Hg and $n/10$ KCl substantially in the same state of uncertainty as before.

§ 4. *Palmaer's Null Solutions.*—When certain conditions, first fully discussed by Paschen, are fulfilled, a mercury electrode immersed in $n/10$ KCl shows a potential about 0·57 volt higher than that of an electrode of mercury streaming into the same solution. Palmaer found that by adding certain substances to this $n/10$ KCl solution, he could not only reduce the observed potential-difference between the still and dropping mercury until it became very small, but could cause it, passing through the value zero, to change in sign. In this way he found two different "null solutions," one containing small quantities of KCN, KOH, and $Hg(CN)_2$, and the other H_2S and a small quantity of acetic acid, for which the potential-difference between the still and dropping electrodes was zero. It is of course well known that a concentrated solution of a mercury salt behaves like a Palmaer null solution in the respect that there is practically no p.d. between still and dropping mercury electrodes immersed in it; but small variations in the composition do not here produce effects of the kind observed by Palmaer. Moreover, the absence of a p.d. in this case has been explained in a way which is not applicable when the amount of dissolved mercury is very small.

§ 5. Experiments of the kind performed by Palmaer were suggested by Nernst (*Ann. d. Physik*, lviii. p. 11, 1896), and it is obvious that the results can be explained, either according to the Helmholtz theory of dropping-electrodes or

to that of Nernst, if we assume that when the observed E.M.F. is zero a solution has been found which exhibits no potential-difference with respect to mercury. In such a case there would be no tendency to produce an E.M.F., whether, in the general case, the double-layer potential is altered by extension of the mercury surface (Helmholtz), or whether the double-layer forms practically instantaneously but is accompanied by concentration changes in the solution (Nernst).

§ 6. Connecting each of his null solutions in turn to a solution of $n/10$ KCl, Palmaer measured the p.d. between mercury in the null solution and mercury in the $n/10$ KCl. Allowing for small contact potential-differences between the electrolytes, he found practically identical values (of which the mean is the number already quoted) for the p.d. Hg | $n/10$ KCl. Strictly, these measurements prove only that the p.d. between mercury and each of the null solutions was the same.

Palmaer found only two satisfactory null solutions; but experiments with several others are described below. From these experiments it will be seen that the p.d. between mercury and a null solution of the type described by Palmaer is not always the same, and hence is never necessarily zero as Palmaer assumes.

§ 7. *A method of search for other Null Solutions.*—In searching for other null solutions we were guided by the relation deducible from Paschen's experiments, that although the p.d. between Hg and an electrolyte when the surface-tension is a maximum need not be zero, it is nevertheless always equal to the p.d. between a dropping electrode of the Paschen type (Wied. *Ann.* xli. p. 42, 1890) and the same electrolyte (*cf.* Phil. Trans. *l. c.* pp. 83 *et seq.*). We have found, in the course of the present experiments, that the statement of this relation, which is one of the most important in electrocapillarity, requires qualification. The relation is true only when the medium through which the mercury falls before it enters the solution does not contain a constituent which interacts chemically with the mercury and the solution. In some cases, for example, the relation does not hold until

the air surrounding the dropping electrode and the solution is replaced by hydrogen, nitrogen, or other non-oxidising medium.

§ 8. Assuming the truth of the above relation, it was obvious that any solution for which the maximum of the electrocapillary curve lay at the origin (applied E.M.F. zero) would be a null solution. If the maximum for a given solution were slightly (*a*) to the right or (*b*) to the left of the origin, then the still electrode for that solution would be slightly (*a*) positive or (*b*) negative to the dropping electrode. The change in sign of the potential of the dropper with respect to the still electrode, with a small change in the composition of the electrolyte, would not correspond necessarily (as assumed by Palmaer) with a change in the sign of the potential of the still mercury with respect to the solution, but only with a change from one side of the origin to the other of the maximum of the electrocapillary curve.

Since it is almost certain that this maximum does not always correspond with zero potential-difference, it seemed equally certain that a null solution of the kind studied by Palmaer could not always, and need not ever, be one exhibiting no contact p.d. with respect to mercury.

§ 9. *Null Solution of* KCN.—It has long been known that moderately concentrated solutions of KCN give electrocapillary curves of which the maxima lie to the left of the origin (applied E.M.F. negative), while weaker solutions give curves of which the maxima lie to the right. If a solution could be found of which the maximum lay at the origin, this would be a null solution of the kind studied by Palmaer. A null solution of KCN was apparently found approximately by Amelung in a research which Palmaer describes (*Zeits. f. physik. Chemie*, lix. p. 164, 1907) but regards as unsatisfactory. This solution used according to Palmaer's method would give a value of roughly 0·7 volt for the contact potential Hg | *n*/10 KCl. It was obtained by gradual dilution of a saturated solution of KCN until the p.d. between the drop electrode and the still electrode became zero.

We have found and examined a null KCN solution in another way. A normal solution of KCN was made up and this, together with weaker solutions of definite strength, obtained from it by dilution, was then examined as described below. As it was found that the electrocapillary properties of solutions, apparently of equal concentration, produced from different samples of KCN frequently differed considerably, the precise constitutions of the different solutions cannot be specified. This, however, is of no importance to the validity of the experiments. For the sample of KCN used in the experiments, the null solution contained about 0·26 *n* KCN; but another sample from another source would probably have given a different result.

The following quantities were observed in the case of each solution:—

1. The electrocapillary curve—the reading for maximum surface-tension, and the E.M.F. required to produce it, being noted as carefully as possible.

2. The electrocapillary curve of a solution of KCl of equal strength.

3. The horizontal distance between the descending branches of the two curves (*cf.* Phil. Trans. *l. c.* p. 69).

4. The E.M.F. of the cell $\mathrm{Hg} \mid \mathrm{KCN} \vdots \mathrm{KCl} \mid \mathrm{Hg}$.

5. The E.M.F. of the dropping electrode circuit $\mathrm{Hg} \downarrow \mid \mathrm{KCN} \mid \mathrm{Hg}$, the measurements being taken when the end of the continuous part of the jet was in the surface of the solution (Paschen).

6. The E.M.F. of the dropping electrode circuit $\mathrm{Hg} \downarrow \mid \mathrm{KCN} \vdots \mathrm{KCl} \mid \mathrm{Hg}$.

Some of the electrocapillary curves are shown in fig. 1, the others being omitted to avoid confusion. The KCN curves are very flat near the maxima, and the E.M.F.s corresponding to these maxima are relatively difficult to determine.

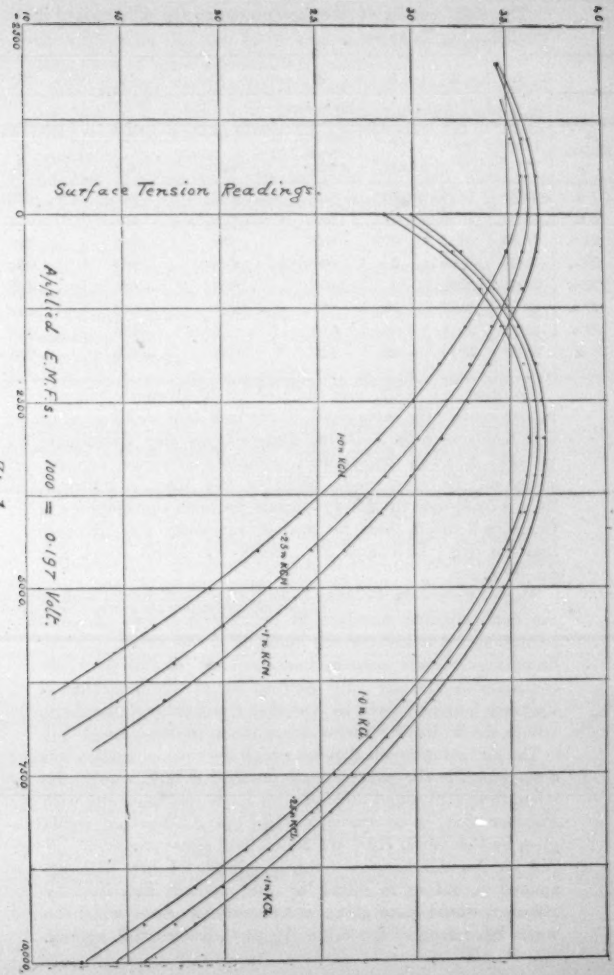

Surface Tension Readings.
Applied E.M.F.s. 1000 = 0·197 Volt.
1·0 n KCN
·25 n KCN
·1 n KCN
1·0 n KCl
·25 n KCl
·1 n KCl

Fig. 1.

The chief results of the measurements are summarized in the following Table:—

	1.	2.	3.	4.	5.	6.	7.
Strengths of solutions.	Max. S.T. KCN (scale-reading).	Max. S.T. KCl (scale-reading).	E.M.F. of Max. S.T. KCN volt.	Horizontal distance between curves. volt.	KCN | KCl | + volt.	KCN | + volt.	KCN | KCl | + volt.
0·1 *n.*	36·65	36·75	·039	·645	·642	·043	·679
·2 *n.*	36·30	36·70	·020	·667	·671	·018	·694
·24 *n.*	36·10	36·83	·010	·687	·687	·012	698
·25 *n.*	36·08	36·35	0	·691	·690	·003	·699
·26 *n.*	36·04	36·30	0	·696	·693	−·001	·697
·27 *n.*	36·00	36·27	−·005	·697	·696	−·005	·693
·50 *n.*	35·81	36·17	−·020	·716	·715	−·016	·703
1·0 *n.*	35·70	36·10	−·059	·748	·745	−·046	·705

Solutions of KCN and KCl of equal strength being of approximately the same conductivity and degree of ionization (*cf.* Kohlrausch & Holborn, *Leitvermögen der Elektrolyte*, pp. 145 & 148), the contact potential-difference between them will be very small, probably less than a millivolt. This is confirmed by the agreement between the numbers in Columns 4 and 5, since the former represents the difference between Hg | KCl and Hg | KCN (*cf.* Phil. Trans. *l. c.* pp. 69 *et seq.*).

Each number in Column 7 should be equal to the sum of the corresponding numbers in Columns 5 and 6. A comparison serves to indicate the limits of the uncertainty of the dropping-electrode measurements. The jet fell freely in contact with the air. By comparison of the numbers in Columns 3 and 6 it will be seen that Paschen's relation held, within the limits of errors of experiment, in every case.

The null solution was that for which the surface-tension was a maximum in the natural state (applied E.M.F. zero). Its strength was, in round numbers, $n/4$ KCN, and assuming with Palmaer that, in consequence, Hg | $n/4$ KCN = 0, it would give, by Column 5, Hg | $n/4$ KCl = 0·69 approximately.

§ 10. As will be seen later, a solution of $n/4$ KCl, examined according to either of the methods described by Palmaer, would have given a null solution from which he would have deduced the value Hg | $n/4$ KCl = 0·565 approx.

The reason of the difference of 0·125 volt between this and the value found by the KCN null solution is suggested by inspection of the electrocapillary curves.

The maxima for the KCN solutions are lower than those for the KCl solutions of corresponding strength, and the "horizontal distance" between the curves after they become parallel is approximately 0·125 volt greater than the distance between the maxima. Consequently (*cf.* Phil. Trans. *l. c.* p. 67) the solution at the maximum surface-tension in KCN is 0·125 volt more positive to the mercury than it is at the maximum in KCl.

§ 11. Every solution for which the maximum of the electrocapillary curve is at the origin is a null solution, but the potential-differences between mercury and different null solutions will be different if the maxima for these solutions are not the same.

If traces of other substances can be added to a $n/4$ KCl solution in such a way as to move the maximum of the electrocapillary curve to the origin without appreciably raising or depressing it, then such a solution will be about 0·125 volt less positive to mercury than the null solution of $n/4$ KCN. This null solution when measured against pure $n/4$ KCl will give an E.M.F. of 0·565 in place of the E.M.F. of 0·69 given by the $n/4$ KCN null solution. Hence, if Palmaer's solutions were produced from $n/10$ KCl without appreciable alteration of the maximum, the difference between his results and ours is immediately explained.

§ 12. *Effect of* Na_2S *on the p.d. between* Hg *and* KCl.—The p.d. between mercury and an electrolyte is controlled, according to the theory of Nernst, by the concentration p of the mercury ions in solution. If π is the potential rise from the solution to the mercury, then

$$\partial\pi/\partial p = kT/p, \quad . \quad . \quad . \quad . \quad . \quad (i.)^*$$

where T is the absolute temperature and k is a constant. The rise can therefore be diminished by decreasing p. From Palmaer's experiments and others (*e. g.*, Behrend, *Zeits f. phys. Chem.* xi. p. 481, 1893), it can be inferred that a large diminution of p is produced by saturating a KCl solution in

* An interesting method of deducing a similar equation was given by Professor J. J. Thomson in the Philosophical Magazine for 1895.

contact with mercury with H_2S. We therefore thought it likely that by addition of Na_2S to the KCl solution it would be possible to move the electrocapillary curve to the right, *i. e.* so that the maximum approached the origin. It remained to determine by experiment the relative amount of Na_2S necessary for the purpose of reaching the origin. Electrocapillary curves were obtained with solutions containing gradually increasing proportions of Na_2S. The general composition of the solutions was

$$x(n\,\mathrm{KCl}) + y(n\,\mathrm{Na_2S},\,9\mathrm{H_2O}),$$

where $x+y=0{\cdot}1$; *i. e.*, considering the sum of the contents, the solution was always 1/10th normal, and the content in kations remained approximately constant. Curves were obtained for a number of different values of y. Some of the results are shown in fig. 2, in which the values of y represented are as follows:—

No. of Solution ...	I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XIII.	XV.
Value of y	0	$0{\cdot}0_2$	$0{\cdot}0_3$72	$0{\cdot}0_3$74	$0{\cdot}0_2$77	$0{\cdot}0_2$1	$0{\cdot}0_2$	$0{\cdot}0_2$5	·01	·03	·08	·10

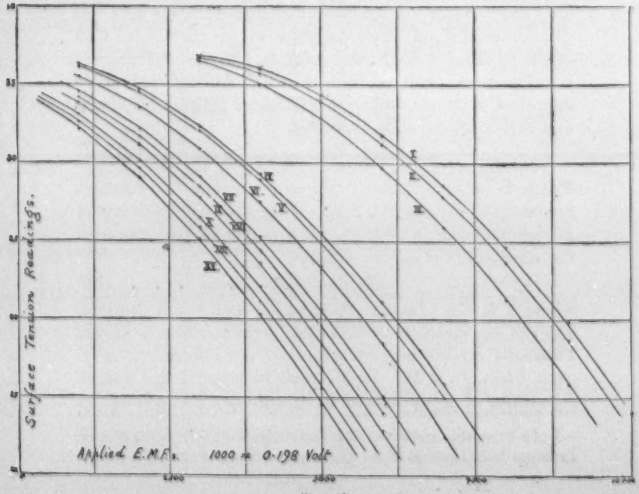
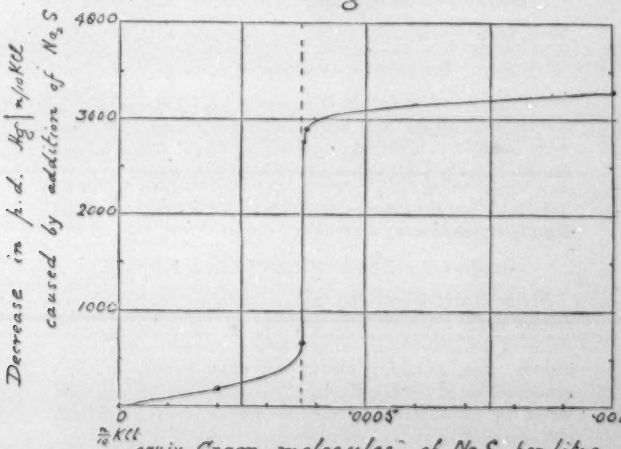


Fig. 2

From these results it will be seen that less than ·001*n* Na_2S added to ·099*n* KCl was sufficient to move the maximum of the electrocapillary curve to the left of the origin. Further, when the value of y was about ·00037, an extremely small variation produced a very large change in the position of the maximum. This is represented in fig. 3, which shows how the displacement of the descending branch of the electrocapillary curve depends on the percentage composition of the *n*/10 solution. Since the concentration in kations was the same for all solutions, this curve represents (Phil. Trans. *l. c.* p. 80) how the potential of the solution with respect to the mercury rises with increase in the amount of the sulphide.

Fig 3



§ 13. The extremely rapid concentration variation of the potential near $y = ·00037$ was very inconvenient for our purpose, for it happened that the solution which we sought, *i. e.* one having its electrocapillary maximum at the origin, would contain an amount of sulphide corresponding approximately to this value of y. The experiments of Behrend

(*l. c.* p. 481) suggested that the rapid variation of potential at a critical value of y occurred near the completion of the conversion of feebly soluble chloride of mercury into still less soluble sulphide. Thus, assuming c to represent the concentration of Hg_2Cl_2 in solution before addition of Na_2S, if the amount of sulphide added were $(c-c')$, *i. e.* less than c, we might have (see however § 15):—

$$cHg_2Cl_2+(c-c')Na_2S=c'Hg_2Cl_2+(c-c')Hg_2S+2(c-c')NaCl.$$

Assuming α and β to represent the fractions of the chloride and sulphide respectively, which are electromotively active, the ionic concentration p of the mercury in solution would be

$$p=2\alpha c'+2\beta(c-c'), \quad \text{and hence} \quad \partial p/\partial c'=2(\alpha-\beta).$$

Then, from equation (i.) above, we should have

$$\partial\pi/\partial c'=kT(\alpha-\beta)/(\alpha c'+\beta c-c').$$

From this it is seen that if β is very small compared with α, the value of $\partial\pi/\partial c'$ will become very large when c' becomes very small, *i. e.* when the amount of Na_2S added is very nearly equivalent to the amount of Hg_2Cl_2 in solution.

§ 14. When the amount of Na_2S added is in excess of the Hg_2Cl_2 in solution we may have

$$cHg_2Cl_2+(c+c')Na_2S=cHg_2S+c'Na_2S+2cNaCl.$$

From this it is seen that the potential of the liquid with respect to the electrode will continue to rise when, after the whole of the chloride is decomposed, more and more Na_2S is added. For, assuming the law of mass action, the ionic concentration of the Hg obtained from the Hg_2S will diminish when, by introduction of further quantities of Na_2S, the concentration of the S ions in solution is raised.

Suppose now that to a solution containing an excess of Na_2S a small quantity $2c''$ of an acid HX is added. Then we shall get

$$cHg_2Cl_2+(c+c')Na_2S+2c''HX=cHg_2S+(c'-c'')Na_2S+c''H_2S+2c''NaX+2cNaCl.$$

Now if we assume the coefficients of ionization of the Na_2S and the H_2S to be γ and δ respectively, the concentration of S ions arising from these two substances will be $\gamma(c'-c'')+\delta c''$; while, before the addition of the acid, the concentration of S ions arising from the Na_2S would exceed $\gamma c'$. The value of δ will in general be less than that of γ, because, in moderately dilute solution, H_2S is a very poor conductor compared with Na_2S. Therefore addition of the acid HX will reduce the concentration of the S ions in the solution containing cHg_2S, and will in consequence cause the concentration of the Hg ions to rise. Thus addition of the acid will tend to reduce the effect of the Na_2S and will cause the electrocapillary curve to move towards the right.

Hence, if too much Na_2S has been added, so that the maximum is to the left of the origin, it may be possible to obtain a null solution by the addition of a small quantity of an acid. Moreover, it may be of advantage experimentally to proceed in this way by overshooting the mark and then adding acid, rather than by attempting to hit it by reducing the amount of Na_2S. For, as can be seen, the rate at which the curve will move towards the right will be less when a small quantity of acid is added than when an equivalent quantity of sulphide is taken away. Thus, when the acid is absent, the concentration p_s of S ions, outside the Hg_2S, is such that $\partial p_s/\partial c'$ exceeds γ; but when the acid is added

$$\partial p_s/\partial c''=-(\gamma-\delta).$$

§ 15. It is not contended for the above equations that they necessarily do more than indicate the course of events. The Na_2S used probably contained traces of higher sulphides, and the chemical relations between the substances concerned are complicated. The final product of the action of Na_2S on the mercury in solution would no doubt be a double sulphide of HgS and Na_2S, and it is worth notice with regard to this that, after the very rapid change at $\cdot00037n$ Na_2S, the potential of the electrode continued to decrease fairly rapidly until about $\cdot002n$ Na_2S had been added. The rate of decrease fell comparatively suddenly at this point, and for subsequent additions of Na_2S (until the whole of the KCl was replaced) remained very small.

The concentration of mercurous ions in a solution of KCl standing over mercury and calomel has been estimated indirectly by Behrend and others, and from the values obtained it would appear that, if the above interpretation of our observations is correct, only a small fraction of the mercury in solution can be in the ionized condition. It is important to note, however, that the comparatively large value of the solubility of Hg_2Cl_2, to which the results of § 12 seem to lead, may on account of the influence of dissolved atmospheric oxygen be more apparent than real. It is commonly agreed that the interaction between Hg and a solution of a chloride, from which the existence of Hg_2Cl_2 in the solution results, occurs through the intervention of the oxygen dissolved in the latter. This interaction will continue (1) until the solution becomes saturated with Hg_2Cl_2, or (2) until the supply of oxygen near the electrode is used up. In the latter event, further formation of Hg_2Cl_2 would cease until, by diffusion, a fresh supply of oxygen approached the electrode. Since, however, the solubility of oxygen in $n/10$ KCl is probably much greater than that of Hg_2Cl_2, it is probable that the interaction terminates in accordance with the first of the alternatives and before he whole of the oxygen per c.cm. near the electrode is removed. If now a small quantity of Na_2S is added to the solution it will precipitate the whole or a part of the Hg_2Cl_2 with which the solution is saturated. But the further solution of Hg_2Cl_2 will be possible by interaction between the mercury, the KCl in the solution and the excess of dissolved oxygen. In fact the complete removal of Hg_2Cl_2 from solution by means of Na_2S, as represented in the equations given in §§ 13, 14, will not be possible until the whole of the dissolved oxygen is used up.

From this point of view, what happens when an aerated solution of KCl containing Na_2S is poured upon mercury may be described, figuratively, as a competition between the salts for the oxygen and mercury in the surface layer. Since any Hg_2Cl_2 formed before the removal of the Na_2S is complete will be at once decomposed by the latter and precipitated as sulphide, the net result of this competition is that the KCl can only interact permanently with such oxygen as

the Na_2S leaves uncombined. The critical amount of the latter is reached when it leaves none *.

In any attempt to form a complete picture of the process, however, it would be necessary to take account of possible difference in the subsequent rates of approach of Na_2S and O to the surface layer, from above, by diffusion and convection. This would lead us too far from our present aim.

The exact composition of our critical Na_2S solution was uncertain, and the further study of the question presented chemical difficulties which it did not seem profitable to investigate with the materials at our disposal.

§ 16. *Null Solution of* KCl.—The argument developed above proved successful as a working hypothesis. An $n/10$ KCl solution containing $\cdot001n$ Na_2S was first made up, and others containing different amounts of acetic acid (suggested by Palmaer's experiments) were then obtained from it. The results of the experiments with these solutions are summarized below:—

	1.	2.	3.	4.	5.	6.	7.
	Strength of acetic acid.	Max. S.T. (scale-reading).	E.M.F. for Max. S.T. (volt.)	Horizontal distance to $\frac{n}{10}$ KCl curve.	sol. $\left\vert \frac{n}{10}\text{ KCl} \right\vert +$ (volt.)	sol. $\left\vert + \right.$ (volt.)	sol. $\left\vert \frac{n}{10}\text{ KCl} \right\vert +$ (volt.)
I....	0	36·82	−·123	·661	·660	−·129	·564
II....	·0011 *n*	36·81	−·051	·622	·618	−·049	·560
III....	·0017 *n*	36·84?	−·020	·588	·582	−·018	·564
IV. ..	·0023 *n*	36·80	−·005	·578	·571	−·009	·564
V....	·0028 *n*	36·80	·010	·568	·562	·005	·562
VI....	·0040 *n*	36·80	·020	·556	·549	·016	·564

36·80 = Maximum S.T. for $n/10$ KCl.

The numbers representing the normality in acetic acid are only approximate. Nothing turns upon the accuracy of their

* Some results of experiments still in progress in connexion with this view were indicated when the paper was read.

determination. Some of the corresponding electrocapillary curves are shown in fig. 4.

The measurements are less exact than in the case of the KCN curves. Owing no doubt to the presence of small quantities of dissolved oxygen, the p.d. between the large Hg electrode and the solution sometimes changed perceptibly during the short time required to determine the electrocapillary curve. It will be seen, however, that a series of curves with practically equal maxima of surface-tension was now obtained. The null solution lies between IV. and V.,

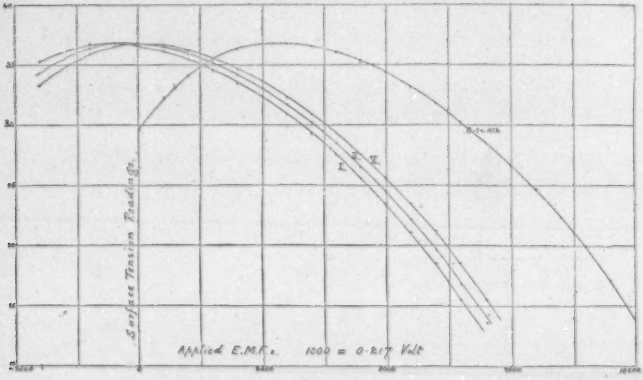


Fig. 4.

and corresponds within the limits of experimental error with a solution of which the maximum of the electrocapillary curve is at the origin. Part of the difference between the numbers in columns 4 and 5 was certainly due to increase with time of the p.d. Hg | solution. The capillary curves were taken first and are not exactly parallel to the decinormal KCl curve. The distance decreases slightly towards the lower ends of the curves. The p.d. between the strongest of the solutions and the $n/10$ KCl solution was probably less than a millivolt (*cf.* Palmaer, *Zeit. p. Chem.* l. c. 1907, p. 155).

By examination of the data for solutions IV. and V. it is seen that a null solution of the present kind, interpreted according to the assumptions of Palmaer, would give, for the p.d. Hg | $n/10$ KCl, a value agreeing with his result. But, as will be seen, his assumption of zero p.d. between Hg and the null solution is not at all necessary. The only necessary conclusion is that the p.d. between the still Hg and the solution is the same as that between the dropping Hg and the solution.

From the data in the Table it is seen that, for reasons already given, the maxima being practically equal in all the solutions, the p.d. between the dropping electrode and the solution should be in every case the same. This result is probably of significance in the theory of electrocapillarity. Its truth is shown by the practical equality of all the numbers in Column 7.

§ 17. *Effect of Oxygen upon Paschen's relation.*—With respect to the data in columns 3 and 6, § 16, showing the fulfilment of the Paschen relation for the present series of electrolytes, it is important to state that the dropping-electrode experiments were performed in an atmosphere of hydrogen (*cf.* § 7, above). A piece of wide glass tubing open at both ends encircled the dropping electrode. The lower end of this tube was immersed in the electrolyte, and the upper was closed by a rubber cork in which were three holes. The drop electrode passed through the central hole: the others served for the inlet and outlet of the hydrogen respectively.

The following data are given as examples of the necessity for the exclusion of air. In one case, the maximum of the capillary curve being +·01 volt, a Paschen electrode against still mercury in the same solution showed −·05 volt, increasing to −·07 when the mercury jet was partially immersed. The solution quickly became cloudy and the mercury tarnished so rapidly that it fell as a dirty powder.

In another experiment the jet was surrounded by hydrogen, the water used in preparing the solution having been recently boiled to decrease the amount of dissolved air. Before complete displacement of the air by hydrogen, the Paschen

electrode E.M.F. was −·024 volt. It fell gradually to −·014 volt and then remained constant. The mercury fell quite clean. Some of the solution was now withdrawn, and its electrocapillary curve was determined. The maximum lay at −·015 volt.

In a third experiment, the jet being surrounded by hydrogn, the electrode E.M.F. was −·024, agreeing with the electrocapillary maximum. Oxygen was now admitted and displaced the hydrogen. The Paschen electrode E.M.F. became −·140. With the jet partially immersed the E.M.F. increased to −·240.

Various experiments with other gases and with insulating liquids were tried, *e. g.*, it was found that the Paschen relation was, under certain conditions, fulfilled when air was replaced by benzene; but further description of these experiments is omitted from consideration of space.

§ 18. *Inferences from results obtained with Null Solutions of* KCN *and of* KCl.—Summarizing the results of the two series of experiments described, it is seen that, with respect to the p.d. Hg | electrolyte as considered by Palmaer, there are two kinds of null solution. One is formed by the addition to the electrolyte of very small quantities of certain substances which leave the shape of the electrocapillary curve unaltered, with the maximum undepressed, but move it parallel to itself towards the left. This kind of null solution, interpreted in the way described by Palmaer, will give the result that the p.d. is zero at the maximum of the electrocapillary curve.

A second kind of null solution is obtained by using a different electrolyte and altering its concentration until the maximum of the capillary curve is at the origin. This kind may have a depressed maximum compared with the original electrolyte, KCl for example, and will in that case give a value, for the p.d. Hg | KCl, greater than that obtained by Palmaer if the measurements are interpreted in the way he describes. If, on the other hand, the curve for the null-solution electrolyte has a higher maximum than the corresponding curve for KCl, then the p.d. Hg | KCl found by Palmaer's method will be less than that which he gives.

We have not attempted to find a null solution exactly of this last kind, although it is known that the electrocapillary maximum of a saturated KOH solution is near the origin and is considerably higher (Phil. Trans. *l. c.* p. 68) than that of concentrated KCl.

We have, however, obtained null solutions by the addition of small quantities of Na_2S (and acetic acid) to $n/10$ KI solution, which has a very depressed maximum, and to $n/10$ KOH solution, which has a higher maximum than $n/10$ KCl. The results of these experiments exhibit the truth of the conclusions we have drawn.

§ 19. *Null Solution of* KI.—The attempt to obtain a null solution from $n/10$ KI succeeded approximately when the composition of the solution was about $\cdot001n$ $Na_2S + \cdot099n$KI. The maximum surface-tension occurred when the applied E.M.F. was $-\cdot01$ volt. It was unnecessary for our purpose to obtain a nearer approximation to a null solution. The E.M.F. between a dropping electrode falling through hydrogen into this solution and a still electrode was at first imperceptible. After some time it was ·006 volt. Mercury in this solution measured against mercury in $n/10$ KCl gave an E.M.F. of ·793 volt. A fresh null solution of KCl, made up by trial for comparison, measured against $n/10$ KCl in the same way gave ·565 volt. The two solutions measured against each other gave ·228 volt. The results of the measurements are tabulated below:—

1.	2.	3.	4.	5.	6.	7.
Null Solution.	Max. S.T. (scale-reading).	E.M.F. for max. S.T.	Horizontal distance to $\frac{n}{10}$ KCl curve.	sol. | $\frac{n}{10}$ KCl | +	sol. | +	null KI | null KCl | +
		volt.	volt.	volt.	volt.	volt.
$\frac{n}{10}$ KI + $\frac{n}{1000}$ Na_2S	28·85	$-\cdot01$	·789	·793	0	·228
$\frac{n}{10}$ KCl + $x$$Na_2S$ ...	30·5	$-\cdot01$	·569	·565	$-\cdot01$	

These experiments show that the p.d. Hg | $n/10$ KCl cannot be 0·57 unless, interpreting them according to Palmaer's method, the p.d. between $n/10$ KCl and $n/10$ KI is nearly a quarter of a volt. Such a potential-difference is impossible according to the theories at present in vogue (*cf.* Phil. Trans. *l. c.* p. 62).

§ 20. *Null Solution of* KOH.—In attempting to obtain a null solution from $n/10$ KOH it was found that the addition of Na_2S (·001n to ·002n) produced a gradual shift of the maximum to the left which continued over a long time. A solution with the maximum at the origin could not be obtained conveniently with Na_2S alone on account of this time effect It was found however that if, after the solution had stood for some time over the mercury, dilute acetic acid was added drop by drop and the solution stirred, the maximum could be brought from the left to zero and the time effect was now negligible. The results of the experiments with this null solution are given below:—

1.	2.	3.	4.	5.	6.
Null Solution.	Max. S.T. (scale-reading).	E.M.F. for max. S.T. volt.	Horizontal distance to $\frac{n}{10}$ KCl curve. volt.	| null sol. | $\frac{n}{10}$ KCl | + | null sol | + volt.	
$\frac{n}{10}$ KOH + Na_2S + $HC_2H_3O_2$	30·61	+·01	·515	·509	·008

In considering the value 0·509 volt in column 5 it has to be remembered that there is now an appreciable p.d. between the solutions. Its value calculated in the usual way is $n/10$ KOH | $n/10$ KCl = ·016 volt at 20° C.; and consequently we now obtain, according to Palmaer's method, Hg | $n/10$ KCl = ·525 volt. But, according to the interpretation we have offered, this result signifies only that the potential reckoned from the solution to mercury at the maximum is about ·04 volt less in the case of $n/10$ KOH

than in that of $n/10$ KCl. A result which, as before, is in agreement with the forms and relative positions of the electrocapillary curves.

The electrocapillary curves for the null solutions of KI and of KOH are shown in fig. 5.

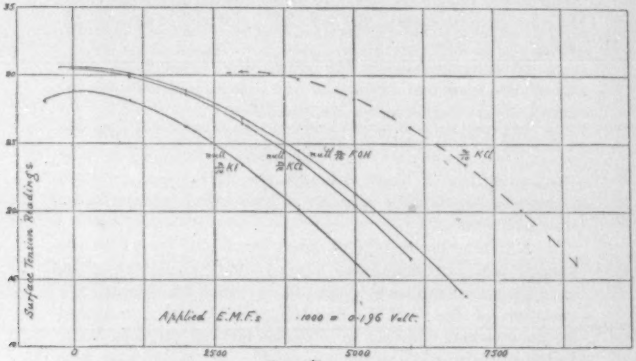


Fig. 5.

The results of our experiments may be summarized as follows:—

§ 21. *Summary of Conclusions.*

1. The potential-differences between different null solutions and mercury are not the same. This is proved, allowing for the contact p.d. between electrolytes, by measuring each p.d. against the p.d. Hg | KCl, and also by measuring one null solution against another. The result is also deducible from the electrocapillary curves alone, without introduction of the question of the p.d. between electrolytes.

2. If the null-solution potential-differences are assumed to be zero, the values obtained for the p.d. Hg | $n/10$ KCl vary from about 0·53 volt to 0·79 volt. Assuming that the p.d. at the electrocapillary maximum is most likely to be zero when the maximum is undepressed, it is probable that the p.d. Hg | $n/10$ KCl does not exceed 0·53 volt.

3. The polarizing E.M.F. required to produce the maximum surface-tension between mercury and a given electrolyte has been proved by Paschen to be equal, in many cases, to the E.M.F. of the corresponding dropping-electrode circuit. This relation is shown to be true in four particular cases in which each E.M.F. is equal to zero. It is also shown that in none of these cases is the p.d. Hg | electrolyte necessarily zero.

4. It is shown that the Paschen relation immediately fails when the chemical action at the drop-electrode due to atmospheric oxygen becomes appreciable.

5. The Paschen E.M.F. between mercury and a solution of KCl remains unchanged on the addition to the solution of small quantities of Na_2S, although the natural p.d. between mercury and the KCl solution is thereby altered by more than half a volt.

6. A critical percentage of Na_2S was found for which the natural p.d. just mentioned altered with extreme rapidity, as in the cases studied by Behrend, in which he observed the variation of the E.M.F., Hg | $Hg_2(NO_3)_2$ | Hg, produced by the gradual addition of KCl or KBr at one electrode. In the present case the change in the p.d. Hg | electrolyte was deduced directly from the electrocapillary curves.

Discussion.

Mr S. Skinner congratulated the Authors upon the accurate series of experiments which they had carried out.

Dr W. Watson expressed his interest in the paper, referring especially to the action of dissolved oxygen present in the solutions.

Mr F. E. Smith referred to the fact that in the paper values for the contact p.d. were given to four places of decimals, and asked if any significance could be attached to the last two figures. He pointed out that the results obtained from drop electrodes were very different from those obtained by Billitzer using a different method. He asked if experiments had been made with jets of various sizes and under various pressures, and whether the form of the curves had been sufficiently studied.

Mr Lewis asked if the Authors had used other substances besides Na_2S, and if they had taken account of chemical actions which might take place at the electrodes.

Dr Erskine Murray referred to experiments he had made some years ago on contact potential-differences between metals in which the

oxygen of the air behaved in a manner analogous to the dissolved oxygen in the Authors' solutions.

Mr S. W. J. Smith, replying to Mr F. E. Smith, said that Palmaer's value for the contact potential-difference between mercury and *n*/10 KCl solution was quoted from his paper. With regard to Billitzer's views, they laboured for the present under the disadvantage that other observers had tried, but failed, to reproduce the effects which he describes. The form of the electrocapillary curve had been carefully studied. Van Laar had formulated a theory based upon the fact that the curve can be considered to consist within the limits of experimental error of portions of two parabolas. His interpretation of the significance of the point at which the curve changes from one parabola to the other had in reality no experimental support. The question of the character of the jet, amount of immersion, &c., had been carefully studied, and certain relations between the E.M.F. and the time of contact with the liquid had been arrived at. His (Mr Smith's) point of view differed from that of Paschen in that he believed the experimental evidence to show that the E.M.F. between a Paschen electrode and an electrolyte was not in general zero. He cited a case in which apparently the p.d. between a Paschen jet and one electrolyte ($ZnSO_4$) differs by nearly half a volt from that between a Paschen jet and another electrolyte ($CuSO_4$). It was only when the electrocapillary maxima for two electrolytes were equal that the Paschen E.M.F.s for the electrolytes were the same. An electrocapillary maximum could not be obtained for $CuSO_4$ solution, probably because of deposition of copper on the mercury, just as a maximum could not be obtained from concentrated H_2SO_4 on account of the evolution of hydrogen. Paschen's experiments did not prove that the p.d. between an electrolyte and a mercury-jet breaking in its surface is zero. It was possible that a certain fraction (in some cases the whole *e.g.* in null solutions) of the steady p.d. might arise practically instantaneously.

In reply to Mr Lewis, Mr Smith remarked that they used Na_2S because it was deducible from electromotive data—in agreement with chemical—that the solubility of mercury sulphide was extremely small even compared with that of calomel. In reference to Mr Lewis's suggestion that certain interactions might take place between the mercury, the dissolved salt and the Na_2S independently of the oxygen, he said that although he had found that the amount of Na_2S required (in the way the experiments were performed) was not chemically equivalent to the amount of oxygen in solution, yet the amounts of Na_2S and oxygen were found to be roughly proportional to one another in experiments they had performed with the help of Mr W. F. Higgins. He indicated to the meeting directions in which further experimental evidence was being sought. The electromotive method of tracing minute chemical change was both delicate and simple.

IX. *An Experimental Examination of Gibbs's Theory of Surface-Concentration, regarded as the basis of Adsorption, with an Application to the Theory of Dyeing.*

(From the Muspratt Laboratory of Physical and Electrochemistry, University of Liverpool.) *By* W. C. M. Lewis, *M.A.**

Contents.

I. Object of the investigation; the characteristics of Adsorption Phenomena.
II. Theoretical Discussion; Gibbs's Theory of Surface Concentration; Milner's Calculations.
III. Experimental Methods and Apparatus; Results.
IV. Application to the Theory of Dyeing.
V. Summary.

I. Object of the Investigation.

Under the term "Adsorption" are grouped phenomena which may be regarded as forming an intermediate stage between chemical combination on the one hand, and true absorption or solution on the other. Different types of Adsorption have been studied, but in general a solid substance (*e. g.* charcoal) has been utilized as the adsorbent body.

The present paper is an account of an experimental attempt at measuring adsorption effects quantitatively and interpreting the values obtained in the light of Gibbs's theory of surface-concentration. The treatment is novel in that the adsorption has been measured at a *liquid-liquid* interface; for it is only when dealing with liquid interfaces (or surfaces) that it is possible to measure interfacial- or surface-tension—a fundamental factor in the thermodynamic consideration of the question.

The Characteristics of Adsorption Phenomena.

The earlier investigations on adsorption dealt in general with the adsorption of gases on the surfaces of various solid materials. The case of vapours is of much more recent date, the most important instances being those of Travers † and

* Read February 28, 1908.

† Travers, Proc. Roy. Soc. series A, vol. lxxviii. p. 9 (1907).

Trouton *. Modern investigators have, however, dealt chiefly with the question of the adsorption of dissolved substances at the surface of some solid body in contact with the solution. Among the numerous recent memoirs on the subject may be mentioned those of Küster †, Schmidt ‡, Walker & Appleyard §, Biltz ‖, Lagergren ¶, and in particular Freundlich **. Freundlich's investigations contain the most systematic and accurate determinations yet recorded of the adsorption at a charcoal surface of a series of organic substances in aqueous solution. The general method was to prepare a solution of a certain substance (say a dye-stuff), and to a known volume of this solution to add a certain quantity of very finely powdered pure blood-charcoal. A certain amount of the dye is removed by the charcoal, the change in concentration being determined colorimetrically.

Among Freundlich's most important results may be mentioned that the equilibrium state is reached exceedingly rapidly—in about five seconds; that, using the same solid body the quantity adsorbed varies with the chemical constitution of the solute, *e.g.* aromatic acids are adsorbed in greater quantity than aliphatic; and finally, that the effects differed with the solvent used.

We may sum up the characteristics of adsorption which differentiate it from chemical combination thus:—

(1) The *order* in which a series of dissolved substances are adsorbed does not differ even when one substitutes as the adsorbing material, bodies as unlike as charcoal, silk, clay, and cotton. This is unlikely on the assumption of chemical combination.

(2) The reaction goes to an end almost instantaneously.

* Trouton, Proc. Roy Soc. series A, vol. lxxvii. p. 292 (1906).

† Küster, *Zeitschrift Phys. Chemie*, vol. xiii. p. 445 (1894).

‡ Schmidt, *Zeit. Phys. Chem.* vol. xv. p. 60 (1894).

§ Walker & Appleyard, Journ. Chem. Soc. vol. lxix. p. 1334 (1896).

‖ Biltz, *Ber. d. deut. chem. Gesell.* vol. xxxvii. p. 1706 (1904); vol. xxxviii. p. 2963 (1905).

¶ Lagergren, *Zeit. Phys. Chem.* vol. xxxii. p. 174 (1900).

** Freundlich, *Zeit. Phys. Chem.* vol. lvii. p. 385 (1906); vol. lix. p. 284 (1907).

(3) Heat effects, even in concentrated solutions, are undetectable. Chemical action, on the other hand, at relatively low temperatures is usually accompanied by evolution of heat.

Characteristics which distinguish Adsorption from Absorption.

The difference existing here (which has been attested by numerous determinations) may be stated thus: There is *not* a direct proportionality between the concentration of the solution or partial pressure of the solute and the amount adsorbed. To take an example from Walker and Appleyard's* paper on the adsorption of picric acid at the surface of silk.

The quantities of picric acid (remaining after the reaction, in the aqueous solution and in the silk itself are estimated) the following being the results obtained:—

TABLE I.

milligrams Picric Acid in 1 c.c. solution.	milligrams Picric Acid in 1 gram silk.	Ratio $\frac{a}{b}$
(a)	(b)	
0·064	13	0·005
1·98	37	0·053
7·0	75	0·094

Assuming the molecular weight of the picric acid to be the same in water and silk, the ratio $\frac{a}{b}$ should have been constant had true *absorption* taken place.

II. THEORETICAL DISCUSSION.

Gibbs's Theory of Surface Concentration.

A theoretical investigation of this subject from the standpoint of thermodynamics forms one of the chapters in Gibbs's† memoir on "Equilibrium in Heterogeneous Systems." He

* Walker & Appleyard, *loc. cit.*

† 'Scientific Papers of J. Willard Gibbs.' English edition, vol. i. p. 219.

first takes up the consideration of the mass or bulk equilibrium of a heterogeneous system, *i. e.* equilibrium in which any surface phenomenon is of insensible magnitude, and then proceeds to examine the case in which the surface area is relatively large and the influence of surfaces of discontinuity upon the equilibrium of heterogeneous masses becomes of importance. To use his own words :—

* "The solution of the problems which precede may be regarded as a first approximation in which the peculiar state of thermodynamic equilibrium about surfaces of discontinuity is neglected. To take account of the condition of things at these surfaces, the following method is employed :—

"Let us suppose that two homogeneous fluid masses are separated by a surface of discontinuity, *i. e.* by a very thin non-homogeneous film. Now we may imagine a state of things in which each of the homogeneous masses extends without variation of the densities of its several components, or of the densities of energy and entropy, quite up to a geometrical surface (to be called the dividing surface) at which the masses meet. We may suppose this surface to be sensibly coincident with the physical surface of discontinuity.

"Now if we compare the actual state of things with the supposed state, there will be in the former in the vicinity of the surface a certain (positive or negative) excess of energy, of entropy, and of each of the component substances. These quantities are denoted by ϵ^s, η^s, m_1^s, m_2^s, etc., and are treated as belonging to the surface. The ^{s} is simply used as a distinguishing mark, and must not be taken for an algebraic exponent.

"It is shown that the conditions of equilibrium already obtained relating to the temperature and the potentials of the homogeneous masses are not affected by the surfaces of discontinuity, and that the complete value of $\delta\epsilon^s$ is given by the equation

$$\delta\epsilon^s = t\delta\eta^s + \sigma ds + \mu_1\delta m_1^s + \mu_2\delta m_2^s + \text{etc.}$$

in which s denotes the area of the surface considered, t the

* Gibbs, 'Scientific Papers,' vol. i. p. 365.

temperature, μ_1 μ_2 etc. the potentials for the various components in the adjacent masses....."

"The quantity σ we may regard as defined by the [above] equation itself or by the following:

$$\epsilon' = t\eta' + \sigma s + \mu_1 m_1' + \mu_2 m_2' + \text{ etc.}$$

from which by differentiation and comparison with the former we obtain

$$d\sigma = -\eta_s dt - \Gamma_1 d\mu_1 - \Gamma_2 d\mu_1 - \text{ etc.},$$

where η_s, Γ_1, Γ_2, etc. are written for

$$\frac{\eta'}{s}, \quad \frac{m_1'}{s}, \quad \frac{m_2'}{s}, \text{ etc.},$$

and denote the superficial densities of entropy and of the various substances. We may regard σ as a function of t, μ_1, μ_2, etc., from which, if known, η_s, Γ_1, Γ_2, may be determined in terms of the same variables. An equation between σ, t, μ_1, μ_2, etc. may therefore be called a *fundamental equation for the surface of discontinuity*."

The final equation obtained above has been simplified and applied by Gibbs * to an actual case, viz.:—

"If liquid mercury meet the mixed vapors of water and mercury in a plane surface, and we use μ_1 and μ_2 to denote the [chemical] potentials of mercury and water respectively and place the dividing surface so that $\Gamma_1 = 0$, *i. e.*, so that the total quantity of mercury is the same as if the liquid mercury reached this surface on one side and the mercury vapor on the other, without change of density on either side, then $\Gamma_{2,1}$ will represent the amount of water in the vicinity of the surface *above that which there would be* if the water-vapor just reached the surface without change of density, and this quantity (which we may call the quantity condensed [*i. e.*, adsorbed] upon the mercury) will be determined by the equation

$$\Gamma_{2,1} = -\frac{d\sigma}{d\mu_2}.$$

In this equation and the following, the temperature is constant and the surface of discontinuity *plane*.

* Gibbs, 'Scientific Papers,' vol. i. p. 235.

"If the pressures in the mixed vapors conform to the law of Dalton, we shall have for constant temperature

$$dp_2 = cd\mu_2 ;$$

p_2 denotes the part of the pressure in the vapor due to the water-vapor, and c the density of the water-vapor. Hence

$$\Gamma_{2,1} = -c\frac{d\sigma}{dp_2} \quad \ldots\ldots "$$

Now applying the gas law $P = RTc$, finally we obtain

$$\Gamma_{2,1} = -\frac{c}{RT}\frac{d\sigma}{dc}.$$

This equation is of fundamental importance in that, if we assume its applicability to adsorption measurements, it contains the relationships between bulk-concentration, surface-tension, and quantity adsorbed in an experimentally determinable form.

The same expression may be obtained more simply as follows* :—

Consider the equilibrium at a surface (say of solid or liquid) in contact with a solution.

Let σ = surface-energy per unit of surface.

s = area of surface exposed to the solution.

m^s = the mass of solute adsorbed at the surface of the solid *in excess* of that normally present.

The temperature is supposed to be constant.

U = total energy of the heterogeneous film per unit of surface.

U can be increased :—

(1) by increasing the surface area, in which case the work required $= \sigma ds$;

(2) by increasing the concentration of the solute in the interfacial layer. This is proportional to dm^s namely μdm^s, where μ is the chemical potential of the solute.

Then $$dU = \sigma ds + \mu dm^s,$$

$$\therefore \quad d(U - \mu m^s) = \sigma ds - m^s d\mu.$$

* I am indebted to Prof. W. B. Morton for this deduction.

And since $d(U - \mu m^s)$ is a complete differential, we have

$$-\left(\frac{d\sigma}{d\mu}\right)_s = \left(\frac{dm^s}{ds}\right)_\mu,$$

but $\frac{dm^s}{ds}$ = mass adsorbed per unit area of surface, *i. e.* = Γ;

$$\therefore \quad \Gamma = -\frac{d\sigma}{d\mu}.$$

Now since $$d\mu = \mathrm{RT}\frac{dc}{c},$$

where c = the concentration of the solute in the bulk of the solution,

it follows that

$$\Gamma = -\frac{c}{\mathrm{RT}}\frac{d\sigma}{dc}.$$

NOTE.—It should be noted that the expression obtained above occurs in Freundlich's paper already referred to. From the nature of his experiments it has been quite impossible to attempt a verification experimentally.

We have now to prepare a solution of certain strength c and measure the quantity adsorbed Γ as well as the value of $\frac{d\sigma}{dc}$. On substitution of the values in the equation we may obtain a verification or otherwise of the expression *. It may be stated at once that the results of experiments to be detailed do *not* show equality on the two sides of the equation—the discrepancy being always in the direction of Γ being many times greater than $\frac{c}{\mathrm{RT}}\frac{d\sigma}{dc}$.

* While this work was in progress, the results of somewhat similar determinations were published by Milner (Phil. Mag. Jan. 1907 [6] vol. xiii. p. 96). These referred to equilibrium at the *air*-liquid surface. He has deduced an identical expression to that already given in this paper, and has applied it to acetic acid in aqueous solution and aqueous sodium oleate. He has found that the actual quantity removed from solution (Γ) is nearly ten times the calculated effect $\left(\frac{c}{\mathrm{RT}}\frac{d\sigma}{dc}\right)$.

Reference might also be made to experiments by Zawidski (*Zeit. Phys. Chem.* xxxv. p. 77 (1900)) on the formation of the foam produced by the addition of saponine to aqueous solutions of hydrochloric and acetic acids. No measurements applicable to the present case were recorded.

III. Experimental Methods.

Use of a Liquid-liquid Interface.

As already stated, previous investigations with the exception of Milner's have dealt solely with adsorption at a solid surface—which does not allow of its surface-extent or surface-tension being determined. As it is necessary to evaluate these quantities, the possibility presented itself of measuring the adsorption of a solute at the interface between its solution and another liquid. This latter liquid must of course be absolutely chemically inert towards the solute itself and its solution. Choice fell on a hydrocarbon oil, since a body such as this is characterized by its chemical inertness. The next object was to obtain a substance, preferably soluble in water, which would possess the property of lowering the interfacial tension between water and oil. From their behaviour at the air-liquid interface it seemed likely that the soaps, saponine, and bile-salts would act in this way at the oil interface. The first experiments to be described were carried out with aqueous solutions of bile-salts.

In order to make certain that no chemical or solubility effects existed between the oil and the solutions, the following tests were carried out :—

(1) A portion of the oil was shaken up with water which was tested with phenolphthalein—neutral reaction, and therefore absence of free fatty acid as impurity.

(2) A portion of the oil was boiled for three hours with reflux condenser with $\frac{N}{10}$ methyl alcoholic potash—excess potash being titrated with standard acid. It was found that the acid required for neutralization was the same in amount as that for a blank experiment in which no oil was present. This proves the absence of fatty esters in the oil.

(3) Some of the bile-salt powder was shaken up with the oil, the latter allowed to stand, filtered, and a "drop-number' taken with the pipette (see later) against distilled water. The same drop-number was obtained as with the untreated oil fresh from the stock. Since the tension is a very delicate test for the presence of the salt, the above shows fairly

conclusively that there was no salt present in the oil after filtration, *i. e.*, the salt is insoluble in the oil.

(4) The oil was shaken up with a solution of the salt, the oil filtered off and a drop-number taken against water. The same result exactly was obtained as with the oil which had been simply shaken up with distilled water filtered off and a similar drop number taken against water.

The above tests point conclusively to the fact that no effects of the nature of chemical combination or solubility take place between the oil and the bile-salt.

Examination of the Bile-salt.

A quantity of "sodium glycocholate" was obtained from Merck. On close examination, however, it appeared that this was far from pure. Besides sodium glycocholate there is also sodium taurocholate and other fatty acid alkali-salts. There were no inorganic substances such as sodium chloride or carbonate. Several determinations of the molecular weight (which is required, as will be seen later) by means of the lowering of freezing-point and rise of boiling-point of water gave as a result 140. Assuming complete dissociation in water, this would give the undissociated molecular weight 280. This latter was confirmed by a determination by the rise of boiling-point of alcohol, the result being 283. The osmotic molecular weight in water (viz. 140) is, however, the quantity required in the subsequent calculations.

Fig. 1.

Method of measuring the Interfacial Tension.

The Drop-Pipette method was employed, the apparatus being of the form shown in fig. 1. This was first filled with oil (by suction at C) up to the mark E. The tap B was then closed during a determination. The aperture F is carefully wiped before using, and the pipette placed in a fixed position relatively to the solution in all determinations.

The bulb D is of about 45 c.c. capacity between E and E_1. The distance E F is about 8 cms. As soon as the tall jar containing the solution is placed in position, tap A is turned full on, the constriction above only allowing very slow entrance of air; and hence the drops of oil rising through the denser solution are formed with exceeding slowness—at 12 to 15 secs. interval. This, as Lord Rayleigh * has pointed out, is of great importance in order to obtain fairly accurate determinations; the approximateness of the method being due to the application of a statical theory to what is really a dynamic phenomenon.

The method consisted simply in counting the number of drops formed while the oil fell from E to E_1.

Theory of the Drop-Pipette Method.

It may be used in the first instance simply to determine the *relative* tensions of solutions of different concentrations, thus:—

Let V = total volume of oil used,

n = total number of drops formed,

then the volume of each drop $= \frac{V}{n}$;

and if ρ = the density,

the weight of a drop $= \frac{V\rho}{n}$.

The tension σ is taken to be proportional to the weight of a drop,

$$\therefore \quad \sigma = K\frac{V\rho}{n}.$$

Similarly for another liquid,

$$\sigma_1 = K\frac{V\rho_1}{n_1}.$$

Hence

$$\frac{\sigma}{\sigma_1} = \frac{n_1\rho}{n\rho_1}.$$

As a result of the exhaustive work of Guye and Perrot †,

* Lord Rayleigh, Phil. Mag. [5] vol. xlviii. p. 321 (1899).

† Guye & Perrot, *Archives d. Sc. Phys.* 4th ser. vol. xi. p. 225 (1901); vol. xv. p. 132 (1903).

Kohlrausch *, and Lohnstein †, a determination of surface-tension in absolute measure by means of the drop-pipette can be obtained as follows :—

The expression giving the weight of a drop in terms of the tension is

$$g = r\alpha\Phi,$$

where α = tension in millgrms./mm.
g = weight of a drop in milligrams.
r = radius of the orifice in mms.
Φ is a function of $\left(\frac{r}{a}\right)$, where
a^2 is given by $\frac{2\alpha}{\rho}$,

ρ being the density of the liquid in the pipette.

This applies to cases of liquid-gas tension, but it may perhaps be extended to the present case by taking into consideration the density of the medium in which the drop of oil is formed, *i. e.* the water.

Calling the density of the water ρ_w, we have

$$a^2 = \frac{2\alpha}{\rho_w - \rho_0}.$$

We have now to evaluate Φ. The ordinary method of using the Lohnstein-Kohlrausch formula is to take some approximate value for α; from this calculate a, and hence $\left(\frac{r}{a}\right)$. Kohlrausch ‡ has given a table showing the values of Φ for different values of $\left(\frac{r}{a}\right)$ and by inter- or extrapolation Φ can be evaluated. We then compare the value of g given by the expression $r\alpha\Phi$ with g actually obtained by experiment; and finally, by a series of approximations, α is given such a value as to make the calculated and observed values of g identical.

Now a determination by Pockels § gives the value 48·3 dynes as the tension between water "and a petroleum oil."

* Kohlrausch, *Ann. d. Physik*, vol. xx. p. 798; vol. xxii. p. 191 (1906).

† Lohnstein, *Ann. d. Physik*, vol. xx. pp. 237, 606; vol. xxi. p. 1030 (1906).

‡ Kohlrausch, *loc. cit.* p. 805.

§ A. Pockels, Wied. *Ann.* lxvii. p. 668 (1899).

Assume in the present case

$$\alpha = 48 \text{ dynes per cm.}$$
$$= 4{\cdot}89 \text{ milligrams per mm.}$$

Then $a^2 = 97{\cdot}8$;

$$\therefore\ a = 9{\cdot}8,$$
$$r = 1{\cdot}5.$$

Hence $\left(\frac{r}{a}\right) = 0{\cdot}152.$

Unfortunately this value of $\left(\frac{r}{a}\right)$ is a little beyond the lower limit of Kohlrausch's table, viz. :—

$\frac{r}{a}$	Φ.
.....	.....
0·300	4·45
0·288	4·48
0·200	4·66

For $\left(\frac{r}{a}\right) = {\cdot}152$, the value of Φ by inspection of these numbers (assuming the values do not change in direction down to $\left(\frac{r}{a}\right) = {\cdot}152$) would be

$$\Phi = 4{\cdot}72;$$

hence $g = 34{\cdot}62$ milligrms.

Now g found experimentally $= 35{\cdot}00$ milligrms.

The value for the interfacial tension was therefore taken to be 48 dynes per cm. To distinguish this oil from another hydrocarbon oil which was found to have a tension of 33·6 dynes, they will be throughout designated oils "A" and "B," viz. :—

Interfacial tension { Water-oil A ..... 33·6 dynes per cm.
Water-oil B ..... 48 dynes per cm. *

* *Note.*—It may be as well to state here that although oil A was as carefully tested as oil B, the writer would lay greater stress on the accuracy of the determinations made with oil B.

Variation of the Interfacial Tension with the Concentration of Sodium Glycocholate Solutions.

First Series.—Oil A.

Aqueous solutions of the sodium glycocholate were prepared of the following concentrations:—

1 %, 0·8 %, 0·5 %, 0·33 %, 0·25 %, 0·18 %, 0·125 %, 0·1 %, 0·05 %, 0·01 %, 0·001 %.

On being freshly prepared, the aqueous solutions of these concentrations were quite transparent, but with considerable rapidity—2 to 3 hours—in the case of the more dilute, a white turbidity manifested itself. All tension measurements were therefore made with fresh transparent solutions so that the values obtained might be quite comparable.

The following table gives the results obtained with oil A. The first column gives the concentration of the solution; the second, the corresponding drop-number; the third gives the relative tension as deduced from the relation already obtained (pp. 159, 160), viz., that the drop-numbers are inversely as the tension; the fourth column gives the absolute values of the tension based on the oil-water determination 33·6 dynes/cm. and by applying the values of column 3.

Table II.—Oil A.

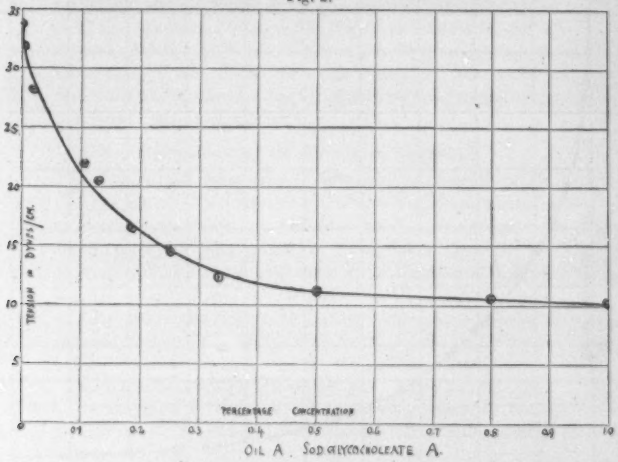
Per cent. Concentration.	Drop-number.	Relative Tension.	Tension dynes/cm.
0	183	1·000	33·60
0·001	191	0·958	32·20
0·01	219	0·836	18·08
0·05	244	0·750	25·21
0·1	280	0·654	21·96
0·125	297	0·616	20·70
0·18	365	0·501	16·81
0·25	418	0·438	14·71
0·33	489	0·374	12·57
0·5	536	0·341	11·48
0·8	558	0·328	11·03
1·0	566	0·323	10·86

The above values in columns one and four are plotted in fig. 2. It will be noticed how exceedingly marked is the lowering effect of the sodium glycocholate upon the interfacial tension.

We are therefore, now, in a position to obtain an experi-

mental value for $\frac{d\sigma}{dc}$ by simply taking the value of the tangent to the curve fig. 2 at the desired concentration.

Fig. 2.



Second Series.—Oil B.

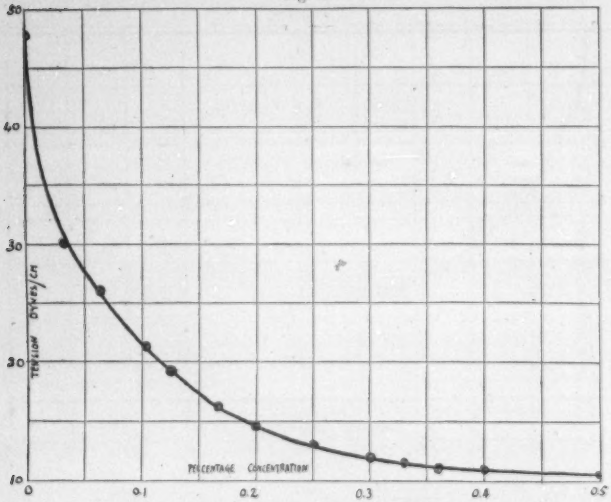
A similar series of determinations with oil B are given in the following table:—

Table III.—Oil B.

Per Cent. concentration.	Drop-number.	Relative Tension.	Tension dynes/cm.
0	128	1·000	48
0·0312	202	0·633	30·38
0·0625	233	0·549	26·35
0·125	317	0·404	19·39
0·165	375	0·341	16·37
0·200	417	0·307	14·73
0·250	471	0·272	13·05
0·300	522	0·245	11·76
0·330	531	0·241	11·57
0·360	557	0·230	11·04
0·400	568	0·225	10·80
0·500	585	0·219	10·51

The above absolute values of Tension and Concentration are plotted in fig. 3.

Fig. 3.



In order to ascertain how far the values for the Tension obtained directly from the drop-number (column 4, Table III.) agreed with those obtained by applying Kohlrausch's formula

$$g = r\alpha\Phi$$

in each case, a comparison is made of the weights of the respective drops, using the values of the tensions for the two concentrations ·25 per cent. and ·5 per cent.

Per cent. Concentration.	Weight of drop $g = r\alpha\Phi$.	Weight of drop actually found.
zero, *i.e.* distilled water	34·62 milligrms.	35 milligrms.
·25 per cent.	8·89 "	8·60 "
·5 "	7·01 "	6·93 "

Having in the foregoing obtained data for the experimental determination of the coefficient $\frac{d\sigma}{dc}$ in the case of two different oils, the next procedure was to determine the value of Γ, *i. e.* the mass adsorbed per cm.^{2} of oil-surface in excess of that which would have been there had no adsorption taken place at all. To express this quantity Γ more briefly, the term "Adsorption Coefficient," following the nomenclature of Ostwald, has been employed.

Measurement of the Adsorption Coefficient.

Although the Adsorption Coefficient is defined as the excess mass of solute adsorbed, yet since the normal amount of solute per square cm. surface (when no adsorption has taken place) is very small in the present instance, this normal amount may be neglected in comparison to the total amount adsorbed per square cm. This is justifiable, as is evident from the following figures from an experiment detailed later:—

The bulk concentration of the glycocholate solution $=\cdot25$ per cent. or $\cdot0025$ grams per c.c.

Assuming the thickness of the surface layer to be of the order of the range of molecular attraction, namely, $13\cdot4\times10^{-6}$ cms. (according to Parks*), while the quantity adsorbed per cm.^{2} in this particular case is $5\cdot4\times10^{-6}$ gram, the layer evidently possesses a concentration of $\cdot403$ gram per c.c. The *surface* concentration is therefore about 160 times the *bulk* concentration.

It has been assumed that the thickness of the layer is of the dimensions of the range of molecular action. Of course the actual case is that there is a gradual "shading off" of the excess from the surface into the bulk of the solution, but *practically all* the excess is in the molecular surface-layer. This follows from the fact that the magnitude of the adsorption depends on the surface-tension, which is a phenomenon whose effects do not extend beyond the range of molecular attraction. As Gibbs† says:—"It is only within very small distances of such a surface that any mass is

* Parks, Phil. Mag. [6] vol. v. p. 517 (1903).

† Gibbs, 'Scientific Papers,' vol. i. p. 219.

sensibly affected by its vicinity—a natural consequence of the exceedingly small sphere of sensible molecular action."

The experimental determination of the adsorption coefficient of sod. glycocholate at the surface of a hydrocarbon oil was carried out, employing two distinct methods according as the adsorption took place under one of the two following conditions:—

(1) Adsorption at a very curved surface.
(2) Adsorption at an approximately plane surface.

(1) *Adsorption at a very curved surface.*

A certain volume of sod. glycocholate solution of known strength is shaken (by a motor-driven shaker) for several hours, with a known volume of oil so as to form a uniform emulsion. The bulk concentration of the solution after emulsification is estimated, and the fall in concentration gives the total quantity of glycocholate adsorbed. To measure the adsorbing area, the droplets of the emulsion are examined under a microscope having a scale of known value in the eyepiece, and the average diameter of a droplet taken. Hence we obtain the radius r and the volume of a droplet $=\frac{4}{3}\pi r^3$. Knowing the total volume of oil emulsified, we obtain the total number of droplets *, and since each drop has a surface area $4\pi r^2$, we finally obtain the total adsorbing area. Dividing the total quantity adsorbed by this area, the adsorption coefficient is determined.

The following are typical examples:—

Determination with Oil A.

One litre of sod. glycocholate solution approximately ·33 per cent. was made up, and of this 500 c.c. were shaken with ·447 c.c. oil for 12 hours and allowed to stand 18 hours. The volume of oil was obtained by weighings—the density of the oil being ·907. A drop of the emulsion was examined under the microscope.

* Assuming that the density of an oil-droplet in the emulsion will sensibly coincide with the density of the oil in bulk.

Average radius of a droplet $= \cdot 0000425$ cm.

Hence surface area of one droplet $= 4 \times 3 \cdot 1416 \times (\cdot 0000425)^2$.

and volume of one droplet $= \frac{4}{3} \times 3 \cdot 1416 \times (\cdot 0000425)^3$.

Total volume of oil emulsified $= \cdot 447$ c.c.

$\therefore$ Total number of drops formed $= 1 \cdot 3 \times 10^{12}$.

$\therefore$ Total adsorbing surface $= 31{,}553$ cm.^{2}

To estimate the quantity adsorbed:—

A drop-number was taken with the pipette through a portion of the original solution. Drop-number = 483, corresponding to a tension 12·80 dynes, which on fig. 2 indicates a concentration 0·318 per cent. After adsorption had occurred, a drop-number through the emulsion gave 459 corresponding to a tension 13·44 dynes, which again corresponds on fig. 2 to a concentration ·295 per cent.

$\therefore$ Fall in concentration $= \cdot 023$ per cent.

This was the general method adopted to determine change in concentration, it being much more delicate than any purely chemical means. It will be noticed that in the above the assumption is made that the emulsion particles themselves would not affect the value given by the pipette for the tension of the solution in which the emulsion is suspended. Justification of this is afforded by the concordance between the results obtained for the adsorption by this method and by an entirely different method to be described later. Continuing:—

Fall in concentration of ·023 per cent. = ·115 gram for the 500 c.c. solution employed.

Hence the mass adsorbed per cm.^{2}, *i. e.*

$$\Gamma = \frac{\cdot 115}{31{,}553}$$

$$\underline{\Gamma = 3 \cdot 6 \times 10^{-6} \text{ gram.}}$$

Similar determination of Γ, *using Oil B*:—

The solution made up gave a drop-number 531, corresponding on fig. 3 to concentration ·317 per cent. 250 c.c.

of solution were emulsified with ·160 c.c. oil (density ·900), and after emulsification the drop-number was 507, corresponding on fig. 3 to a concentration ·290 per cent.

Hence the fall in concentration .....=·027 p. cent.

∴ Total mass adsorbed from the 250 c.c. soln.=·067 gram.

Total adsorbing area .....=11058 cm.²

$$\therefore \underline{\Gamma = 5{\cdot}9 \times 10^{-6}} \text{ gram per cm.}^2$$

A further determination with a solution whose original concentration was ·2 per cent. gave a value for the adsorption coefficient

$$\underline{\Gamma = 4{\cdot}7 \times 10^{-6}} \text{ gram per cm.}^2$$

An estimation of the probable error of these values really depends on the estimation of the radius of the emulsion particle. As great care as possible was taken to obtain these very uniform in size. The radius of a drop is taken as being ·0000425 cm. The maximum value would be represented by

$$r = {\cdot}00005 \text{ cm.},$$

and as a minimum,

$$r = {\cdot}000035 \text{ cm.}$$

Substituting these values in the first determination with oil B, we obtain the following result:—

$r = {\cdot}000035$ cm.,	$\Gamma = 4{\cdot}4 \times 10^{-6}$ grm. per cm.²
$r = {\cdot}000050$ „	$\Gamma = 6{\cdot}9 \times 10^{-6}$ „ „
$r = {\cdot}0000425$ „	$\Gamma = 5{\cdot}9 \times 10^{-6}$ „ „

It is evident from these figures that the method is not sufficiently accurate to detect a different value of Γ corresponding to solutions of strengths ·317 and ·2 per cent. respectively. The different values obtained are within the limits of experimental error.

Comparison of Experimental values of Γ *with Calculated values.*

The calculated value for Γ is, as already pointed out,

$$-\frac{c}{\mathrm{RT}}\frac{d\sigma}{dc}.$$

Let us take as a example the second determination with oil B :—

The concentration of the solution $= c = \cdot 2$ per cent.
$= \cdot 002$ grm. per c. c.

Temperature $T = 289$ abs.

R is the gas constant. For one gram of solute,

$$R = \frac{2 \times 4 \cdot 2 \times 10^7 \text{ ergs}}{\text{molecular weight}}$$

The molecular weight determined as already described is 140 in aqueous solution.

The coefficient $\frac{d\sigma}{dc}$, read from the curve fig. 3 at the point where $c = 2$ per cent., gave

$$\frac{d\sigma}{dc} = \frac{9 \cdot 5 \text{ dynes}}{\cdot 002 \text{ grm. per c. c.}} = 4750.$$

The true value of this probably lies between the limits 4800–4700, *i. e.*, an error of about 2 per cent.

Hence

$$-\frac{c}{RT}\frac{d\sigma}{dc} = \frac{\cdot 002 \times 4750 \times 140}{2 \times 4 \cdot 2 \times 10^7 \times 289}$$

$$= \underline{5 \cdot 5 \times 10^{-8}} \text{ grm. per cm.}^2$$

$$\Gamma \text{ (found)} = \underline{4 \cdot 7 \times 10^{-6}} \text{ grm. per cm.}^2$$

That is to say, the actual quantity adsorbed is about 85 times the calculated amount. This is far beyond experimental error, and there can be no doubt that a real discrepancy does exist. To confirm this, however, a different method of experimentally determining Γ was resorted to in which emulsions were dispensed with, as it was thought possible that the radius of the drops might be too near the value for the range of molecular forces.

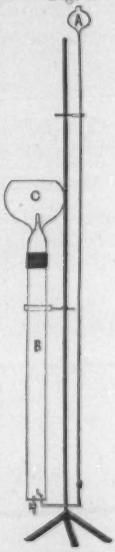
(2) *Adsorption at an approximately plane surface.*

In the previous emulsion method we have been dealing with tension and adsorption at very curved surfaces. Owing to the discrepancy obtained in the above method, it is

necessary to carry out determinations at a practically plane surface—*i. e.*, one of very small curvature in comparison with the curvature of the sphere of molecular action.

For this purpose the apparatus shown in fig. 4 was devised. It consists of a bulb A, of about 170 c. c. capacity, which is filled with oil. This bulb is connected to a long vertical narrow tube (1½ metres high), having a rubber joint and pinchcock near the base so as to regulate the flow of oil. The upwardly directed nozzle of this tube is inserted through a cork into a wider tube B (1 cm. radius), which contains the solution through which the oil rises in the form of large drops. The tube B becomes constricted near the top, and then opens into a large cup-like vessel C. The sod.-glycocholate solution (about 1 litre) is poured into B and C until the constricted end of B is about 3 cms. below the surface of the solution. The upwardly moving drops of oil adsorb on their surface some of the solute and carry it through the constriction, the drops eventuallycoalescing. The object of the constriction is to prevent the sod. glycocholate, which has once more been partially returned to solution (by the coalescence), from being carried back into the lower part of the tube. When A is empty, the tube B (which has a rubber joint near the top) is nipped at this point and the upper portion disconnected. The lower solution (which is 250 c.c. in volume) is run off from below, and a drop-number through it taken with the pipette. The fall in concentration is read off by means of fig. 3—or, rather, a somewhat enlarged form of fig. 3; the enlargement being necessary for accurate readings on account of the very small changes in concentration produced by this method, the fewer large drops offering a very much smaller surface than the emulsion particles.

Fig. 4.



Example: *Oil* B.

One litre of ·25 per cent. sod. glycocholate was prepared. The solution was poured into B and C as described, and after the drops have been passing for about 2 hours the reservoir A was empty and the contents of tube B run off. Through this solution a drop-number gave 466, *i. e.*, a tension of 13·2 dynes, corresponding on the curve to a concentration ·243 per cent.

Hence the fall in concentration = ·007 per cent.

∴ for 250 c.c. (the volume of B) the total quantity removed = ·0175 grm.

The volume of the oil-reservoir = 168 c.c.

The time of formation of 50 drops was taken at the beginning, in the middle, and near the end of the experiment. It was found that the middle value was the mean of the beginning and end, viz. 50 drops in 40 seconds.

Total time of emptying A . . = 7940 secs.
∴ total number of drops . . = 9925.
Hence volume of a drop . . = ·0169 c.c.
∴ radius of a drop . . . . = ·160 cm.
and hence total adsorbing area = 3192 cm.^{2}

∴ Adsorption coefficient = $5{\cdot}4 \times 10^{-6}$ gram per cm.^{2}

A second determination was made in which three times the former quantity of oil was used, and hence a greater fall in concentration was observed, which is in favour of a more trustworthy result:—

Total volume of oil used . . = 504 c.c.
Total time of dropping . . = 20520 secs.
Total adsorbing area . . . = 7414 cm.^{2}
Total mass adsorbed . . . = ·022 grm.

∴ Adsorption coefficient = $3{\cdot}1 \times 10^{-6}$.

It is worthy of note that these values are of quite the same order as those obtained at the very curved surface in the emulsion method.

The calculated amount for this particular concentration (0·25 per cent.) gave

$$\frac{c}{RT}\frac{d\sigma}{dc} = 3{\cdot}6 \times 10^{-8} \text{ gram per square cm.}$$

We therefore find in this case also, that the experimental value for Γ far exceeds the calculated value. Thus over 98 per cent. of the total observed effect is *un*accounted for in the theoretical calculation.

This result is unexpected, especially when one remembers that in this latter method of determining Γ the condition of planeness of surface is fulfilled to a very high degree of approximation; for the drops at the surface of which adsorption took place were of practically the same diameter as those produced by the drop-pipette in carrying out the measurements of $\frac{d\sigma}{dc}$; *i. e.*, the drops are very large in comparison to the sphere of molecular action. One would have expected, therefore, that the calculated result (dependent as it is on the value of $\frac{d\sigma}{dc}$) would be in good agreement with the experimental value of Γ.

One point of considerable interest lies, however, in the good agreement observable in the *experimental* values of Γ obtained by the two methods. On the one hand it appears to point (at least as far as one may rely on the accuracy of the determinations of Γ) to the fact that the tension even at very great curvatures does not differ appreciably from that at an approximately plane surface. On the other hand, since the portion of the adsorption which is unaccounted for is so much greater than the value required by theory, it is possible that even considerable variations in this latter would be undetected owing to the "swamping" effect of the excess values.

Discussion of Results.

The net result of the foregoing experiments is to show that between observed and calculated values there is a very great discrepancy—the actual quantity adsorbed being about eighty times the calculated amount. The most obvious explanation lies in the existence of some assumption in the theory which has been overlooked in practice.

The following are Gibbs's assumptions in the deduction of the equilibrium equation :—

(1) The adsorbing surface is plane.
(2) The solvent is to show no concentration at the interface.
(3) For the particular equation used in the present paper, there is supposed to be only one component capable of being adsorbed.
(4) This component and the solvent in which it is dissolved are supposed to form a single phase.

(1) *Planeness of the surface.*—It may be readily assumed that adsorption measurements made at the surfaces of oil-drops of sensible magnitude approximate exceedingly closely to those at a plane surface.

(2) *Absence of surface-concentration of the solvent.*—This was simply assumed to be the case, no means of testing its validity having as yet suggested itself.

(3) *Adsorption of a single component.*—This follows from the general experimental conditions, the solute, the sodium glycocholate, being the only substance whose adsorption is measured.

(4) *Adsorption from a monophase system.*—This assumption is implied from the actual example of surface-concentration given by Gibbs as exemplifying his theory, viz.:—A mixture of mercury- and water-vapours meeting at a liquid mercury surface, the water being the component which suffers surface-concentration. Mixtures of vapours are essentially monophase systems; and the question is, are we dealing with a monophase system in the case of an aqueous solution of sod. glycocholate? The evidence given by its osmotic behaviour in raising the boiling-points of water and alcohol and lowering the freezing-point of water, is strongly in favour of its being a true electrolyte, and hence of its solution being a monophase system.

Temperature Effects accompanying Adsorption.

All attempts at a direct determination of heat evolution or adsorption at the oil surface were negative. It may be shown, however, indirectly that a small *evolution* of heat must occur, from the observed variation of the interfacial tension with temperature. Thus, with a solution of concentration ·3 per cent. and oil B, the following results were obtained:—

Temperature.	Tension in dynes per cm.
16° C.	11·76
41° C.	12·96
67° C.	14·4

We have here the remarkable phenomenon of the tension *increasing* with the temperature. But in similar experiments, using water alone instead of solution, there was a marked fall in the tension, as was to be expected, viz.:—

Temperature.	Tension in dynes per cm.
16° C.	48
41° C.	40·8
67° C.	37·1

The increase in tension with temperature in the case of the glycocholate solution is to be accounted for, therefore, by partial "desorption" having taken place; that is, an increase of temperature in the case of the glycocholate solution decreases the quantity adsorbed, and hence, in accordance with the principle of Le Chatelier, adsorption must be accompanied by heat *evolution*.

Further experimental determination may make it possible to calculate the quantity of heat (Q) evolved per gram-mol. adsorbed by applying the equation

$$\frac{d \log K}{dt} = \frac{Q}{RT^2},$$

where K = ratio of concentration in surface layer to the bulk concentration.

IV. Application to the Theory of Dyeing.

The various theories which have been advanced to account for the process of dyeing may be roughly divided into

(1) Purely chemical combination or solid solution.
(2) Purely physical surface effects (*i. e.*, Adsorption).
(3) Partially surface effects and partially chemical combination.

A great mass of evidence has been brought forward in support of these different views, but it is not proposed in this place to discuss them *. Mention should be made, however,

* For an account of these various theories one may consult the work of W. P. Dreaper—'Chemistry and Physics of Dyeing' (1906).

of the most recent contribution, namely that of Freundlich *, whose results point very strongly to the absence of chemical combination, but leave it an open question as to whether dyeing is an example of adsorption in the sense of Gibbs's theory.

It is evidently essential to this view that dye-stuffs in solution should possess the property of lowering the interfacial tension. This, of course, cannot be determined in Freundlich's experiments owing to the presence of the solid phase, but considerable support would be given to the adsorption theory of dyeing if it were shown that dye solutions actually did lower the tension at the interface between the solution and an inert liquid. This liquid must of course be such as to exclude both chemical combination and solution.

With this object further measurements were made with the hydrocarbon oil B already experimented with, which was to function as the substance to be "dyed."

Aqueous solutions of various concentrations of the following dyes were prepared:—Congo red (sodium salt), methyl orange (sodium salt).

It is hardly to be expected that in any of these cases there would be solution in, or chemical combination with, the oil.

Congo Red.

A preliminary trial with this substance showed that there was a distinct lowering of interfacial tension. The following table gives the results obtained for solutions of different concentrations.

TABLE IV.

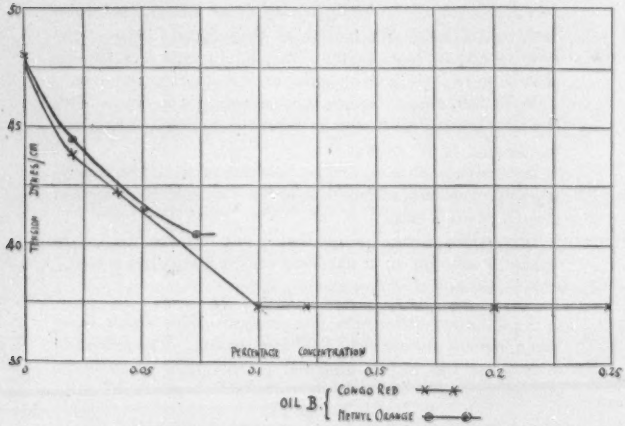
Per cent. Concentration.	Drop-number.	Relative Tension.	Tension in dynes/cm.
0·0	128	1·0	48
0·02	140	0·914	43·87
0·05	148	0·865	41·52
0·1	165	0·775	37·20
0·12	165	0·775	37·20
0·2	165	0·775	37·20
0·25	165	0·775	37·20

* Freundlich, *Zeitschr. Phys. Chem.* vol. lix. p. 284 (1907).

The results of Table IV. are plotted in fig. 5 (lower curve).

It will be noticed that after the concentration has reached 0·1 per cent. there is no further effect on the tension. This probably represents the maximum true solubility of the dye; and the minimum value for the tension might possibly be made use of as a practical method of ascertaining the optimum concentration for the dye-bath.

Fig. 5.



From the lower curve (fig. 5) we obtain the value of the coefficient $\frac{d\sigma}{dc}$, for the curve itself is practically a straight line. We may proceed, therefore, to apply the adsorption equation already obtained and calculate the adsorption coefficient (Γ).

Thus, taking as an example the solution of concentration 0·1 per cent.,

$$c = \cdot 001 \text{ grm. per c.c.},$$

$$\frac{d\sigma}{dc} = \frac{8 \text{ dynes}}{\cdot 001 \text{ grm./c.c.}}.$$

$$\therefore \frac{c}{RT}\frac{d\sigma}{dc} = \frac{\cdot 001 \times 8 \times 690}{289 \times 2 \times 4{\cdot}2 \times 10^7 \times \cdot 001} = \underline{2{\cdot}3 \times 10^{-7}} \text{ gram per cm.}^2$$

where 690 = the molecular weight of the undissociated Congo red, assuming it to be a monosodium derivative. Of course there is doubt as regards the value, since we are dealing with the substance when dissolved.

Thus, assuming complete dissociation in solution, the value of the molecular weight would be 345, and the resulting value for the adsorption coefficient would be $1{\cdot}15 \times 10^{-7}$ grm. per cm.^{2} On the other hand, if the dye is colloidal the molecular weight might possibly be much higher; though the behaviour of sod. glycocholate, by analogy, is rather against this.

Experimental result:—

250 c.c. of 0·1 per cent. Congo-red solution were shaken for fifteen hours with ·175 c.c. of oil. The resulting emulsion particles were examined under the microscope. It was found that $r = \cdot 0000435$ cm.

Hence, calculating exactly as in the sod. glycocholate solutions, we obtain the Total adsorbing area $= \frac{3 \times \cdot 175}{\cdot 0000435}$ cm.^{2}

The quantity adsorbed was determined as usual by taking a drop-number with the pipette through the emulsion. The drop-number was found to be 158, = that is, a tension 38·73 dynes/cm., corresponding on fig. 5 to the concentration ·082 per cent.

$\therefore$ Fall in concentration $= \cdot 045$ grm.

$\therefore$ Adsorption coefficient $\underline{\Gamma = 3{\cdot}7 \times 10^{-6} \text{ grm. per cm.}^2}$

It will be observed that this is *very much the same value as was* obtained in the case of the sod. glycocholate, and shows a large discrepancy as regards the calculated value. The fact, however, of a lowered tension shows that adsorption effects in the sense of Gibbs's theory cannot be absent.

Methyl Orange.

The solubility of this substance in water is small—a saturated solution being about 0·073 per cent. The following results were obtained:—

TABLE V.

Per cent. Concentration.	Drop-number.	Relative Tension.	Tension in dynes/cm.
0	128	1·0	48
0·018	138	·927	44·49
0·036	143	·892	42·81
0·073	152	·842	40·41

These values are plotted in fig. 5 (upper curve).

It will be noted that fig. 5 has a marked difference in appearance to figs. 2 & 3. This is of course due to the solubility limit being reached in the former case before the tension has fallen much in value.

Experimental determination of Γ :—

250 c.c. of ·07 per cent. solution were shaken with oil and formed an emulsion.

The Fall in concentration = ·032 per cent.

$\therefore$ Total mass adsorbed = ·08 grm.

Adsorbing surface area $= \dfrac{3 \times \cdot 204}{\cdot 0000435}$ cm.^{2}

$$\therefore \Gamma = 5{\cdot}5 \times 10^{-6} \text{ grm. per cm.}^2$$

Calculated value $= 1{\cdot}2 \times 10^{-7}$ " "

We have here a further repetition of the observed anomaly.

In considering the behaviour of these dyes in relation to the adsorption theory, it may be admitted that despite the discrepancy between the observed and calculated results, substantive dyeing, at any rate, must be more or less an adsorptive process. Possibly the actual mechanism of dyeing consists *first* in adsorption and *afterwards* coagulation possibly due to "colloidal neutralization." There may finally be some degree of chemical combination, which varies from case to case.

Note on Rosaniline hydrochloride and Crystal Violet.

A saturated aqueous solution of Rosaniline hydrochloride (about ·06 per cent.) gave a drop-number of 140—*i. e.*, a Tension of 43·87 dynes/cm. The decrease in interfacial tension is less marked than in methyl orange or Congo red. No further experiments were carried out with this substance. Also Crystal Violet dye concentration 1 per cent. gave a drop-number 230, and hence also shows a lowering of tension.

V. Summary.

1. The object of the investigation has been to examine from an experimental standpoint Gibbs's theory of surface-concentration, as the chief cause of the phenomenon of Adsorption. By modifying Gibbs's fundamental expression for surface-concentration, an equation is obtained giving the mass of solute adsorbed per cm.^{2} in terms of the concentration of the solution and the change in tension at the interface, aqueous solution—hydrocarbon oil.

2. The material employed consisted of aqueous solutions of bile-salts, the solute being adsorbed at the hydrocarbon-oil surface. The mass adsorbed per cm.^{2} oil surface was determined by two methods. The corresponding tension measurements were carried out by the drop-pipette method.

3. The results show a considerable discrepancy between the actual amount adsorbed and that calculated on Gibbs's theory—the actual amount measured being always in excess to the extent of twenty to eighty times the theoretical values.

4. Experiments with dye-stuffs show similar behaviour, the discrepancy being of the order stated.

5. As regards the discrepancy noted, no suggestion is as yet offered. Further experiments are in progress employing other electrolytes and non-electrolytes as material for measurements of a similar nature.

In conclusion, I would express my indebtedness to Professor Donnan for his advice and criticism during the course of this work.

Discussion.

Mr S. W. J. Smith remarked with reference to the Author's opinion that possible concentration changes at the surface of the mercury might have some bearing upon contact potentials, that Warburg had developed a theory of electrocapillarity based upon this assumption. It had been

shown that mercury drops falling through a solution of a salt produced changes of concentration, just as in the Author's experiments concentration changes were produced in a solution by rising drops of oil. There were at least two different interpretations of the mercury experiments.

Mr A. Campbell asked if the absorption of moisture from the air by cellulose was a case of adsorption, and also if the amount of adsorption in dyeing depended upon the temperature.

Mr Lewis said there was little doubt that the deposition of moisture on cellulose was a case of adsorption. Adsorption is accompanied by an evolution of heat, and a rise of temperature decreases adsorption.

X. *Observations on Recalescence Curves.* By Walter Rosenhain, *B.A., B.C.E., of the National Physical Laboratory.*

[Plate VI.]

The importance of the recalescence phenomena which occur in the cooling of steel and other alloys renders it desirable that the methods of observation used in the study of these phenomena, and the interpretations of the indications derived from these methods, should be fully discussed and understood. This is the more necessary since various workers have adopted different methods of observing recalescence phenomena, and hold somewhat divergent views as to the relative values of these methods. The author has therefore undertaken an investigation of two of these methods which, in various modifications, are widely used at the present time. This investigation has taken the form of a theoretical discussion of the physical facts which are represented by the cooling-curves obtained, and a consideration of the experimental means requisite for an approximate realization of the conditions indicated by the theoretical considerations. Examples of experimentally obtained cooling-curves are then given and discussed with a view to illustrating the possibilities of the various methods. Finally, an observation of a recalescence in crystalline silica, which was observed in the course of the experiments made in connexion with the present investigation, is described.

* Read January 24, 1908.

The two methods of obtaining cooling-curves, to which reference has already been made, may be called the "time-temperature" and the "differential" methods respectively. These names are justified by the fact that in the first class of measurements, observations of time and temperature are alone used for obtaining the data from which the cooling-curves are plotted; while in the second class time measurements are not employed, observations being made of the temperature of the body under experiment and, simultaneously, of the difference of temperature which exists between this body and a "blank" or neutral body cooling at approximately the same average rate. As will be seen below, the data obtained by each method may be represented graphically in several different ways.

For the study of all cooling phenomena, we may regard as fundamental the curve which is obtained when observations of time and temperature are plotted in a simple time-temperature curve, on which any given ordinate represents the temperature of the cooling body at the time corresponding with the abscissa. If the body in question

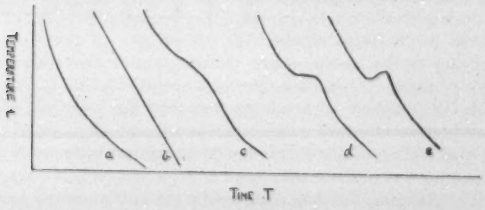


Fig. 1.

cools steadily, *i.e.* without undergoing any changes involving an evolution or absorption of heat or a change in specific heat, this curve will take a form somewhat like fig. 1 (*a*), where temperature (*t*) is plotted vertically and time (T) is plotted horizontally. The form of this curve will, of course, depend on the law of cooling followed by the body in question; for the case of a constant rate of cooling the curve will be a straight line, as in fig. 1 (*b*). If, during

the cooling process, the body undergoes a change involving an evolution of heat, the resulting curve on this diagram will take the form of fig. 1 (*c*), (*d*), or (*e*), according to the quantity of heat evolved and to the rate at which it is evolved. As there is a connexion between the shape of the curve and both the quantity of heat evolved and the rate of its evolution, it is interesting to enquire whether the form of the curve, or of a curve derivable from it, can be utilized to give quantitative data regarding the heat evolutions.

It is evident that at any point A of a time-temperature curve (fig. 2) the body under observation is losing heat at a definite rate. In the absence of any recalescence this rate of loss of heat would result in a rate of cooling indicated by a curve whose shape would be the same as that of the cooling-curve of a neutral body of the same thermal constants. At the moment of the beginning of recalescence (A) the body is losing heat at a rate proportional to the angle which the tangent to the cooling-curve at that point makes with the axis of time. In the case of a small evolution of heat, which does not produce a very large hump on the curve, it will not involve any very great error to suppose, first, that the rate of cooling of the ideal equivalent neutral body, cooling along the line of the dotted curve ABC, would have remained constant during the time occupied by the recalescence; and, secondly, that the actual body is losing heat at this same rate during the entire time of the recalescence. With these assumptions made, it follows that the quantity of heat generated is proportional to the greatest vertical distance between the two curves ABC and ARC, which is indicated by the line BR in the diagram. This line represents the greatest amount by which the temperature of the recalescent body exceeds that of an ideal equivalent neutral body. We may illustrate this from the special case where heat is added to a body which is not losing any appreciable amount of heat.

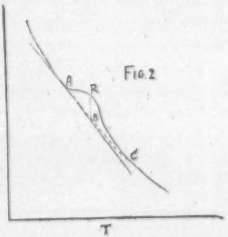


Fig. 2

In that case the curve ABC is a horizontal line, and the rise of temperature produced is proportional to the quantity of heat entering the body. In the case of a body losing heat at a *constant* rate, we may approximately write "excess of temperature" in place of "rise of temperature" in the adiabatic case.

In practice, however, two errors, corresponding to departures from the conditions assumed above, make themselves seriously felt. In the first place, if the recalescence extends over a considerable time, either because the quantity of heat evolved is very large, or because the evolution does not take place at one definite temperature but is spread over a considerable range, the rate of cooling can no longer be regarded as constant. This departure from the assumed conditions will result in making the line BR shorter than it should be if strictly in proportion to the quantity of heat evolved. Further, during the time that the recalescent body is passing along the portion of the curve from A to R its conditions of cooling, or rather the rate at which it is giving out heat, are not identical with its condition in this respect when at the point A. At the latter point the body is still cooling uniformly with its immediate environment, *i. e.* the furnace etc. in which it is contained, but as soon as the fall in the recalescing body is arrested, a difference of temperature is set up between the recalescing body and its environment which results in a loss of heat from the body at a greater rate than that denoted by the tangent to the curve at A. This also will result in shortening the line BR, more particularly in the case of rather rapid and vigorous evolutions of heat. It must, however, be borne in mind that a full discussion of the phenomena must also take into account the fact that the temperature of the furnace will also be affected by the more rapid transfer of heat from the recalescing body—a process which tends to lessen the amount of error introduced by the effect previously considered. All that we are therefore justified in concluding is, that there is a relation between the amount of heat generated and the length of the line BR, but that this relation only approaches proportionality under conditions of gradual cooling at a nearly constant rate and

for evolutions of heat which do not produce very large humps in the curve.

With the same degree of approximation just indicated for the proportionality between the quantity of heat evolved and the length of the line BR (*i.e.*, the vertical distance between the two time-temperature curves), the shape of the curves can also be used as a guide to the rate of evolution of heat, *i.e.* the intensity of the reaction producing the recalescence. At any instant up to the end of recalescence, the length of the line BR represents, approximately, the quantity of heat generated, and therefore the rate at which this length increases measures the rate of evolution of heat. If the curve ABC is approximately a straight line, the rate of increase of BR will be a maximum where the curve ARC approaches most nearly to the horizontal.

In practice it is not convenient to plot the observations of cooling, even when taken in the form of time and temperature measurements, in the form of a simple time-temperature curve such as we have so far discussed. The observations taken are generally those of the time intervals which elapse while the temperature of the body under observation falls through a definite small range. These intervals are then plotted as abscissæ with the temperature of the body as ordinate. In this way the well-known inverse-rate curves are obtained. It will be seen at once that this curve simply represents the differential coefficient with respect to temperature of the simple time-temperature curves which we have just discussed. In those curves time (T) was plotted against temperature (t), while in the inverse-rate curve the time dT, occupied by a fall of temperature (dt), *i.e.* $d\mathrm{T}/dt$, is plotted against temperature (t). The relationship between the two curves is now readily understood.

In fig. 3 let the line LMN represent an inverse-rate cooling curve. The area enclosed by an infinitesimal element of the curve and its abscissæ with the axis of t is given by $d\mathrm{T}/dt \,.\, dt$, and therefore the area included between the curve and the axis of t between values t_a and t_b of t is $\int_a^b d\mathrm{T}/dt$ and the value of the integral is $\mathrm{T}_b - \mathrm{T}_a$; *i.e.*, it represents the time

which has elapsed during the cooling of the body from the temperature t_a to the temperature t_b. Now, if we find the value of this time both for the recalescing body and for an ideal equivalent neutral body, and subtract one from the other, the difference represents the retardation in time which has resulted in the cooling of the body under observation in consequence of the evolution of heat. This retardation in time, multiplied by the rate of loss of heat from the body under observation during that time, measures the difference between the quantity of heat lost by this body and an ideal neutral body cooling at the same rate, and this excess of heat is obviously the quantity generated by the recalescence.

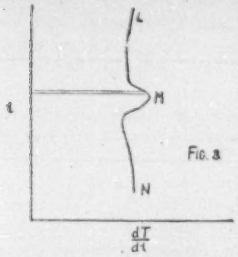


FIG. 3.

At first sight, therefore, such a curve appears to give much useful information; but the difficulty of knowing the rate of loss of heat of the recalescing body introduces exactly the same errors here as we have already considered in the case of the line BR (fig. 2), and a further difficulty remains to be considered. This arises when we have to decide between what limits the time, *i. e.*, the area of the curve in fig. 3, is to be taken. The correct limits are obviously the beginning and end of the recalescence; but while the beginning of the recalescence may be well marked, the end is not readily ascertained. Owing to the fact that during the recalescence the fall of temperature of the recalescing body lags behind that of the furnace, as soon as the recalescence has ceased, the body under observation for a time undergoes much more rapid cooling, ultimately—more or less according to circumstances—completely recovering its equilibrium with the furnace in which it is contained. The result is, on the time-temperature curve, a steep portion following the flattening due to the recalescence, while the inverse-rate curve approaches considerably nearer to the zero than does the curve of an ideal neutral body. From the present point of view it follows that if the time T in the inverse-rate curve

be taken over too wide a range, the acceleration of cooling just described will materially affect the result: indeed, this is almost obvious when we consider that the total time occupied by the cooling of a small specimen of steel, for example, contained in a fairly large furnace is not very much affected by the small amount of heat generated by the recalescence of the steel. The choice of the limits between which the value of T shall be taken on the curve in fig. 3, therefore, becomes of serious importance. It will not be correct to take the point where the time-temperature curve of the recalescing body becomes parallel to that of the ideal neutral body as indicating the end of the recalescence. The recalescing body is losing heat rather faster than the neutral body would do at the same instant, so that when the rate of fall of temperature in the hotter body becomes the same as the rate of fall in the colder (neutral) would be, there must still be an evolution of heat making good the difference in the rates of loss of heat. The end of the recalescence therefore lies just beyond the point where the actual observed curve and the ideal blank become parallel. In the inverse-rate curves this is represented by a point just beyond that at which the observed curve crosses the ideal blank. It is from the beginning of the recalescence up to this point that the time (T) must be measured by means of the area of the peak of the inverse-rate curve if a value proportional to the amount of heat generated, divided by the mean rate of loss of heat from the body during recalescence, is to be obtained. We have already indicated that this cannot be done with any degree of certainty. All that can be said of the inverse-rate curve therefore, is that, provided the rate of cooling be the same in the two cases, a peak of larger area on such a curve, indicates the evolution of a greater quantity of heat, but this comparison cannot be relied upon to any degree of accuracy unless the general shape of the two peaks to be compared is similar. Where recalescences extend over long ranges of temperature, the quantitative indications of these curves are very vague indeed. Perhaps the most important case in which the comparison of the areas of several peaks would be of interest, if they could be correctly interpreted

in terms of heat evolution, is that of a single cooling-curve showing several recalescences at various temperatures, such as are met with in the cooling of mild steel. Here the difficulty arises that the rate of cooling when the body is at a temperature near 800° C. is different from that when the same body is near 700° C.; and, unfortunately, the change in the rate of cooling of an actual blank body such as a piece of platinum or of asbestos, cannot be applied directly to the case of the recalescing body, for the reasons already indicated. In cases of this kind, the state of affairs is further complicated by the fact that frequently a second recalescence commences long before the end of the rapid cooling which follows the cessation of the previous recalescence, with the result that—roughly speaking—a fresh hump is superimposed on the backward-sloping portion of the time-temperature curve. In such cases it is evident that the conditions are too complicated to allow of any quantitative evaluation of the phenomena, and the indications of the curves can be regarded as qualitative only.

As regards the intensity of a recalescence, *i. e.* the rate of evolution of heat, we have seen above that this is represented by the rate of change of the vertical distance between the observed and the ideal blank time-temperature curves—this representation being understood with the same limitations that apply to the other indications of these curves. This rate of change of vertical distance is simply equal to the difference between the slopes of the two curves. Assuming that the slope of the ideal curve is constant during the recalescence, this difference becomes inversely proportional to the slope of the observed time-temperature curve. But the slope of the observed curve is the rate of cooling of the observed body, and its inverse is the "inverse rate" (dT/dt), which is plotted in the inverse-rate curves under discussion. We thus see that—to a rough degree of approximation—the height of the apex of the peak of an inverse-rate curve is proportional to the intensity of the heat evolution producing the recalescence, and its position on the temperature-scale indicates the temperature at which the reaction proceeds most vigorously. Here also the limitations that attach to the estimation of the quantity of heat evolved apply with

full force, so that it is only under special conditions that any quantitative value can be attached to the height of a peak. As a rule nothing more can be said than this, that under reasonably similar conditions of cooling a tall peak represents a rapid and a flat peak a gradual evolution of heat.

We turn now to a consideration of the differential method of obtaining cooling-curves. As already indicated, this method is independent of time measurements; what is actually determined is the difference in temperature between the body under observation and a blank or neutral body cooling at approximately the same rate, the observations of this difference of temperature being plotted as abscissæ against the actual temperature of the body under observation as ordinate. In the ideal case, which although not realizable in practice we may use as the basis of our discussion, the blank body is supposed to cool at the same rate as the recalescing body would have done in the absence of recalescences, and the time-temperature curves of the two bodies would then be represented by the two curves ABC and ARC in fig. 2. In that case, the observed differences of temperature would simply represent directly the values of a succession of vertical lines such as BR. In that case, the interpretation of the differential curve would be very simple, since—again with the limitations previously insisted upon—the maximum value of BR is proportional to the quantity of heat evolved, while the rate of evolution of heat is proportional to the rate of change of BR. In the ideal differential curve, therefore, the maximum height of the peak representing a recalescence would be proportional to the quantity of heat evolved, while the maximum rate of change of angle of the curve would measure the rate of evolution of heat during the change. The actual form of such an ideal differential curve is indicated in fig. 4, where, in the absence of recalescences, the curve is vertical. This condition is, however, not attainable in practice, since it involves the necessity for a similarity of properties as regards specific heat, thermal conductivity, &c. between the neutral or blank body and the body under observation. A case which is obtainable in practice is illustrated diagrammatically in fig. 5. Here,

quite apart from the occurrence of recalescences, there is a steady divergence between the time-temperature curves representing the cooling of the blank body and the observed body respectively, with the result that the differential curve shows a steady slope in one direction upon which the peak due to recalescence is superposed.

FIG 4

Provided that the slope of the differential curve is not too great, *i. e.* provided that the blank body does not cool much more rapidly or much more slowly than the body under observation, recalescences superposed on such a slope appear

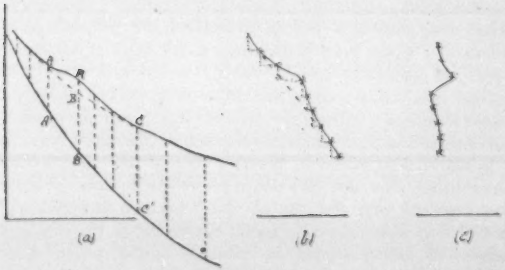


FIG. 5.

as perfectly definite peaks and are capable of the same interpretation as before. In fig. 5 (*a*), using the same notation as in fig. 2, we find that the abscissæ of the differential curve represent the lengths AA′, BB′, CC′, &c. If the recalescence

does not extend over a very great range of temperature, and if the divergence between the curves ABC and A′B′C′ is not very great, then we may take AA′=BB′=CC′, and the height of the peak of the differential curve still represents the length BR+ a quantity which is simply the average difference between the temperatures of the two bodies at the beginning of the recalescence. Even if the rate of divergence between the temperatures of the two bodies is considerable, we can arrive at an approximate estimate of the value of this average from the general slope of the differential curve. Graphically this is equivalent to drawing an assumed blank as indicated by the dotted lines in fig. 5 (*b*), and measuring the height of the peak of the differential curve from this assumed line. The error involved in making such a correction is in most cases smaller than the inaccuracy which, as we have seen, our ignorance of the exact cooling process renders unavoidable in connexion with the quantitative interpretation of all forms of cooling-curve.

In practice, particularly where rapid cooling is employed, it is sometimes found that the general slope of the differential curve becomes excessive, thus rendering the detection of the beginnings of small peaks difficult and uncertain; sometimes, indeed, this difficulty is felt in reading the indications of differential curves even when the general slope is not excessive. This difficulty will be readily understood when it is realized that fundamentally the differential curve is merely a reproduction on a somewhat different scale and on another base-line of the time-temperature curve. Therefore, just as it is found convenient in practice to plot the inverse-rate curve rather than the time-temperature curve, so it occurred to the author that the use of the "Derived Differential" curve *," in which the differential coefficient of the ordinary differential curve should be plotted, might prove more intelligible than the first differential curve itself (fig. 5 *c*). The examples shown in the second part of this paper have justified this supposition, and it therefore becomes desirable

* The curves referred to in this paper as "differential" cooling-curves would perhaps be more correctly described as "difference" curves, but since the method of obtaining these curves is widely known as the "differential method," it was thought desirable to retain the term.

to consider the exact interpretation of these "derived differential" curves.

Returning to the fundamental time-temperature curves of the two bodies, we may write the equations of their cooling-curves as follows:—$t_1 = f(T)$ and $t_2 = F(T)$, where $f(T)$ and $F(T)$ are two different but unknown functions of T, except that in the case of a constant rate of cooling $f(T)$, which represents the cooling-curve of the neutral body, takes the simple form $t_1 = kT$ where k is constant. The abscissæ of the ordinary or "first" differential curve are the values of $t_2 - t_1$ for various values of T, and we have

$$t_2 - t_1 = F(T) - f(T);$$

calling this R, we have

$$\frac{dR}{dt_2} = 1 - f'(T)\frac{dT}{dt_2},$$

where $f'(T)$ is the differential of $f(T)$ with respect to T.

Now dR/dt_2 is the quantity that is plotted as abscissa in the derived differential curve. It is interesting to note that, for a constant rate of cooling, for which $f'(T)$ becomes either unity or a constant, the quantity dR/dt_2, plotted against t in the derived differential, becomes a linear function of the quantity dT/dt_2, which is the quantity plotted in the inverse-rate curve (see above).

We thus see that under the only conditions which render the curves interpretable at all, the derived differential and the inverse-rate curves take exactly the same form and are capable of exactly the same interpretation. Where the rate of cooling is not constant, the curves will be somewhat different, but qualitatively their indications must be identical, and—so far as they are capable of quantitative interpretation at all—this must be applicable to both with the same degree of accuracy. The essential difference between the two systems of measurement and representation, lies in the fact that the retarding effect of recalescences upon the cooling of a hot body is measured and represented in one case as a retardation in time, and in the other case as a retardation in temperature. In the time method the rate of cooling is measured against an assumed uniform rate of

cooling, while in the differential method the temperature is measured against an actually uniformly falling temperature. Since time and temperatures are dependent variables whose relation over short ranges of temperature at all events does not differ widely from a linear one, it is only natural that the curves obtained by the two methods, if properly compared, should yield identical results.

Returning for a moment to the derived differential curve, we find for the area of a peak the value $\int \frac{dR}{dt_2} dt_2$, and this is equal to R between the limits of the beginning and end of the recalescence. R, as we have seen, measures the quantity of heat evolved, so that, with the same limitations as already indicated for the inverse-rate curves, the area of a peak on the derived differential curve is also proportional to the heat evolved by the recalescence. The rate of cooling is supposed known and taken into account in both cases, since this factor affects both the maximum temperature difference attained and the time of retardation to the same extent. Further, in the same way as it was shown for the inverse-rate curve, the area in question must be measured to that point of the curve representing the end of the recalescence, and this is equally difficult to ascertain in both curves.

From a fairly full consideration of the physical facts which determine the shape of cooling-curves as found by either the time or the differential method, we are led inevitably to the conclusion that as far as either qualitative or quantitative interpretation, and in fact as far as the detailed form of the curves is concerned, there is no distinction between the inverse-rate and the derived differential curves, both proving equally valuable as qualitative guides to the existence of recalescence, but both proving equally vague as far as quantitative results are concerned, except under very special conditions which are not readily realized in practice. The answer to the further obvious question as to which of the two methods is preferable in practice, must depend upon experimental considerations alone. These considerations are next to be discussed.

The experimental conditions to be realized in taking time-

temperature observations for the purpose of drawing inverse-rate curves are very simple in character. An accurate means of measuring the temperature of the body under observation and of recording the times occupied by small decrements of temperature, are all the appliances required; but these must be made of sufficient delicacy and accuracy to detect very small retardations in the cooling-process. On the other hand, the cooling-process itself must be conducted with perfect steadiness, *i. e.* the rate of cooling, apart from recalescences, must be perfectly steady and regular and remain unaffected by external circumstances. Unfortunately in practice these two conditions are difficult to reconcile, since it is found that when sufficiently delicate means of registering time and temperature are employed, slight irregularities in the rate of cooling become apparent even in the case of a body free from recalescences. This in fact constitutes the most serious disadvantage of the time-temperature method, since variations in external conditions translate themselves directly to the resulting cooling-curve. Where it is desired to accurately locate small evolutions of heat, it therefore becomes necessary to repeat the series of observations several times; and this is not only laborious, but is in some cases open to the objection that repeated heating and cooling, even *in vacuo*, affects the constitution of metals, particularly of those in which meta-stable equilibria are met with.

As regards the degree of sensitiveness required in the time and temperature measurements for the observations by this method in order to arrive at a degree of sensitiveness similar to that attainable by the use of the differential method, we may consider a case of cooling under typical conditions. If we consider a body cooling at the rate of say one degree Centigrade in 12 seconds, *i. e.* at a rate of about 300° C. per hour, which is a reasonably slow rate, small recalescences are frequently met with which cause a prolongation of the time required for a fall of temperature of 1° C. from 12 seconds to 20 seconds at the maximum. Taking a case of this kind, we have the following comparison:—

	Temperature.		Time of falling. 1° C.		Difference of temperature on differential scale *.
From	651° C.			(say)	10·0° C.
	650	to	12″		10·4
	649		18″		10·9
	648		20″		11·2
	647		16″		10·8
	646		8″		10·5
	645		10″		10·4
	644		12″		10·3

If we plot the derived differential curve and the inverse-rate curve from these observations, we may find that they are—as we should expect—very similar in character (see fig. 6). The effect of errors of observation on the two curves will, however, be very different. In the case of the time observations, an error of 0°·1 C. in the estimation of

Fig. 6.

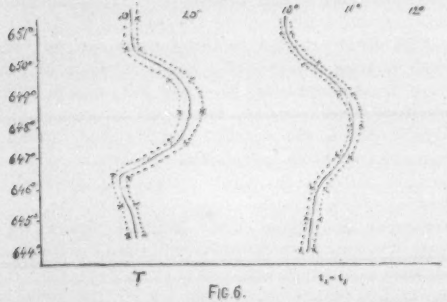


FIG 6.

the temperature would result in an error in the observed time of one-tenth of the total interval, and larger errors in the temperature measurement will result in proportionately larger variations in the observed time intervals. In the case of the differential readings, a similar statement applies—

* The differences of temperature given in column III. are such as would correspond to the time observations of column II., assuming a constant rate of loss of heat. The calculations have been made by drawing the curves on a large scale and assuming an initial difference of 10° C. between the bodies.

an error of 0°·1 in the estimation of the temperature difference between the two bodies will affect the readings for the derived differential curve to the extent of from 10 to 20 per cent. The extreme variations in the two curves resulting from absolute errors of 0°·1 C. in the observations are shown by the crosses on the two sides of the two curves. With observations taken on instruments whose limit of accuracy in both cases is represented by 0°·1 C., the observed curves would show irregularities lying within the two dotted curves in each case, and in both cases these irregularities would not suffice to mask even the small recalescence here considered.

The actual temperature of the cooling-body in the case of the inverse-rate observations, as also the difference of temperature between the two bodies in the differential method, is very frequently measured by the deflexion of a sensitive galvanometer. Assuming that the scale is divided in millimetres and that the position of the moving spot of light can be read at any instant accurately to one-fourth of a scale-division, then to attain the degree of accuracy referred to above, it becomes necessary that the sensitiveness of the galvanometer, scale-distance, &c., shall be so arranged that in both cases 0°·1 C. is represented by a deflexion of not less than 0·25 millimetre. Now in the case of the direct-reading inverse-rate method, the same galvanometer is required to indicate temperatures of some 800° C., and with the above requirement to be satisfied this would demand a scale length of about 2 metres. The galvanometer of the differential apparatus, on the other hand, is only required to indicate differences of temperature which rarely exceed 40° C., and for this purpose a scale-length of only 100 millimetres would be required under the same conditions as were assumed for the direct method. While the latter condition is obviously quite easily fulfilled by a simple deflexion instrument, the requirement of adequate sensitiveness for the direct method cannot be met by a simple deflexion method even on the ground of length of scale alone. In fact, the only manner in which the direct or inverse-rate method can be rendered as sensitive as the differential method, is by the use of a potentiometer method in which the E.M.F. of the couple is partly balanced by that of a constant (standard)

cell and the deflexions used only over very short ranges of temperature. When this is done, the spot of light from the galvanometer moves across the scale as the temperature falls, but the entire scale only represents some 10° C.; and when a deflexion of that amount has been obtained, the spot of light is brought back to the opposite end of the scale by altering the variable resistance of the potentiometer, whereby the opposing E.M.F. from the standard cell is altered. This sensitive method has been adopted for the preparation of all the inverse-rate curves shown in the latter part of this paper. This method, however, exhibits certain very decided disadvantages. In the first place, the zero position of the galvanometer must be accurately known if time observations are to be made at more than one point of the scale, otherwise alternately long and short intervals will be systematically obtained. Unfortunately, the extremely delicate galvanometers required to give the requisite sensitiveness are very liable to zero creep, so that towards the end of long curves, some such regular alternations are nearly always to be found. Further, this sensitive method is unsuited to rapid cooling, since in that case the time taken for the galvanometer to become steady after the insertion of the plug in the potentiometer circuit is too long compared with the time-interval to be measured, and the next observing point is liable to be crossed while the galvanometer is still oscillating. From all these disadvantages the differential method is entirely free, for although a sensitive galvanometer is still required to give the differential reading, it is used over only a relatively short length of scale, and a slight amount of zero-creep does not affect the shape of the derived differential-curve materially. The practical consequence of this difference between the two methods is that while all "inverse-rate" curves show quite notable irregularities due to observational error, the readings of the differential method usually yield smooth and regular curves which require no smoothing. Since, at all events in the ordinary less sensitive forms of inverse-rate determinations, the experimental irregularities are frequently comparable in size with the peaks due to the smaller recalescences, it follows that these irregularities must be eliminated by repetitions of the same cooling-curve.

A serious drawback to both these methods of taking cooling-curves is the very laborious character of the observations. The inverse-rate method is, in this respect, more troublesome than the differential, for in the latter the observer has merely to record the deflexions of one galvanometer at the moment when another galvanometer shows a certain deflexion, and the differences between these successive observations are plotted directly as the "derived differential" curve. In the case of the inverse-rate method, on the other hand, the time observations, for accurate work, must be made by means of a chronograph indicating seconds, and before the curve can be plotted, many yards of tape must be counted in order to obtain the actual values of the observed intervals.

From this point of view, as also from that of obtaining objective evidence of the actual nature of the observed cooling-curves, the desirability of autographic means of obtaining these curves becomes evident. Apart from the use of the Schneider-Carpentier induction-galvanometer, which cannot be made adequately sensitive except for very rapid cooling, the only means of autographically recording the observations of the direct method lies in obtaining simple time-temperature curves, from which the inverse-rate curve would have to be obtained by graphic differentiation*. The fact that adequate sensitiveness involves the use of a very long scale, makes it impossible to carry this out in practice for accurate work. In the case of the differential method, the method devised by M. Saladin of causing the motion of one galvanometer to displace the beam of light in a vertical direction while the movement of a second galvanometer moves the beam horizontally, renders the production of autographic curves of adequate accuracy at all events possible.

* When the inverse-rate curve is plotted from time observations at finite intervals of temperature, the resulting curve may depart considerably from its true form. Thus for a strictly horizontal part of the time-temperature curve, $dt/d\Gamma$ is zero and therefore dt (which is the abscissa of the inverse-rate curve) is infinite. If the time-temperature curve actually rises at any point of a recalescence, the inverse-rate curve should pass through infinity and reappear on the negative side. These considerations show the necessity of taking the inverse-rate observations at very short intervals of temperature in order to obtain close approximation to the true curve.

Referring to the quantitative comparison of the two methods given above, we see that with a given size of paper in an autographic recorder of any kind, the differential method is capable of from 10 to 20 times the sensitiveness of the direct method. At the present time, the only consideration adverse to the complete adoption of autographic apparatus for the determination of differential cooling-curves lies in the risk of error from zero-creep of the galvanometer. In the Saladin apparatus, one galvanometer gives the actual temperature of the body under observation, and the other galvanometer the differential reading corresponding to that temperature. A slight amount of creep in the latter instrument would not materially affect the result, but even a small amount of creep in the former instrument would vitiate the result by causing an error—not in the amount or magnitude of the recalescence—but in the temperature at which it would be indicated as taking place *. With ordinary moving-coil galvanometers the errors from this cause—with sensitive recording paper of moderate size—might amount to from 5° to 10° C., and this is too large an error to be admissible. By the use of a special galvanometer, having a bi-filar suspension which is entirely free from zero-creep, it is hoped to overcome this source of error and then to effect much saving of time and labour by the use of an autographic apparatus for obtaining differential curves. The possibility of doing this without sacrifice of the necessary sensitiveness is an advantage of the differential method which must not be underrated †.

In comparing the direct and differential methods of obtaining recalescence curves, several other experimental questions remain to be considered. In the first place, there is a very

* Zero-creep is, perhaps, responsible for some peculiarities of certain cooling-curves taken with the Saladin apparatus, which have been published. Thus Dr. Guillet was led to suppose that certain recalescences in copper-aluminium alloys took place at higher temperatures on cooling than the corresponding absorptions of heat on heating. A displacement of the zero of the temperature-galvanometer would account for this anomaly.

† The curves obtained by the Saladin apparatus are, of course, first differential curves, and would require conversion into the derived differential curves by graphic differentiation.

considerable difference between the two methods in respect of their sensitiveness to external disturbing causes. Serious external changes of temperature are, of course, liable to affect both methods, since the change in the rate of cooling may easily affect one of the two bodies in the differential method somewhat more rapidly than the other. On the other hand, smaller disturbances, such as are almost inevitable in most laboratories, are liable and in fact do affect the readings of the direct method while they leave the differential curve unaffected, since the difference in their effect upon the two cooling bodies is too minute to be traced upon the curve. From the point of view of the experimental refinements required, this is a great advantage in favour of the differential method.

An objection sometimes taken to the differential method arises from the fact that it involves the use of a neutral or "blank" cooling body, which is generally a cylinder of platinum. In order to approximate to the ideal condition of equal rates of cooling of the two bodies in the differential method, it would be desirable to use as the neutral body a substance having physical constants as nearly like those of the body under observation as possible. Owing, however, to the necessity of avoiding substances possessing recalescences of their own, the choice in this respect is very limited. In practice, platinum is found to have such different constants from those of iron and steel at all events, that the departure from ideal conditions is very marked, although it has already been indicated, and will be further proved by examples from actual curves, that the use of the derived differential curve eliminates the uncertainties thus introduced into the results. With a view to entirely avoiding the use of a neutral body in the differential method, the author proposed to apply the differential apparatus to the case of two pieces of the same material, cooling at approximately the same rate, but one somewhat retarded in time (and temperature) relatively to the other. This is readily accomplished by placing the two bodies in parts of the furnace which attain slightly different temperatures. One specimen of the material under observation will then commence its cooling from a somewhat lower temperature, and

will therefore arrive at a recalescence point before the other; the result will be a deflexion of the differential galvanometer in the direction indicating that the temperatures of the two specimens are approaching one another. As both temperatures continue to fall, the recalescence will be completed in the first body and will then begin in the second body, with the result that the deflexion of the differential galvanometer will be rapidly reversed, since while one body will temporarily remain at a stationary temperature, the other will be cooling particularly rapidly. Each arrest-point in the cooling-curve thus taken will therefore be represented by a double swing of the differential galvanometer. An actual curve of this kind is shown (fig. 12, Pl. VI.), but the usefulness of the method is, unfortunately, greatly impaired by the fact that in metals where several recalescences follow somewhat closely upon one another, their effects become superposed and the indications become confused and useless. From the point of view of the theory of cooling-curves, however, the method and example given in fig. 12 are instructive.

With regard to the general experimental details, such as the manner of inserting the thermo-electric junctions into the specimens, the arrangement of the specimen in the furnace and its protection, when required, from oxidation or other chemical actions, it need only be remarked here that the same precautions are required for both methods of determining cooling-curves.

In order to test by actual results the correctness of the reasoning outlined above, a number of experiments were carried out for the purpose of illustrating the capacity of the differential method of taking recalescence curves. The first of these consisted in taking a series of "blank" curves, the two cooling bodies being in some cases identical and of materials supposed to be free from recalescences of any kind. In the first experiment, the two thermo-junctions connected to the terminals of the differential galvanometer were simply placed centrally within the hot tube of an electric furnace, which was then heated to a temperature of about 1100° C., this temperature being measured by means of a third thermocouple attached to a delicate potentiometer. The resulting curve, in which deflexions of the differential galvanometer are plotted against

readings of the actual temperature of the furnace, is given in fig. 7 (*a*), Pl. VI. As would be expected under these circumstances, the curve is very nearly a vertical straight line; it must be pointed out, in view of observations to be described below, that such a result can only be obtained if the furnace itself is of suitable construction, for while *small* irregularities in the cooling of the furnace are obliterated by the fact that they affect both differential couples in the same way and at the same time, this ceases to be true if there are any *large* irregularities in the cooling of the furnace, since if these should affect one of the couples before the other they would produce irregularities on the differential curve.

In the second experiment, two similar platinum cylinders were cooled together, one of the cylinders, however, being some 10 per cent. heavier than the other. As a result, the upper part of the differential curve (fig. 7 *b*) shows a slight slope which, however, dies away for the lower portions, where the cooling is more gradual. This is an example of a marked characteristic of the curves obtained by the differential method; *i. e.*, that they are more nearly vertical, and therefore depart less from the ideal conditions of equal cooling of both bodies when the cooling is gradual than when it is rapid.

In the third experiment, a cylinder of platinum was cooled against one of fire-clay in order to show the effect of very wide differences in the physical constants of the two bodies. As would be anticipated, the curve (fig. 7 *c*) shows a decided slope, the cooling of the fire-clay lagging considerably behind that of the platinum, but the curve is nevertheless satisfactory as a blank because it is quite free from irregularities, and a "derived differential" curve (fig. 7 *d*) derived from it in the manner indicated above is perfectly free from peaks. It should perhaps be noted that it would be quite possible so to arrange matters as to overcome the obliquity of the first differential curve in such a case by placing the cylinder which normally tends to cool more slowly in a part of the furnace which cools rather more rapidly; but this would involve several experiments before the best position could be found and would therefore be too laborious, while the device of using the derived differential curve makes a general slope of the ordinary differential curve a matter of indifference.

In the following figures differential cooling-curves of some carbon-steels are given, the ordinary and derived differential curve being given in each case.

Fig. 8 (Pl. VI.) shows the differential and derived differential curves for a steel containing approximately 0·20 per cent. of carbon, the usual impurities being present in very small amounts only. This steel was kindly supplied to the Laboratory by Professor J. O. Arnold of Sheffield, for the purpose of comparing the results obtained in taking cooling-curves of this steel at the Laboratory of the Sheffield University with those obtained in the present experiments. Other cooling-curves of the same steel are given in figs. 13, 14, and 15 A described below. In fig. 8 it will be seen at once that the derived differential curve is far more definite in its indications than the first differential curve. This curve as well as the others given of this steel very closely resemble the curves obtained by Professor Arnold except that a doubling of the highest recalescence-point is described by Professor Arnold, but its existence is not shown in the present curve. In figs. 9, 10, and 11 similar first and derived differential cooling-curves are given for a series of very pure carbon-steels of increasing carbon content (0·326, 0·991, and 1·263 per cent. respectively). These steels were supplied to the Laboratory through the kindness of Mr. J. E. Stead, F.R.S., and although originally intended for another purpose, they have proved very useful for the present series of experiments. These curves, and particularly the derived differential curves, very clearly indicate the recalescence-points which are well known to occur in steels of this composition. In reading these curves, the temperature at which any given recalescence is to be regarded as taking place is that at which the point or apex of the peak of the derived differential curve is found to lie, since this is the temperature at which the most rapid evolution of heat occurs. If we compare the curves here given with those described in the paper by Carpenter and Keeling on "The Range of Solidification and the Critical Ranges of Iron-Carbon Alloys," * it will be found that the observed differential curves themselves are in most cases in very close agreement, but the interpretation adopted by the

* Journal of the Iron and Steel Institute, 1903, i.

authors of that paper differs somewhat from that which the present author deduces from the considerations here advanced. Messrs. Carpenter and Keeling appear to regard the temperature at which the apex of the differential curve is found as the "critical point"—in so far as they admit the existence of such "points" as distinct from ranges at all—while it now appears more correct to interpret the apex of the *derived* differential curve in this sense. There can, of course, be no doubt that in many of the transformations found in these and other alloys, the change occurs gradually and not entirely at one temperature; but for the purpose of arriving at an approximately correct equilibrium diagram, the author considers that the "points" as indicated by the apices of the derived differential curves should be used.

In order to further illustrate the use of the "derived differential" curve, differential cooling-curves were taken of specimens of one particular variety of steel under conditions of more rapid cooling than that adopted in the ordinary course. The author is inclined to regard a rate of cooling in which the temperature of the steel specimen falls from about 1000° C. to the ordinary temperature in the course of two hours as more or less "normal," since this rate of cooling appears to be slow enough to allow most of the transformations to take place completely, at all events in the case of the softer steels. Rates of cooling considerably more rapid than this are used by some workers, but since the object is to determine the *equilibrium* conditions, any discrepancies that arise between rapid and slow cooling must be interpreted in favour of the slower cooling method, since while too rapid a rate of cooling can, and often does, result in the partial suppression or displacement of thermal transformations, leaving the metal temporarily or permanently in a meta-stable state, too slow a rate of cooling cannot result in anything beyond the complete attainment of equilibrium conditions. It is therefore especially interesting to compare the curves shown in figs. 13 and 14 with those of fig. 8. These all refer to the cooling of samples of the same steel, but while in the case of fig. 8 the time occupied by the cooling process was about 1¾ hours, in the experiments represented by the curves of fig. 13 the apparatus was arranged to allow of more rapid cooling, and

the time occupied was accordingly only 35 minutes, while in the case of fig. 14 the cooling was arranged to take place almost in the free air and therefore only occupied about 12 minutes. It will be seen at once, on comparing these three figures, that the ordinary differential curves of figs. 13 and 14 are not at all readily interpreted by themselves, since the rapid cooling has resulted in a wide departure from the ideal conditions of equal cooling of the two bodies in the differential apparatus, the resulting first differential curves being extreme cases of the type illustrated in fig. 5. The derived differential curves of both 13 and 14 are, however, perfectly clear and readily interpreted, and they will be found, on comparison, to be in striking agreement both with one another and with fig. 8. The only discrepancy appears in the fact that there seems to be a difference in the relative intensities of the three transformations as shown by these three curves; as the relative size of the peaks on these curves is, however, considerably dependent on the rate of cooling in each case, the comparison of the three curves from this point of view cannot carry much weight.

For the purpose of comparison, cooling-curves of some of the same steels were also taken by means of the "inverse-rate" method, this method being applied by means of the very sensitive potentiometer installed at the Laboratory. In this instrument a fall in the temperature of the cooling body through 12° C. causes the spot of light on the galvanometer-scale to traverse 104 scale-divisions of 2 mm. each. At the end of this swing, the spot is brought back to the left-hand end of the scale by moving a plug on the resistance-box, and the galvanometer is sufficiently dead-beat for all oscillations to have died out long before the spot of light crosses the centre of the scale. Observations are taken every time the spot of light crosses the centre of the scale and also when a deflexion of 52 divisions is indicated, these points representing successive decrements of 200 micro-volts in the E.M.F. of the thermo-couple. The time observations are made by depressing a key which makes a mark on a chronograph-tape, accurate seconds being marked on the same by means of a relay actuated by the standard clock of the Laboratory; the resulting curves are shown in fig. 15, A & B, which should

be compared with figs. 8 and 9. The rate of cooling adopted for this method was the same as that used for obtaining the ordinary differential curves; and a comparison of the two sets of curves will show that their results are in substantial agreement, except that the "inverse-rate" method produces curves which suffer from much greater irregularities due to experimental imperfections than do the differential curves: this is undoubtedly due to the fact that in the case of direct time-temperature observations every external irregularity is translated to the resulting curve, and the more faithfully, the greater the sensitiveness of the apparatus employed.

The general conclusion which is, in the author's opinion, to be drawn from the results is that for most purposes the differential method of obtaining cooling-curves is eminently satisfactory; it has the advantage of being largely independent of external conditions and of the physical constants of the body under observation, and it has been shown to be applicable over a wide range of rates of cooling. The observations themselves are easily made and are not liable to any serious error, given reasonable care and skill on the part of the observer. The most serious criticism that could, perhaps, have been advanced against the method was the difficulty of correctly interpreting the curves in cases where the two bodies cooled at somewhat different rates; but the theoretical considerations laid down in the earlier part of the present paper, and the use of the derived differential curves here advocated, entirely remove that difficulty. On the other hand, while it is certainly possible to obtain "inverse-rate" curves showing all the small recalescences indicated by the differential method, this can only be done by using great experimental refinement, a very open temperature-scale, and moderate rates of cooling, and under these circumstances the method becomes very laborious and also subject to error from external disturbing causes.

For the purpose of the experiments made for the purpose of the foregoing investigation, some observations were made which not only are of interest in connexion with the experimental arrangements of cooling-curve apparatus, but are of some intrinsic importance, and are therefore to be described here.

The earlier "blank" curves taken with either bare thermo-junctions or two platinum cylinders, were obtained in an electric resistance-furnace of a type that has been introduced and described by Dr. H. C. H. Carpenter; this furnace consists essentially of a porcelain tube (of a material made for this purpose by Messrs. MacIntyre, of Burslem) wound with nickel wire and embedded in a thick lagging of finely powdered quartz. The blank curves thus obtained all showed, more or less clearly marked, a recalescence-point at a temperature of 580° C., and it proved a matter of considerable difficulty to trace the source of this evolution of heat. The platinum cylinders, the thermo-junctions themselves, and the nickel-wire winding of the furnace were suspected in turn, but each of these was found to be free from the actual recalescence. Finally, it was found that by using two bare couples and placing one close against the wall of the furnace-tube and the other well in the middle of the tube, a greatly increased recalescence could be detected. The curve obtained is shown in fig. 16 *a*. As it had already been shown that nickel showed no recalescence at this low temperature, the material of the tube itself was suspected, and a similar experiment was tried in an electric-resistance furnace by Heraeus, in which the platinum-resistance foil is wound on a tube of special Berlin porcelain; in this furnace the mysterious recalescence at 580° C. disappeared entirely. Messrs. MacIntyre then kindly furnished the author with some small cylinders of their porcelain, and these, cooled against a platinum cylinder in the Heraeus furnace, showed the recalescence clearly (see curve, fig. 16 *c*), but not so vigorously as would be expected if the whole mass of the cylinder were taking part in the reaction. The author therefore ascribed the recalescence to one constituent of the porcelain, and this view was confirmed when chemical analysis showed that the English porcelain contains a considerable proportion of free silica, whilst the special Berlin porcelain contains an excess of alumina. Differential cooling-curves of various kinds of silica were then taken (figs. 17 and 18) and in crystalline varieties, such as sand and quartz, very vigorous recalescences were obtained, but fused (vitrified) silica and precipitated silica were found to be entirely free from the reaction. The author accordingly

ascribes the recalescence found at 580° C. to a polymorphic change in crystalline silica, the occurrence of the recalescence being in fact a very sensitive test for the presence of free silica in ceramic materials. As to the nature of the change which silica undergoes at this temperature, both on heating and cooling, the author is unable to offer any suggestion; but it is interesting to note that in their recent paper on the Silica-Lime series, Messrs. Shepherd and Day * describe a polymorphic transformation in Silica (Quartz = Trydimite) as occurring at a temperature of 750°–800° C. The author's curves show no trace of any evolution of heat at that temperature, while the recalescence at 580° is so marked and constant both on heating and cooling, that there can be no doubt as to its occurrence at that temperature.

In another connexion, Le Chatelier and others † have noticed that in the heating of fire-clays and other similar bodies there are marked retardations at or near 600° C. The authors quoted have ascribed these retardations to one or more steps of the dehydration process; but there can be little doubt that one at least of these phenomena must be ascribed to the same cause as the recalescence in the author's curves of silica, viz., a polymorphic change in crystalline silica itself.

The same phenomena appear also to have affected the observations of Roberts-Austen, of Carpenter and Keeling, and of others on the lower recalescence-points of iron and steel. In the well-known differential cooling-curve of electro-deposited iron published by Roberts-Austen ‡ and throughout the diagram of the Iron-Carbon alloys as given by Carpenter and Keeling §, recalescence-points are indicated at or very near the temperature of 580° C.; and there can, therefore, be very little doubt that this recalescence is due to some silica contained in the walls or lagging of the furnaces used by those workers, and that therefore this point must be deleted from the Iron-Carbon diagram.

* Shepherd and Day, Amer. Journ. Science, October 1906.

† Le Chatelier, *Comptes Rendus*, tome civ. pp. 1443–1446, 1517–1520 (1887); W. C. Hancock, Journ. Soc. Chem. Ind., 1906.

‡ W. C. Roberts-Austen, "Fifth Report to the Alloys Research Committee," Proc. Inst. Mech. Engineers, 1899.

§ Carpenter and Keeling, Journ. Iron & Steel Inst., 1903, i.

The experimental portions of the above research were carried out in the metallurgical laboratories of the National Physical Laboratory, the author having been assisted, in the earlier portions of the work, by Mr. C. A. Edwards. The author also wishes to express his indebtedness to Mr. F. J. Selby, M.A., for help in connexion with the mathematics of this paper, and to the Director, Dr. R. T. Glazebrook, F.R.S., not only for the warm interest he has taken in the progress of the work, but also for valuable suggestions in reference to the theory of cooling-curves as discussed in the earlier part of this paper.

Discussion.

Mr F. J. Selby remarked on the skill with which the Author, by direct reasoning from certain approximate assumptions, had arrived at his interesting results as to the points of importance in connexion with recalescence curves. Mr Rosenhain had shown, very clearly and graphically, that from the theoretical point of view all the methods of representing the cooling-curves must give identical information, and that therefore the decision as to method to be adopted depended simply on the experimental accuracy attainable; while, in this respect, he had brought forward very strong evidence in favour of the Roberts-Austen differential method. Mr Selby suggested that it might be worth while to draw out theoretical cooling-curves based on certain simple assumptions as to the conditions of cooling. He showed how this might be done for the parts of the curve preceding and following the recalescence; and that by making a suitable supposition as to the variation in the rate of evolution of heat during the recalescence, such as $\frac{dQ}{dT} = (a - bT)^2 e^{cT}$, a complete theoretical curve might be drawn by the comparison of which with the experimental curve the constants might be determined and all the quantities of importance in connexion with the recalescence estimated with greater accuracy. In particular, and without any assumption as to the form of $\frac{dQ}{dT}$, some of the results in the paper could be verified; it could be shown that the evolution of heat was not completed at the peak on the derived differential curve (the point of inflection on the Roberts-Austen Curve), and the point of maximum evolution of heat on the experimental curve could be fixed. The value of the derived differential curve advocated by the Author becomes apparent from such calculations.

Mr A. Campbell remarked that the end of the paper was the part which would most likely be of interest to physicists. It was most interesting to find that the Author has shown (by the cooling-curve method) a change-point in silica near 580° C. Quite a number of years ago Le Chatelier showed that between 460° and 500° C. the coefficient of

expansion of quartz showed a quick increase, and he predicted that a change-point should occur at 270° C. Randall's experiments in 1905 on quartz showed a similar result so far as the expansion is concerned. Experiments which he (Mr Campbell) had made with nickel showed no abnormality in coefficient of expansion at temperatures which gave strongly marked thermo-electric change-points. For the direct reading of rates of cooling, Dejean not long ago devised an ingenious arrangement in which the thermocouple is connected to one coil of a differential moving-coil galvanometer whose other coil is connected to a second galvanometer: the readings of the latter will be a function of the rate of cooling. It occurred to the speaker that a transformer would give a simpler method of obtaining di/dt, but the slowness of change of i makes it too insensitive. The assumptions made in the paper that the specific heat, the thermal conductivity, and the emissivity of any material remain approximately constant even over a small range of temperature near recalescence, seem somewhat uncertain to the physicist, and must be regarded merely as working conventions by the metallurgist.

Mr S. W. J. Smith expressed his interest, from the consideration of magnetic data, in the Author's opinion that there is no critical point near 580° C. in the cooling-curves of iron and steel. It was curious that in the paper in which Roberts-Austen describes the critical point found at about 580° C., he cites the magnetic experiments of Morris in support of the fact that a change occurs in iron at that temperature. Morris puts the temperature of the magnetic change at about 550° in one specimen, but at about 500° in another; while in a third the effect was scarcely observable. Even in the specimen most studied, the magnitude of the effect seemed to depend upon the annealing temperature. In some experiments on permeability-temperature variation in nearly pure iron (·06 % C., ·1 % Mn) annealed at about 850° C., Mr Satterly and the speaker found a distinct change in the slope of the μ–θ curve (H=0·43) slightly below 500° during cooling, and a distinct minimum value of μ near 250° during heating. But with the field-strength used there was no apparent break in the slope of the μ–θ curve near 580° C. There was thus apparently no conclusive evidence from magnetic data of a critical point at 580° C. The effect observed at 580° by Roberts-Austen was very feeble compared with another effect which began near 500° and continued over a considerable range. This effect was obvious as a more or less pronounced "hump," of which the maximum lies near 400° on practically all the permeability-curves. It was even traceable in the curves of Hopkinson, although his observations below 500° were very few in number. Their own magnetic experiments agreed with those of Morris and with the Roberts-Austen cooling-curve in showing another critical point near 250°, although in their case the effect was only detected during heating.

XI. *The Distribution in Electric Fields of the Active Deposits of Radium, Thorium, and Actinium.* By SIDNEY RUSS, *Demonstrator of Physics, Victoria University, Manchester* *.

Introduction.

THE distribution in an electric field of the excited activity produced by thorium and radium emanations has been studied by several observers. Working with thorium emanation, Rutherford † has shown that the amount of activity imparted to a rod charged negatively decreases as the pressure in the containing vessel is reduced after a certain pressure is reached; while experiments made by Makower ‡ show that similar effects are obtained with the excited activity produced from radium emanation. Some further experiments by Rutherford § with the emanation from radium indicate that while at atmospheric pressure the greater part of the excited activity is directed to a cathode by moderate electric fields, yet a small fraction (about 5 per cent.) goes to an anode. Reasons are then (*loc. cit.*) given for supposing that some of the active deposit particles carry a negative charge, thus accounting for their transmission to the anode.

It seemed of interest then to find out whether the quantity of active deposit that goes to an anode changes when the pressure is varied.

A comparison has therefore been made over a range of pressure extending from ·001 cm. to 10 cms. between the amount of activity imparted to a rod charged positively and then negatively when exposed for the same interval of time to equal quantities of the radium emanation.

It was found that whereas the activity of the cathode decreased as the pressure was diminished, the anode showed a corresponding increase in activity.

The rod which had served as cathode or anode was then made a neutral pole by connecting it to the containing vessel,

* Read March 13, 1908.

† Rutherford, Phil. Mag. Feb. 1900.

‡ Makower, Phil. Mag. Nov. 1905.

§ Rutherford, Phil. Mag. Jan. 1903.

and what may be called diffusion experiments, *i. e.* with no electric field existing in the gas, were then made. Observations showed that there was a marked decrease in the amount of active deposit that diffused on to the rod as the pressure was diminished.

After a slight modification of the apparatus, a comparison of the activity obtained on a cathode and anode over a range of pressure extending from ·1 mm. to 1 mm. was made with three gases having widely different densities, namely, hydrogen, air, and sulphur dioxide.

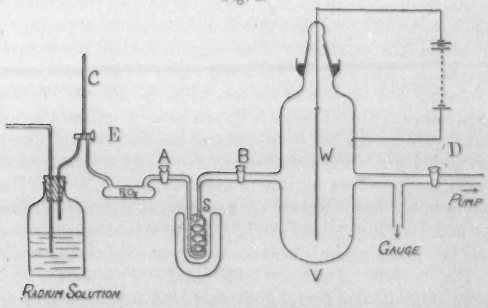
The results in the case of air show that the activity of the cathode is considerably greater than that of the anode, but that they approach equality as the lower pressure is reached.

This effect is more marked in sulphur dioxide, while for hydrogen no such difference in activity is observed, just as much active deposit being transmitted to the anode as to the cathode over this range of pressure.

Methods of Experiment.

A comparison was first made between the amounts of excited activity obtained on a wire maintained at 200 volts acting as cathode and then as anode over a range of pressure extending from ·01 mm. to 1·2 mm.

Fig. 1.



The apparatus used is represented in fig. 1. A brass wire W 10 cms. long, ·55 mm. diameter, on which the active

deposit was obtained, was suspended so as to hang symmetrically within the glass vessel V which was silvered on the inside, connexion to a battery of cells being made by means of a piece of platinum fused in the side of the vessel.

The usual method of experimenting was as follows:—

The whole system, with the exception of the vessel containing the radium emanation evolved from a solution of radium bromide, was exhausted firstly by means of a Fleuss pump and then by a charcoal tube immersed in liquid air (not shown in diagram). When the desired pressure had been obtained the tap D was closed, and by turning the tap E the small capillary tube C (·2 c.c. volume) was filled with the emanation at atmospheric pressure.

The tap B was then closed and the glass spiral S surrounded by liquid air, E was then turned and the emanation contained in the capillary C was allowed to pass over a tube containing P_2O_5 for drying purposes, after which it condensed in the spiral S.

A few trials showed that nearly all of the emanation had condensed in 15 minutes. The tap A was then closed, B opened, the liquid air removed from the spiral, and the emanation allowed to diffuse into the vessel V and a McLeod gauge which was in connexion with it. The resulting pressure was then read.

As the volume of the P_2O_5 tube was several times that of the spiral, quite low pressures were obtainable in the spiral although the capillary C had been filled at atmospheric pressures. The wire W was made either a cathode or anode as desired and exposed to the emanation for 1 hour. It was then removed from the vessel V, and its activity tested in the usual way by making it the central electrode of a cylinder connected with a Dolezalek electrometer and measuring the ionization produced by the α rays emitted from it. The electrometer leak obtained in a measured interval of time 14 minutes after the removal of the wire from the emanation was taken as a measure of the activity obtained on the wire.

The variation in activity with change of pressure is seen from Table I., also from fig. 2, where the abscissæ represent pressures and the ordinates the corresponding activities of the wire. It will be seen from the diagram that there is a

TABLE I.

Positive Electrode.		Negative Electrode.	
Pressure in mms.	Activity.	Pressure in mms.	Activity.
·011	69	·006	142
·013	79	·008	189
·137	84	·102	173
·202	81	·125	157
·250	62	·476	425
·608	63	·739	736
·792	67	1·165	1118
·865	56	**Neutral Electrode.**	
·922	56	Pressure in mms.	Activity.
1·102	57	·152	72
		·435	64
		·722	57
		1·140	62

very large decrease in the activity of the cathode as the pressure is diminished, while the anode shows if anything a small increase; the amount obtained by diffusion alone, *i. e.* to a neutral rod, being practically constant over this range of pressure.

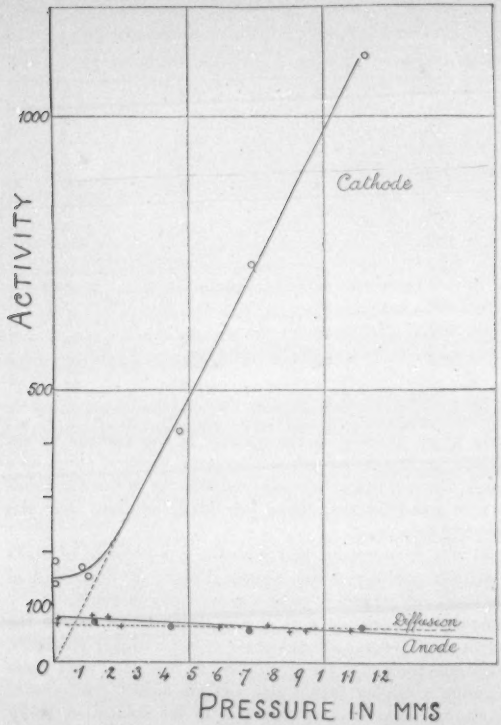
It will be observed that whereas at a pressure of about 1 mm. the activity of the cathode is about 20 times that of the anode, at ·01 mm. it is only about twice as much.

At this stage of the work it was decided to compare the activities of the cathode and anode at much higher pressures.

Several trials with the apparatus just described were made at pressures higher than 1 cm., but discordant results were obtained. This was probably due to the emanation being blown back into the spiral on allowing air to enter the vessel V, which was initially well exhausted as in the previous experiments.

For this reason observations were made with a modified system consisting simply of a brass cylinder down the centre of which passed a brass rod (provided with end-pieces to ensure uniformity of field) which was connected to cells giving

Fig. 2.



the same voltage as before. The brass cylinder was connected on one side with a small vessel which was filled with the emanation at atmospheric pressure, and on the other with a pump and pressure-gauge.

The small vessel containing the emanation was provided with metal electrodes, and the precaution was taken of applying a strong electric field to the emanation before letting

it into the brass vessel; this removed any active deposit or dust particles.

The system was initially exhausted to about 1 mm. pressure, the emanation then let in and the small vessel isolated from the cylinder, the pressure in which was then adjusted to whatever value was required. The activity obtained on the brass rod after one hour's exposure was measured in exactly the same way as before. Observations were made for a range of pressure extending from 2 mms. to 10 cms. The numerical data are contained in Table II., and the results shown graphically in fig. 3, where abscissæ denote pressures and ordinates the corresponding activities.

TABLE II.

Positive Electrode.		Negative Electrode.	
Pressure in cms.	Activity.	Pressure in cms.	Activity.
·13	152	·13	780
·27	140	2·04	1360
·55	163	2·70	1400
·63	148	7·22	1540
1·10	120	7·80	1470
3·78	94	**Neutral Electrode.**	
6·24	79	Pressure in cms.	Activity.
7·30	77	·36	219
8·94	77	·40	198
10·96	75	3·14	287
		3·74	300
		8·44	447
		8·82	437

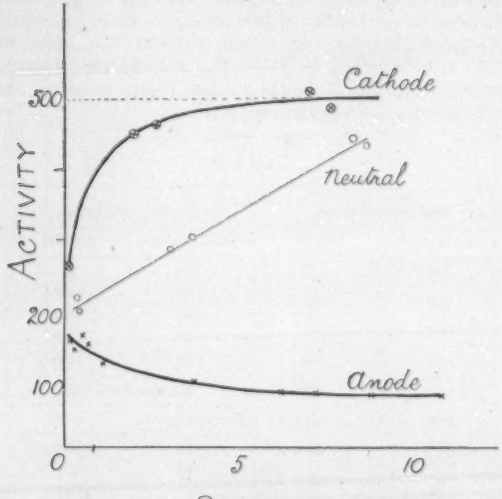
It will be seen that whereas the activity of the cathode *decreases* with diminishing pressure, there is a very marked *increase* in the activity of the anode, while the amount of active deposit that diffuses on to the neutral rod decreases very considerably as the pressure is reduced.

Diffusion experiments similar to those just cited are at

present being made, in which, however, other metals than brass are used as containing-cylinder and rod.

It will be seen from fig. 3 that at a pressure of about 2 mms. the quantity of active deposit transmitted to the anode is approximately equal to that diffusing on to a neutral pole.

Fig. 3.



The activity of the cathode being much larger than the anode, the scale in fig. 3 for the cathode is one third that for the anode and neutral pole.

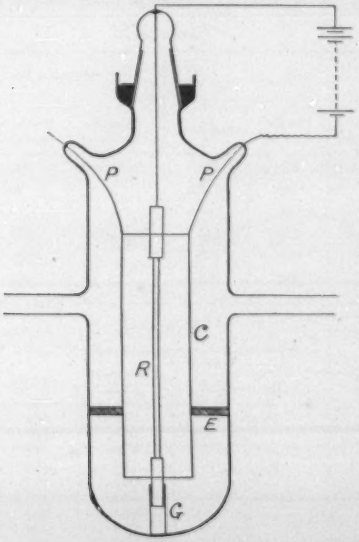
Now the active deposit that diffuses on to a neutral pole we have seen to be in the main positively charged (the bulk of it going to a cathode in an electric field). Very little of this positively charged matter can make its way to the anode. In order to account for the quantity obtained on the anode we are led to the conclusion that some at least of the active deposit particles that make their way to the anode are negatively charged.

The above results then go to confirm the view held by

Rutherford already referred to, that the active deposit particles which are directed to the anode have a negative charge.

It has been suggested by Rutherford*, in explanation of his results with thorium emanation, that the decrease in activity of the cathode as the pressure is reduced is due to a decreasing number of collisions between the active deposit

Fig. 4.



particles and the gaseous molecules with which they are mixed.

At the moment of expulsion of an α particle from the emanation atom, the residue, *i. e.* the active deposit particle, acquires a velocity in the opposite direction, which, though small compared with that of the α particle, is high compared with the velocity of an ordinary gaseous molecule.

* 'Radioactivity,' p. 319.

On the view above cited a certain number of collisions with the gas molecules is necessary to sufficiently reduce the velocity of the active deposit particles in order that moderate electric fields may direct them to cathode or anode.

Hence we might expect the effectiveness of a molecular encounter to depend on the nature of the gaseous molecules with which the active matter is associated. Experiments to test this point were therefore made with three gases differing considerably in molecular constitution, namely, air,

TABLE III.

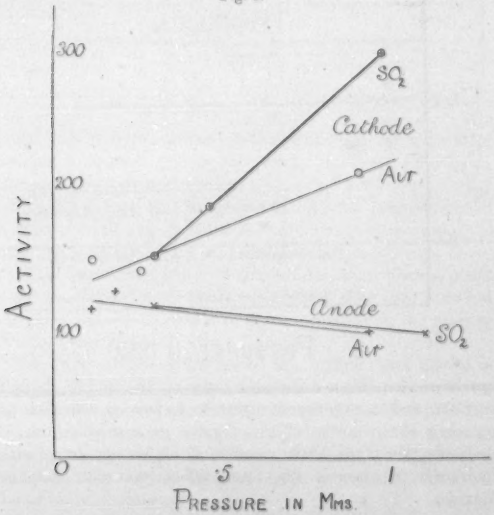
Air.			Sulphur Dioxide.		
Sign of Electrode.	Pressure in mms.	Activity.	Sign of Electrode.	Pressure in mms.	Activity.
Positive.	·115	108	Positive.	·302	111
,,	·185	122	,,	1·120	93
,,	·931	95	,,	...	...
Negative.	·115	146	Negative.	·297	148
,,	·260	137	,,	·462	184
,,	·902	211	,,	·970	208

Hydrogen.		
Sign of Electrode.	Pressure in mms.	Activity.
Positive.	·084	184
,,	·481	200
,,	·828	194
Negative.	·116	212
,,	·131	138
,,	·462	192
,,	·925	202
,,	·970	239
,,	1·080	188

hydrogen, and sulphur dioxide, and a comparison between the activities of cathode and anode over a range of pressure extending from ·1 mm. to 1 mm. was made with the low pressure system already described. The silvered vessel was, however, replaced by another glass vessel containing two concentric cylinders as seen in fig. 4 (p. 217).

The inner one R, 7 cms. long and ·36 cm. diameter, acted as the rod upon which the active deposit was obtained, C the outside cylinder, 8·95 cms. long and 2·7 cms. diameter, being suspended by two platinum wires PP which were sealed into the glass vessel and connected to either pole of the battery as required. The ebonite ring E kept the outer cylinder steady, and the rod R was supplied with a guard-ring at each end, one of which fitted into a glass tube G, thus serving to keep the rod central within the cylinder C.

Fig. 5.

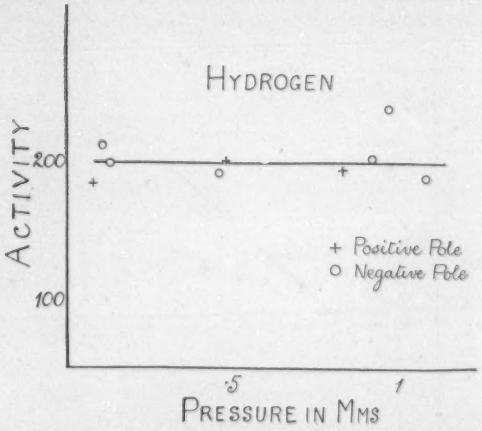


These modifications were adopted so as to ensure a uniform electric field between the rod and cylinder, and also to render the problem more accessible to mathematical treatment, the areas of the rod and cylinder being not very different. With this arrangement experiments with the gases already mentioned were made, and the results obtained are collected in Table III. and reproduced graphically in figs. 5 and 6.

It will be seen that for SO_2 there is a much greater difference in the activities of cathode and anode than is the case for air, while for hydrogen practically no difference in activity over the same range of pressure was obtained.

We should expect that the curves for cathode and anode in the case of hydrogen would show a divergence at higher

Fig. 6.



pressures, for if an encounter between the active deposit particle and a hydrogen molecule is not so effective in reducing the velocity of the former as a sulphur dioxide molecule, then we shall require a higher pressure with hydrogen to observe the same effects as with sulphur dioxide.

Thorium and Actinium.

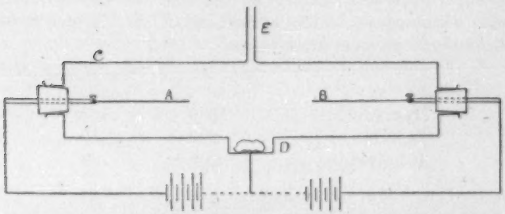
A few experiments made with the emanations from thorium and actinium show a rather striking difference in the distribution of their active deposits in an electric field.

The experimental arrangement seen in fig. 7 was used.

Two thin brass wires 7·5 cms. long and ·7 mm. diameter were made to lie along the axis of a brass tube C 30 cms.

long and 4·2 cms. diameter. This tube was fitted with a small capsule D, into which was placed a small quantity of thorium oxide, or of a preparation of actinium kindly lent me by Professor Rutherford.

Fig. 7.



The wire A was connected to the positive, B to the negative pole of a battery giving 320 volts, the brass tube being connected to the middle point.

The exit-tube E was connected to a Fleuss pump and a set of tubes containing calcium chloride and cotton-wool, thus ensuring a supply of dry and dust-free air.

When experimenting with thorium the wires were exposed to the emanation for 24 hours, after which they were removed and their activities tested with an electrometer in the manner already described.

The ratio of the activities of the cathode and anode at atmospheric pressure was found to be about 200 : 1; on reducing the pressure to 2 mms. the ratio was diminished to 25 : 1, this reduction being mainly due to the decrease in activity of the cathode already observed by Rutherford (*loc. cit.*). No certain increase in the very small activity of the anode was observed.

In the experiments with actinium the wires were exposed usually for about 2 hours.

At atmospheric pressures the ratio of the activities of the cathode and anode was about 2 to 1, while at 2 mm. pressure it was 22 to 1, the *increase* in the ratio being mostly due to an increase in the activity of the cathode. The active deposit on the two wires decayed at the same rate, namely, that of actinium A.

These results show that while the greater part of the active deposit from actinium emanation has a positive charge, yet at amospheric pressure a very considerable fraction has a negative charge, and is therefore directed to the anode.

For purposes of comparison the activities obtained on a wire when exposed to the emanations of Ra, Th, and Ac at atmospheric pressure are tabulated.

	Wire charged negatively.	Wire charged positively.
Thorium	200	1
Radium	200	10
Actinium	200	100

Conclusion.

In view of the different distributions in an electric field of the active deposits of thorium, radium, and actinium, it is difficult to think of the whole series of events between the formation of the active deposit and its transference to either electrode, as taking place in an exactly similar fashion.

It has been shown by Miss Slater* that slowly moving β particles accompany the α particles which are ejected from thorium and radium emanations. If there are two such β particles to every α particle, the remainder of the emanation atom, *i. e.* the active deposit, would be left with a positive charge.

This supposition would explain the almost complete transference of the active deposit of thorium to the cathode; but the small, though quite definite, quantity of the active deposit of radium that is directed to the anode still presents a difficulty. It may be that a few of the active deposit particles *gain* negative ions from the gas in which they are moving, which must be present in very large numbers.

In the case of actinium, although at a pressure of a few millims., far more activity is observed on the cathode than on the anode, as has already been found by Debierne†, yet we have seen that at higher pressures this inequality is very

* Phil. Mag. Oct. 1905.

† *Comptes Rendus*, vol. cxxxvi. p. 671.

much reduced, there being almost as many negative carriers as positive.

It is hoped that a more extended series of observations may throw some light on the apparently anomalous behaviour of actinium.

In conclusion, I desire to thank Professor Rutherford for several suggestions during the latter part of the work. The research was begun with Mr. Makower, who, however, could not continue the collaboration owing to pressure of other work. I wish to express my thanks for the very considerable help he has given me in these experiments.

Note added March 17th.

Some recent experiments made with actinium tend to explain the anomalous behaviour referred to above. A preparation of actinium was placed at the bottom of a cylinder, and two parallel plates were suspended so as to hang vertically above it. These plates were connected to the terminals of a battery of 300 volts, and after an exposure at atmospheric pressure usually lasting several hours the activities of the two plates were compared by means of an electroscope. The distance of the plates from the actinium preparation could be altered as required.

As already observed in the experiments described, the activity of the cathode was always greater than that of the anode.

It was found that when the plates were fixed at a distance of 4 cms. from the actinium, the ratio of the activity of the cathode to that of the anode was about 5 to 1; on bringing the plates nearer to the actinium this ratio rapidly increased, and when the distance was reduced to 2 mms. the activity of the cathode was more than a hundred times that of the anode.

These observations and others of a similar nature at different pressures (details of which it is hoped will appear in a future number), indicate that the sign of the electrical charge exhibited by the active deposit particles is some function of the distance which they have travelled through the containing gas, before reaching the electrodes.

Discussion.

Dr C. H. Lees expressed his interest in the paper, referring especially to the fact that the amount of active deposit which diffused onto a neutral rod varied with the pressure.

The Chairman referred to the radioactivity of the atmosphere, and remarked that the Author's experiments might be of use to meteorologists in enabling them to settle the question of the source of the radioactivity. He expressed his interest in the results obtained with actinium.

Mr Russ, in reply to Dr Lees, said that the fact that the quantity of active deposit on a neutral rod varied with the pressure was being investigated mathematically by Mr Bateman. With regard to the Chairman's remarks he observed that the source of the radioactivity of the atmosphere was an interesting question. The radioactivity due to thorium presented more difficulties than that due to radium because of the long life of the radium emanation. He suggested that the results might be due to the active deposits themselves which had longer lives than the emanations.

XII. *Note on certain Dynamical Analogues of Temperature Equilibrium.* *By* Prof. G. H. Bryan, *M.A., F.R.S.* (*Bangor*) *.

In the *Archives Néerlandaises* for 1900 (Livre jubilaire dédié à M. H. A. Lorentz) I described under the title of "Energy Accelerations" a method of studying problems dealing with the partition of energy in systems of particles in which some kind of statistical equilibrium exists. This method consists essentially in writing down expressions for the second differential coefficients with respect to the time of the squares and products of velocities or momenta of the system considered.

The method in question appears to have been previously employed by Mr. Burbury, who, however, did not employ the term "energy-accelerations" in connexion with the second differential coefficients in question.

I was, and still am, of opinion that a further study of energy-accelerations must have an important bearing on all

* Read March 13, 1908.

problems in the Kinetic Theory dealing with questions of temperature-equilibrium, and the fact that no attempt seems to have been made to follow the problem up, may be taken as evidence of the large congestion of unsolved problems which presses heavily on the shoulders of the mathematician of to-day. The present note deals with two simple applications which happened to remain unnoticed when the paper was written.

Consider in the first place an ideal medium formed of material particles uniformly distributed in space, both as regards position and as regards direction, and attracting or repelling one another according to any law of force as a function of the distance between them.

If we confine our attention to one particular particle, the effect of the other particles will be to produce varying fields of force acting on the particle in question. Take now the equation of energy-accelerations which may be written in the more general form :—

$$\frac{d^2}{dt^2}(\tfrac{1}{2}mu^2) = \frac{1}{m}\left(\frac{dV}{dx}\right)^2 - u\frac{d^2V}{dt\,dx} - u^2\frac{d^2V}{dx^2} - uv\frac{d^2V}{dx\,dy} - uw\frac{d^2V}{dx\,dz},$$

where u, v, w stand for velocity components, V the potential of the field due to the other particles.

The assumption that the field produced by the other particles is independent of the motion of the particle under consideration, and that this field has on an average no directional properties, shows that the mean values of the last two terms vanish, and the assumption that a stationary state exists requires that the mean value of the term $u\,d^2V/dt\,dx$ should also vanish, reducing the energy-acceleration equation in this case to the ordinary standard form, brackets denoting mean values :—

$$\left[\frac{d^2}{dt^2}(\tfrac{1}{2}mu^2)\right] = \frac{1}{m}\left[\left(\frac{dV}{dx}\right)^2\right] - [u^2]\left[\frac{d^2V}{dx^2}\right].$$

For energy equilibrium the left-hand side must vanish. This is only possible when the mean value of d^2V/dx^2 is positive. Now the assumed absence of directional properties,

or in other words the assumed isotropic character of the medium, requires that the average values of d^2V/dx^2, d^2V/dy^2, and d^2V/dz^2 shall be equal ; therefore each is equal to one-third the average value of ΔV. We conclude that statistical energy equilibrium cannot exist in a system of particles possessing the assumed properties unless ΔV is positive.

For the Newtonian law of force $\Delta V=0$. In this case the mean value $[d^2V/dx^2]$ therefore also vanishes, and the energy-acceleration equation takes the form :—

$$\left[\frac{d^2}{dt^2}\left(\tfrac{1}{2}mu^2\right)\right]=\frac{1}{m}\left[\left(\frac{dV}{dx}\right)^2\right].$$

This shows that the only kind of statistical equilibrium is the statical state of unstable equilibrium defined by $dV/dx=0$, $dV/dy=0$, and $dV/dz=0$ on each particle, and in the absence of this state an acceleration of kinetic energy will take place.

It is thus impossible with the Newtonian law to build up a medium of material particles, either attracting or repelling one another, and possessing properties of energy equilibrium analogous to those of a system of gas molecules.

It need not be pointed out that this investigation does not preclude the possibility under the Newtonian Law, of stationary motions such as those occurring in the solar system.

Exactly the same conclusions hold good if the law of force is such as to make ΔV negative. The equations of energy equilibrium here require the further condition that $[u^2]$ $[v^2]$ $[w^2]$ should be zero, which can only happen if the system always remains at rest.

The necessary condition $\Delta V>0$ becomes, for the particular case in which the field is due to an attracting particle placed at the origin,

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right)>0,$$

showing that r^2dV/dr must increase with r. In other words, the force if repulsive must vary according to a higher power

of the inverse distance than the square (the inverse fifth power would represent a possible law of force if repulsive). If the force is attractive it must vary according to a lower power of the inverse distance than the square.

The second point to be noticed is that unless the expression for the kinetic energy of a dynamical system is a quadratic function of the velocities with *constant* coefficients, the equations of energy equilibrium no longer assume the form of linear relations connecting the mean squares and products of velocities, but they also involve terms of the fourth degree in these velocities. An illustration of this fact is afforded by the motion of a particle in a plane when referred to polar coordinates. If we write down the expressions for the energy-accelerations

$$\frac{d^2}{dt^2}(\tfrac{1}{2}\dot{r}^2) \quad \text{and} \quad \frac{d^2}{dt^2}(\tfrac{1}{2}r^2\dot{\theta}^2)$$

in terms of the velocities and coordinates, using the equations of motion

$$\ddot{r} - r\dot{\theta}^2 = -\frac{d\mathrm{V}}{dr} \quad \text{and} \quad \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = -\frac{d\mathrm{V}}{rd\theta}$$

we shall obtain expressions involving the velocity components up to the fourth degree, and the energy components therefore up to the second degree. The conditions of energy equilibrium between the transversal and radial components will therefore no longer take the form of linear relations connecting the mean values of the energy components in question.

On the other hand, when we refer the motion to x and y coordinates, the equations of energy equilibrium are linear in the energy components.

It appears, therefore, that if dynamical systems are to represent the phenomena of temperature-equilibrium consistently with the commonly accepted hypothesis that temperature is a quantity of the nature and dimensions of molecular kinetic energy, the expression for the kinetic energy must in general have constant coefficients or must at least satisfy certain conditions which are fulfilled in the case of constant

coefficients. While the analogy of kinetic energy with temperature may hold good in the case of a system of particles, there must exist dynamical systems for which this analogy does not hold good. As a purely negative conclusion this result is not inconsistent with Stefan's law in which we have the energy of radiation proportional to the fourth power of the energy of molecular motion. But whether it is possible to construct a dynamical model whose energy-partition is analogous to Stefan's Law, must be a question for future investigation.

Discussion.

Prof. Cassie discussed the method and results of the paper, and remarked that the results of experiments appeared to indicate that the forces between particles varied with the distance according to some law more rapid than the inverse fifth power.

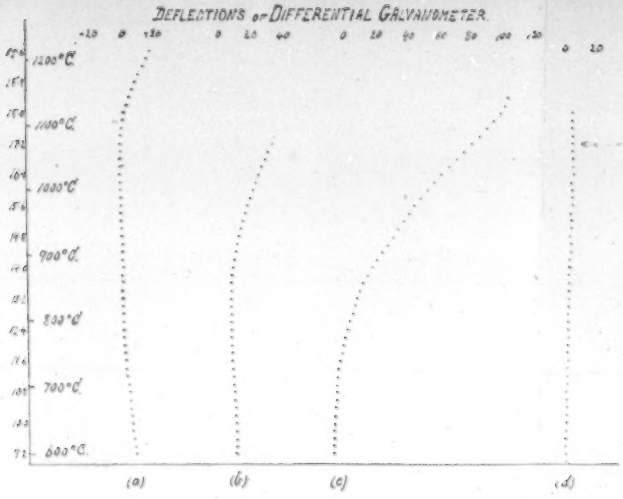
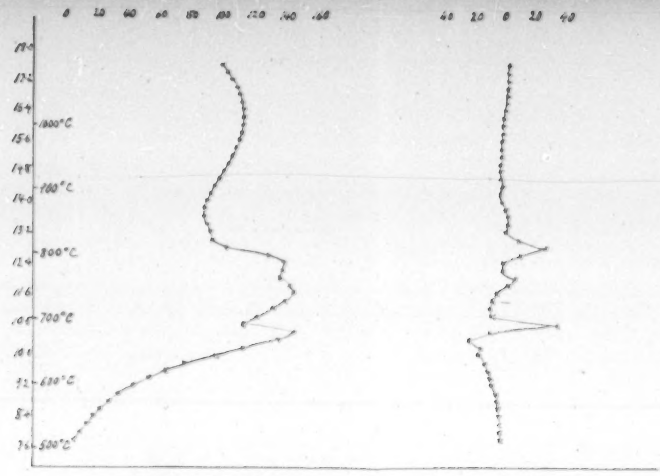


FIG. 7.

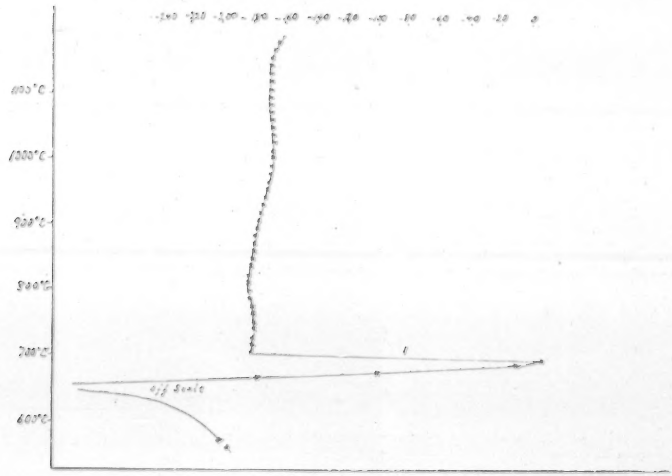


DIFFERENTIAL AND DERIVED DIFFERENTIAL COOLING CURVES OF STEEL
(0·20% CARBON)
SLOW COOLING (1 HOUR 45 MINUTES).

FIG. 8.

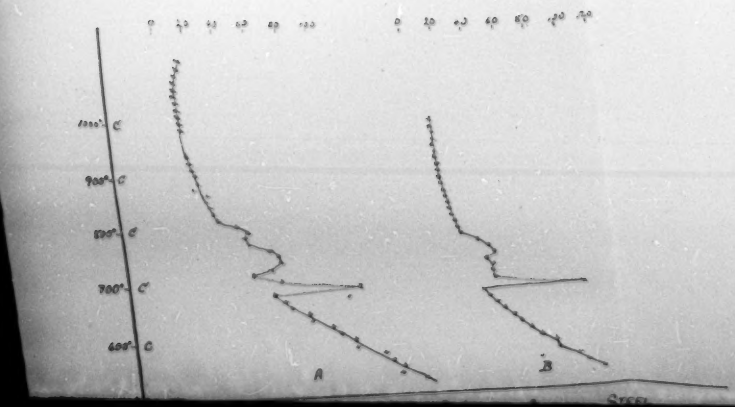
DIFFERENTIAL AND DERIVED DIFFERENTIAL CURVES OF STEEL
(1·26% CARBON)

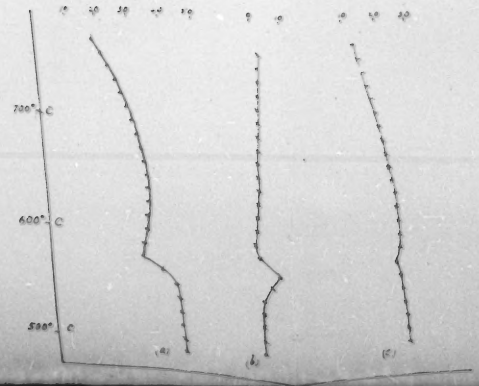
FIG. 11.

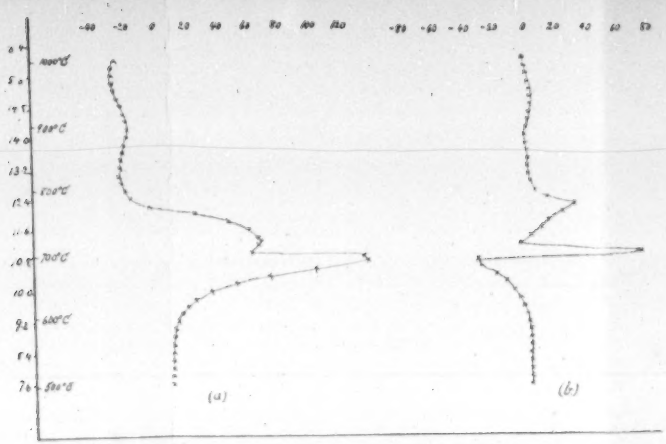


DIFFERENTIAL CURVE OF TWO PIECES OF STEEL
(CARBON 0·99%)

FIG. 12.

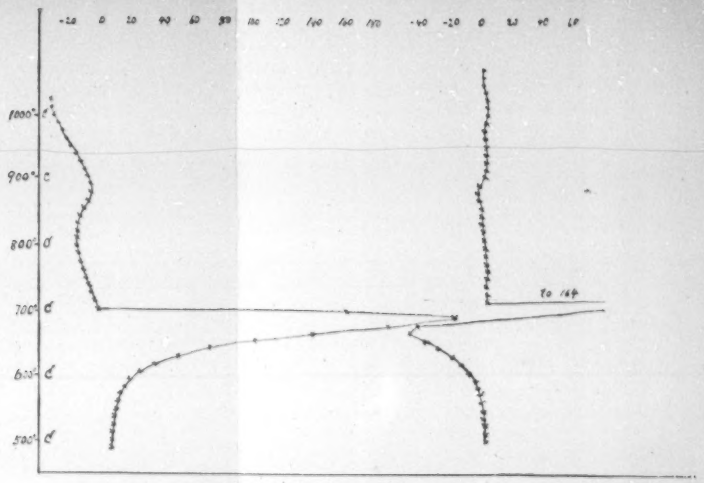






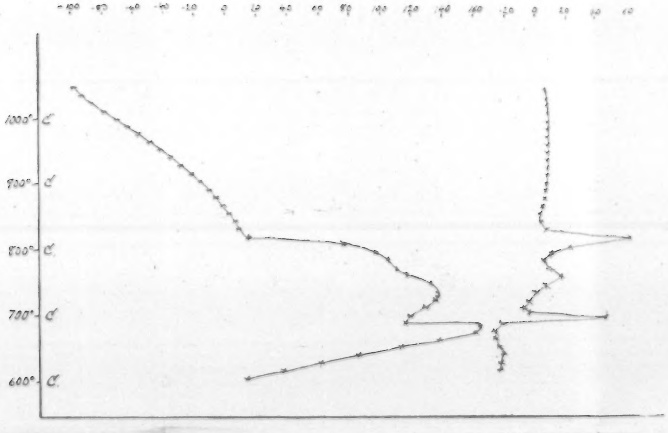
DIFFERENTIAL AND DERIVED DIFFERENTIAL CURVES OF STEEL
(0·326% CARBON)

FIG. 9.



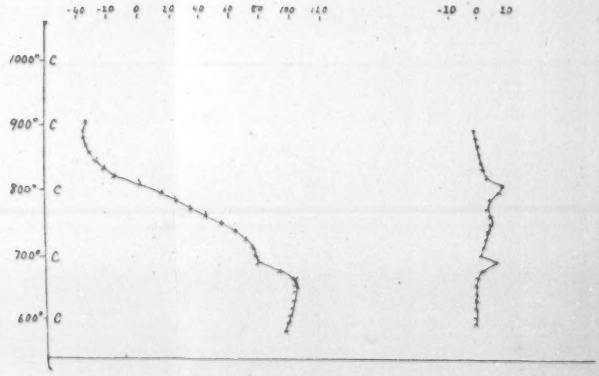
DIFFERENTIAL AND DERIVED DIFFERENTIAL CURVES OF STEEL
(0·99% CARBON)

FIG. 10.



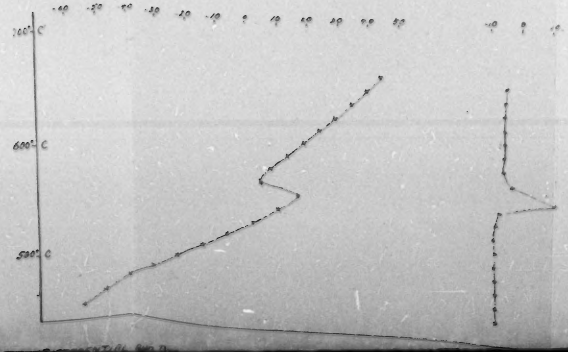
DIFFERENTIAL AND DERIVED DIFFERENTIAL CURVES OF STEEL
RAPID COOLING - 2·5 MINUTES (CARBON 0·20%)

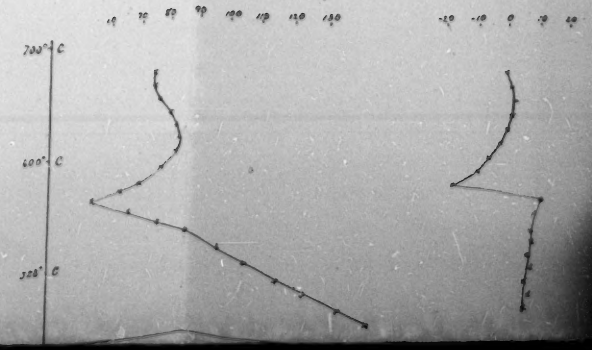
FIG. 13.



DIFFERENTIAL AND DERIVED DIFFERENTIAL CURVES OF STEEL
VERY RAPID COOLING - 12 MINUTES (0·20% CARBON)

FIG. 14.





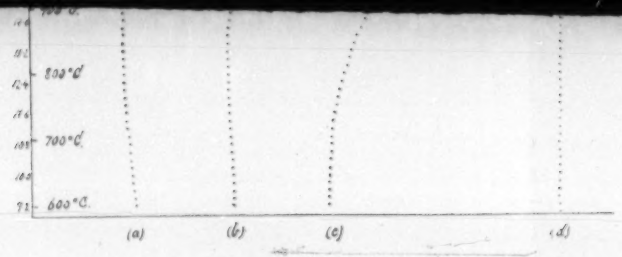
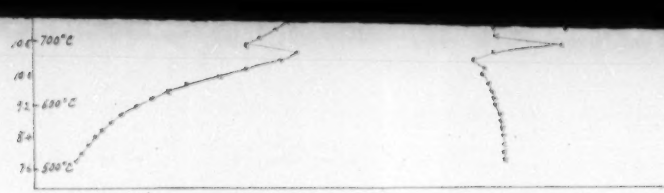
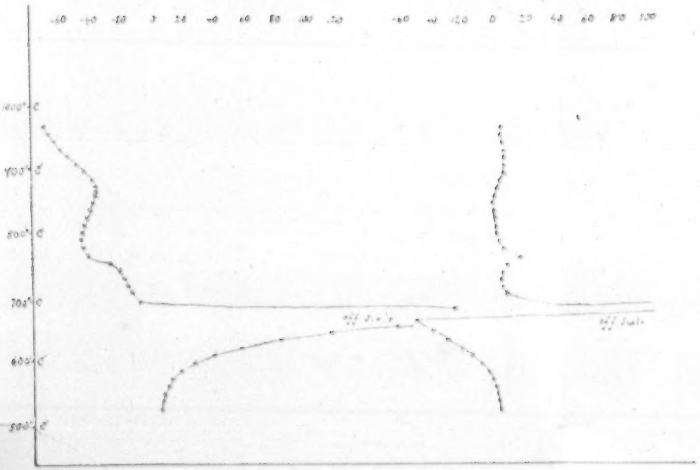


FIG. 7.



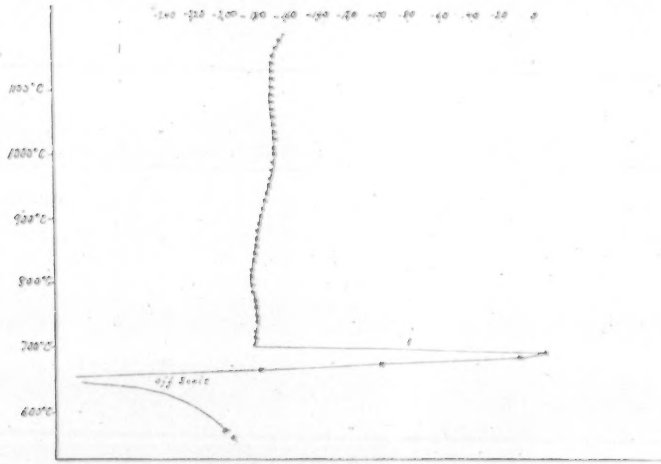
DIFFERENTIAL AND DERIVED DIFFERENTIAL COOLING CURVES OF STEEL
(0·20% CARBON)
SLOW COOLING (1 HOUR 45 MINUTES).

FIG. 8



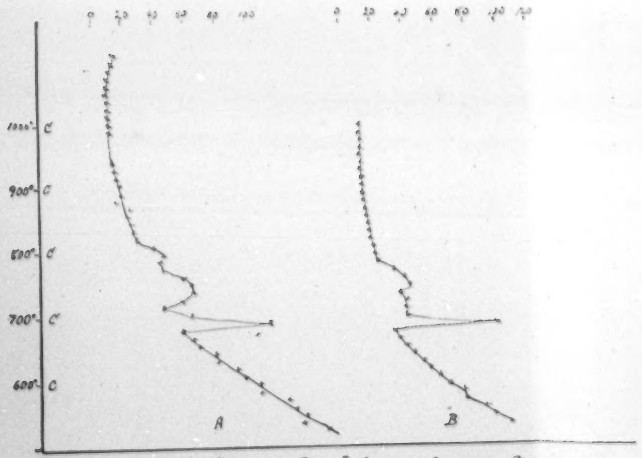
DIFFERENTIAL AND DERIVED DIFFERENTIAL CURVES OF STEEL
(1·26% CARBON)

FIG. 11



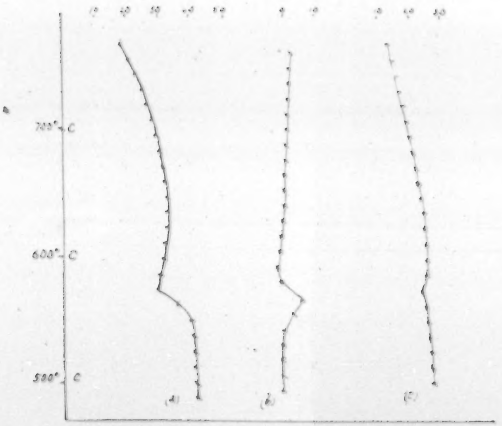
DIFFERENTIAL CURVE OF TWO PIECES OF STEEL
(CARBON 0·99%)

FIG. 12.



"INVERSE RATE" COOLING CURVES OF STEEL
(CARBON:- A·0·20% B·0·31·6%)

FIG. 15.

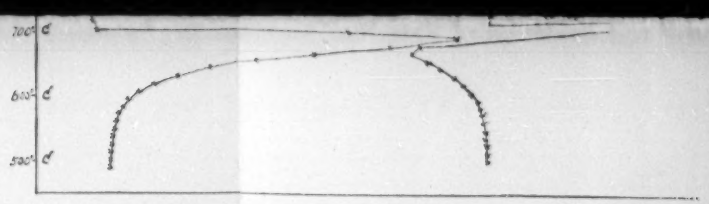


DIFFERENTIAL HEATING (a) AND COOLING (b) CURVES OF N.P.L. ELECTRIC FURNACE
(c) DIFFERENTIAL COOLING CURVE OF MACINTYRE PORCELAIN

FIG. 16

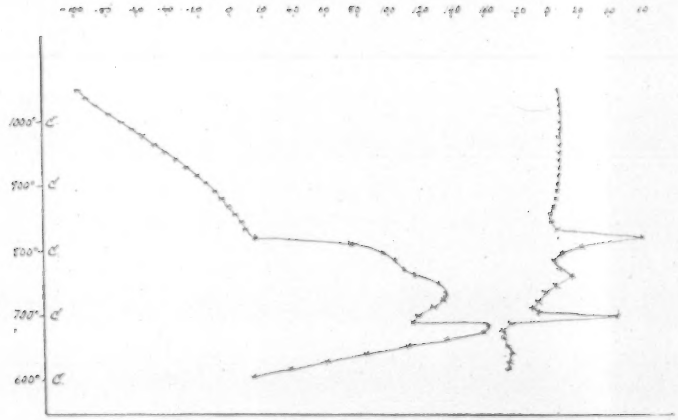
DIFFERENTIAL AND DERIVED DIFFERENTIAL CURVES OF STEEL
(0·326% CARBON)

FIG. 9.



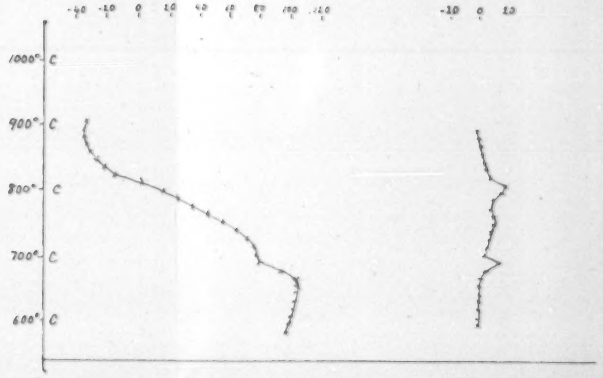
DIFFERENTIAL AND DERIVED DIFFERENTIAL CURVES OF STEEL
(0·99% CARBON)

FIG. 10.



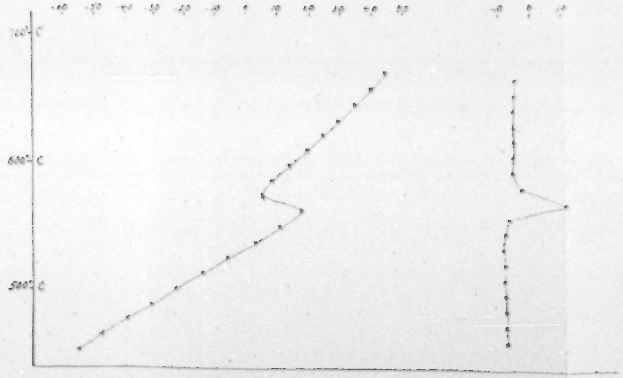
DIFFERENTIAL AND DERIVED DIFFERENTIAL CURVES OF STEEL
RAPID COOLING - 2.5 MINUTES (CARBON 0·20%)

FIG. 13.



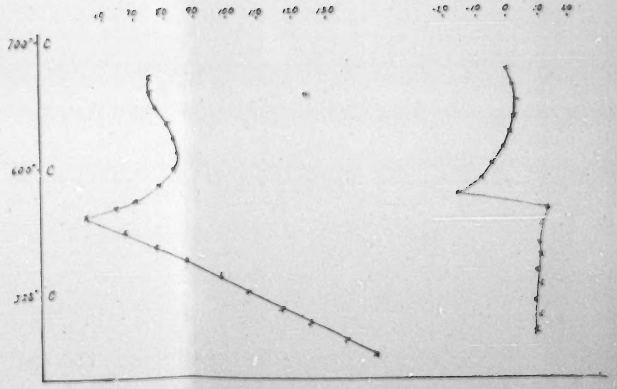
DIFFERENTIAL AND DERIVED DIFFERENTIAL CURVES OF STEEL
VERY RAPID COOLING - 12 MINUTES (0·20% CARBON)

FIG. 14.



DIFFERENTIAL AND DERIVED DIFFERENTIAL HEATING CURVE OF "SILVER SAND"

FIG. 17



DIFFERENTIAL AND DERIVED DIFFERENTIAL COOLING CURVE OF "SILVER SAND"

FIG. 18.

XIII. *Notes on the Plug Permeameter.*

By CHARLES V. DRYSDALE, *D.Sc.**

[Plates VII. & VIII.]

IN 1901 the writer described a permeameter†, devised with the object of enabling permeability-tests to be made on castings and forgings for dynamo-work. The apparatus consisted of a special drill, which could be employed to bore a hole in the casting or forging, leaving, however, a small piece of the metal in the form of a rod or pin standing in the centre of the hole. A split iron plug, arranged to fit both the conical sides of the hole and over the pin, was employed to complete the magnetic circuit; and this plug carried a bobbin wound with magnetizing and search coils. A special portable testing set was also designed with the object of making permeability-tests by direct reading of H and B.

It was noticed at the time that the magnetization-curves obtained by this method fell below those obtained by measurements on a long bar of the same material. This would naturally be attributed to a bad magnetic joint between the pin and plug; but experiments made at the time seemed to point rather to the effect being due to the reluctance of the return path at the point of entrance of the flux, due to the concentration of the lines of force at these points. This seemed probable from the fact that the readings were remarkably consistent when the plug was removed and replaced.

This hypothesis rendered it probable that there might be a more or less definite relation between the value of the true magnetizing-force H in the specimen, and the apparent magnetizing-force $H' = \frac{4\pi}{10}\frac{Cn}{l}$; and arrangements were made to test this relation. The most perfect method was suggested by the double-yoke method of Prof. Ewing. It consisted in finding the relation of the induction density and the magnetizing current for the ordinary drilled specimen, and afterwards when the length of the specimen was doubled by drilling

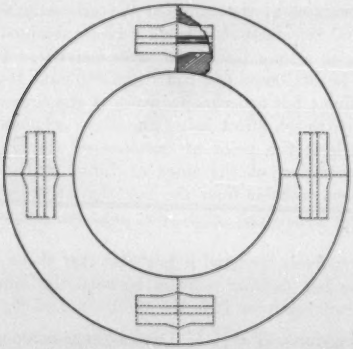
* Read March 27, 1908.

† Proc. Inst. Electr. Eng. vol. xxxi. p. 283.

to a greater depth with a second drill. The increase of the magnetizing current for the same induction should give the true H. A large number of tests were made in this manner, but proved of little value, as, although the first drilling presented no difficulties, it was found impossible to perform the second drilling accurately enough to preserve parallelism of the pin.

Recourse was therefore had to the more simple device of obtaining some specimens of cast and wrought iron and steel, in the form of rings; testing them by the ordinary ring method, and afterwards cutting them up and drilling, for tests by the plug. Rings of this kind were made for us by Messrs. G. Wailes of Euston Road, and were of ordinary cast iron (O.C.I.), special cast iron (S.C.I.), wrought iron (W.I.), and mild steel (M.S.). Two rings of each material

Fig. 1.



were tested, and each ring was cut into four quadrants. Each of these quadrants was drilled in both ends, so that eight drillings were made in each ring. Readings of B were then made with the plug method, at nominal values of H of 30 and 100 on the eight drillings, and the drilling selected for final test, the readings on which corresponded to the mean of the eight. It may be mentioned that the whole of the

64 drillings on the eight rings were made with the same drill, which was none the worse after. None of the drillings gave a bad fit with the plug. Fig. 1 shows the form and mode of cutting and drilling the ring, which was 5 in. external diameter and 1 in. square in section; while Table I. shows an example of the readings on the various drillings of one ring, and the amount of variation usually found. It should be specially noted that these variations are not apparently due to variation in the fit of the plug, but rather to actual differences in the magnetic quality of the iron.

TABLE I.

Readings on Wrought Iron Ring.

Hole No.	Fit.	H=30. B.	H=100. B.
1	Fair.	13,140	15,600
2	Moderate.	12,880	15,820
3	Good.	12,400	15,130
4	Good.	12,040	14,550
5	Fair.	11,730	14,550
6	Good.	12,930	15,080
7	Good.	11,?00	14,190
8	Fair.	11,670	14,550
1 (Check-reading)		.....	15,600
Mean		12,250	14,900

Figs. 2–9, Pl. VII., show magnetization-curves taken on the ring and by the plug on the mean drilling for each specimen. It will be noted that the curve obtained by the plug lies below that for the ring in every case; but there is no evidence to show that this is due to an air-gap. If such a gap were present, of approximately equal amount in each of the specimens, the shearing-over of the curves of the wrought-iron and mild-steel specimens would be much more marked in comparison to those of the cast-iron specimens, owing to the much greater permeability and induction-density in the former.

Table II. is obtained from the curves by finding the value of H′, the nominal magnetizing force in the case of the plug-test, for each value of H from the ring-test, for a given induction-density; and fig. 10, Pl. VIII., gives curves showing

the relation of H′ to H for each of the materials and for the mean of all. It will be observed that although there are considerable differences between the values in the various columns, they appear to have no relation to the permeability of the specimen, and must therefore be regarded as accidental. In fact, the difference between H and H′ is greater in the case of the special cast-iron than for either the wrought-iron or mild-steel specimens, which is exactly contrary to the effect which would be produced by an air-gap. The difference must consequently be attributed to the shortness of the specimen; and the results therefore may have some general value in indicating the amount of the end effect in the case of yoke-permeameters, in which the specimen is only five diameters long.

TABLE II.

Relation of H to H′.

H.	Mean H′ for O.C.I.	Mean H′ for S.C.I.	Mean H′ for W.I.	Mean H′ for M.S.	Mean.
0	0	0	0	0	0
5	8·5	9·7	8·5	8·0	8·7
10	17·2	19·2	18·75	18·5	18·4
15	26·2	29·4	28·2	29·0	28·2
20	34·0	39·0	35·7	37·7	36·6
25	42·2	47·7	42·2	46·5	44·7
30	49·5	56·0	48·0	54·2	51·5
35	57·0	65·25	54·0	61·0	59·3
40	63·5	73·75	60·5	66·5	66·0
45	69·5	81·75	64·5	72·2	71·9
50	76·5	90·75	69·7	77·5	78·5
55	83·5	98·25	75·0	82·0	84·7
60	90·0	106·0	79·5	88·5	91·0
65	96·5	114·0	83·7	93·2	96·9
70	103·0	121·5	89·0	97·5	102·8
75	109·2	129·0	94·2	102·2	108·6
80	115·0	135·7	99·0	108·5	114·6

In order to show the result of correcting the indications of the instrument, the curves figs. 2–9 have points marked by crosses, in which the readings of the plug-permeameter are shifted backwards along the H axis by amounts corresponding to the mean curve in fig. 10 (Pl. VIII.). The agreement with the ring-tests then becomes fairly good. Fig. 11, Pl. VIII., shows readings taken by one of the portable testing sets, in which the H scale has been marked off in this way. The

curves are those given by the ring method, while the points are taken by the testing set. The agreement is as good as in most of the recognized permeameters.

Fig. 12, Pl. VIII., shows two curves taken by the plug permeameter: one on a cast-steel pole-piece for a large generator, which was sent us by the dynamo-builders under the impression that it was of poor magnetic quality. The test shows the permeability to be very good. The other curve is for another specimen of steel, and indicates the remarkable variations of quality which are found in practice.

It should be mentioned that when the plug used in these tests was replaced by a second one, the results agreed within an accuracy of $1\frac{1}{2}$ per cent.

The general conclusion to be drawn from these experiments is that the plug permeameter, in reasonably careful hands, gives as good results as any other method of iron testing, so far as permeability tests are concerned. In a recent paper by Mr. Murdoch* this permeameter was criticized as employing a very small specimen; but this is precisely what has been aimed at in its design. It appears to be of great importance that permeameters for castings or forgings should be capable of being employed on the bulk of the metal itself, without extra machined specimens, and up to the present no other instrument has been devised which meets this requirement.

The writer's hearty thanks are due to Mr. A. C. Jolley for the considerable amount of work involved in carrying out the tests. Mr. C. M. Dowse has also rendered considerable assistance in the later tests.

Discussion.

Prof. S. P. Thompson said the Author's instrument commended itself for its usefulness and ingenuity. He would like to know how far its results were comparable with those obtained from ring and double yoke methods. He was not sure that the Author's explanation that the discrepancies were due to end effects was the correct one. In drilling the hole in the casting the magnetic properties of the iron were altered and the hysteresis increased.

Mr A. Campbell remarked that from his experience of several

* Electrician, vol. lx. p. 245.

varieties of permeameters he was not at all surprised to find that Dr Drysdale had had to calibrate his permeameter by the purely experimental method of comparison with the trustworthy ring method. By averaging the different calibrations given by various types of material he appeared to be able to obtain fairly good results, but there were other permeameters which gave as satisfactory results without any empirical calibration.

Dr A. Russell said he appreciated the ingenuity displayed in the Drysdale permeameter. He suggested the following as the probable cause of part of the discrepancy between the magnetization curves got by the ring method and by the permeameter. In the former method the mean flux and the mean magnetizing force over the cross section of the ring were measured. The ring experimented on was of appreciable radial depth, so that the magnetic force on the inner circumference of the ring was about 50 per cent. greater than that on the outer circumference. The permeability of the iron at the greater force would probably be very different from that at the smaller force. The mean value, therefore, of the flux density over the cross sectional area of the ring might possibly be something quite different from the value it would have if the magnetic force were constant and equal to the mean magnetic force. The errors due to this cause in the ring method, even when the ring was narrow, were sometimes very appreciable. The criticism also applied to many cases in which electricians apply what they call the fundamental magnetic equation.

Mr W. Duddell asked if it was assumed that all the pins had the same diameter.

Dr C. V. Drysdale, in reply to Prof. Thompson, said that the end effect was not of the same nature as an air-gap or it would have had a more marked influence at high permeabilities When the instrument was first introduced, a few tests had been made by drilling a specimen and testing it, and afterwards annealing and re-testing. No change had been observed, but possibly a greater number of tests would have shown variations. The size of the pin seemed not to vary by more than a mil whatever metal was drilled. As to Mr Campbell's remarks, he quite agreed that other permeameters were equally or more accurate, but this was the only one that could be used on the actual forging or casting, and it was surprising and gratifying to find that it was as accurate as it seemed to be. The thickness of the ring mentioned by Dr Russell was an important point. It was necessary in order to permit of drilling, but he was inclined to think that there would be little error from this cause except at low inductions.

XIV. *The Use of Shunts and Transformers with Alternate Current Measuring Instruments.* *By* CHARLES V. DRYSDALE, *D.Sc.**

[Plate IX.]

THE growing demand for instruments for alternate-current measurement, of considerable range and accuracy, has called attention to some of the difficulties attending their production, especially to their limited range. As alternate-current instruments measure the root-mean-square values of the P.D. or current, it necessarily follows that the deflecting torque falls very rapidly as the P.D. or current is decreased, For most commercial instruments, it may be taken that a ten-fold range of torque is the utmost possible for accurate reading, so that when the deflexion is proportional to the torque the range of an A.C. instrument can only be of the order of 3 to 1 except by the employment of double coils or auxiliary devices. The auxiliary devices which may be employed are transformers for electromagnetic ammeters, voltmeters, and wattmeters; shunts for ammeters and wattmeters, series resistances for electromagnetic voltmeters, and condensers and split resistances for electromagnetic voltmeters. Of these, the use of shunts and transformers may give rise to serious errors, and it is proposed here to investigate the amount of these errors and the conditions for their elimination.

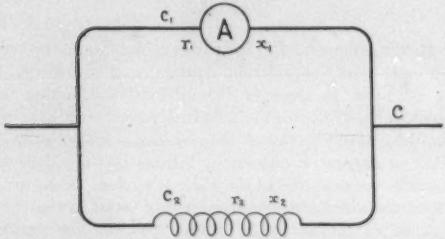
In dealing with any shunting or transforming device we must bear in mind that it may cause errors in two ways:—(*a*) by the multiplying power of the shunt or the ratio of the transformer being affected by frequency, &c.; and (*b*) by a phase displacement being introduced between the current or P.D. on the instrument and the main current or voltage. Ammeters and voltmeters are of course only affected by the first error, but the phase-displacement may be of much greater importance in wattmeters.

* Read March 27, 1908.

Shunts.

Multiplying Power.—In fig. 1, we have an instrument or resistance r_1 and reactance x_1 shunted by a circuit of resistance r_2 and reactance x_2. Then, if V is the P.D. between

Fig. 1.



the terminals, and C_1 and C_2 are the currents in the instrument and shunt respectively, we have

$$C_1 = \frac{V}{I_1} = \frac{V}{r_1 - jx_1} \quad \text{and} \quad C_2 = \frac{V}{I_2} = \frac{V}{r_2 - jx_2}.$$

Hence the total current in the circuit $C = C_1 + C_2$

$$= V\left\{\frac{1}{r_1 - jx_1} + \frac{1}{r_2 - jx_2}\right\} = V\frac{r_1 + r_2 - j(x_1 + x_2)}{(r_1 - jx_1)(r_2 - jx_2)}, \quad . \quad (1)$$

and the virtual current $\overline{C} = \overline{V}\frac{I}{I_1 I_2}$, where I is the impedance of the whole circuit formed by the instrument and shunt *in series*, and I_1 and I_2 are the impedances of the instrument and shunt respectively.

Consequently, since $\overline{C_1} = \frac{V}{I_1}$ the multiplying power of the shunt

$$M' = \frac{\overline{C}}{\overline{C_1}} = \frac{I}{I_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence

$$M' = \sqrt{\frac{(r_1 + r_2)^2 + (x_1 + x_2)^2}{r_2^2 + x_2^2}} = \sqrt{\frac{\left(\frac{r_1 + r_2}{r_2}\right)^2 + \left(\frac{x_1 + x_2}{r_2}\right)^2}{1 + \left(\frac{x_2}{r_2}\right)^2}}$$

But the multiplying power of the shunt for direct currents $M = \frac{r_1 + r_2}{r_2}$ from which $r_1 = (M-1)r_2$. In addition $\frac{x_1}{r_1} = T_1 p$, where T_1 is the time constant of the instrument and p is 2π times the frequency as usual. Similarly $\frac{x_2}{r_2} = T_2 p$, where T_2 is the time constant of the shunt.

Putting these values in the expression above we have

$$M' = \sqrt{\frac{M^2 + \{(M-1)T_1 + T_2\}^2 p^2}{1 + T_2^2 p^2}},$$

which after a little further simplification reduces to

$$M' = M\sqrt{1 + \frac{M-1}{M^2}\,\frac{(M-1)T_1 + (M+1)T_2}{1 + T_2^2 p^2}\,(T_1 - T_2)p^2}. \quad (3)$$

This formula at once shows that if $T_1 = T_2$ the shunt has the same multiplying power for both direct and alternate currents, as is well known. It further shows that if

$$(M-1)T_1 + (M+1)T_2 = 0 \quad \text{or} \quad \frac{T_2}{T_1} = -\frac{M-1}{M+1}$$

the shunt is again correct. This would be the case if the instrument is shunted with a resistance r_2 and capacity K such that $Kr_2 = \frac{M-1}{M+1}T_1$*.

If the instrument is inductive, and is shunted with a non-inductive resistance, $T_2 = 0$ and the formula reduces to

$$M' = M\sqrt{1 + \left(\frac{M-1}{M}\right)^2 T_1^2 p^2}. \quad . \quad . \quad . \quad (4)$$

As an example, a Kelvin centiampere balance was found on test to have a resistance of 62·4 ohms and an inductance (by secohmmeter) of 161 millihenrys. The time constant of this instrument was therefore ·00258 second. The instrument was then shunted with a non-inductive resistance, and it was found that a current of ·878 amp. gave a reading of ·17 amp. on the balance with direct current, thus giving a multiplying

* Mr. Alexander Russell has cast doubts on this formula, and I have found that it is only true in special cases. It was based on the assumption that a shunted condenser could be treated as an impedance with negative time constant, which is not strictly true.

power M of 5·16. Hence at 50 periods, which is the usual frequency of the supply, M′ should be

$$5{\cdot}16\sqrt{1+\frac{4{\cdot}16^2\times{\cdot}00258^2\times314^2}{5{\cdot}16^2}}=6{\cdot}17;$$

while on testing a current of ·927 amp. gave a reading of ·15 amp. or a multiplying power of 6·18. It should be noticed that the formula may be expressed in the form $T_1=\frac{\sqrt{M'^2-M^2}}{(M-1)p}$, which gives us a convenient method of testing the inductances of ammeters, &c.

For commercial work, therefore, we see that the only legitimate method of employing shunts is to make the time constants of the instruments and shunts either negligible or approximately equal. If T_1 and T_2 are nearly equal since $\sqrt{1+h}=1+\frac{h}{2}$ nearly when h is small,

$$M'=M\left\{1+\frac{M-1}{2M^2}\frac{(M-1)T_1+(M+1)T_2}{1+T_2^2p^2}(T_1-T_2)p^2\right\}. \quad (5)$$

It will be noticed that for a given difference in the time constants the correcting term is greater the greater the value of M and the less that of T_2. Consequently, if we take in our formula M so large that M, (M+1) and (M−1) may be regarded as equal and at the same time neglect T_2p in comparison with unity, we have:

$$M'=M\left\{1+\frac{T_1+T_2}{2}(T_1-T_2)p^2\right\}=M\{1+(T_1-T_2)Tp^2\}, \quad (6)$$

where T is the average value of the two time constants T_1 and T_2.

This may be written in the form $\frac{\Delta M}{M}=T.\Delta T.p^2$, and finally

$$\frac{\Delta M}{M}=T^2p^2\frac{\Delta T}{T}. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Consequently, to ensure that the ratio M shall not be altered by more than x per cent. with alternating current of given frequency, the time constants must be adjusted to equality within $\frac{x}{T^2p^2}$ per cent.

In the case of the Kelvin centiampere balance before mentioned, $T = \cdot00258$ at a frequency of 50, therefore $Tp = 314 \times \cdot00258 = \cdot810$ and $T^2p^2 = \cdot655$. Consequently the time constant or the self-induction of the shunt must be adjusted to $\frac{1}{\cdot655} = 1\cdot53$ per cent. for an accuracy of 1 per cent. in the multiplying power. The simplest method of adjusting or checking the shunt is of course to test the ratio with both D.C. and A.C. or with A.C. of two different frequencies.

Phase Displacement.—Returning to formula (1) and rationalizing the denominator, we get

$$C = \frac{V}{I_1^2 I_2^2}\left\{(r_1+r_2) - j(x_1+x_2)\right\}\left\{(r_1+jx_1)(r_2+jx_2)\right\}$$

$$= \frac{V}{I_1^2 I_2^2}\left\{(r_1+r_2)(r_1r_2-x_1x_2) + (x_1+x_2)(r_1x_2+r_2x_1)\right.$$

$$\left. + j\left[(r_1+r_2)(r_1x_2+r_2x_1) - (x_1+x_2)(r_1r_2-x_1x_2)\right]\right\},$$

and consequently

$$\tan\phi = \frac{(r_1+r_2)(r_1x_2+r_2x_1) - (x_1+x_2)(r_1r_2-x_1x_2)}{(r_1+r_2)(r_1r_2-x_1x_2) + (x_1+x_2)(r_1x_2+r_2x_1)}$$

where ϕ is the angle of lag of the main current behind V.

Inserting the values of M, T_1, and T_2 as before, we get

$$\tan\phi = \frac{T_1 + (M-1)T_2 + \{(M-1)T_1 + T_2\}T_1T_2p^2}{M + \{(M-1)T_1^2 + T_2^2\}p^2}\,p.$$

Similarly $\tan\phi_1 = T_1p$ where ϕ_1 is the lag in the instrument. Hence

$$\tan(\phi_1-\phi) = \frac{\tan\phi_1 - \tan\phi}{1+\tan\phi_1\tan\phi}$$

is the tangent of the angle of lag of the current in the instrument behind the main current. Putting in the values of $\tan\phi_1$ and $\tan\phi$ and simplifying, we have as the result

$$\tan\psi = \tan(\phi_1-\phi)$$

$$= (M-1)(T_1-T_2)p \frac{1+T_1^2p^2}{M + \{MT_1^2 + (M-1)T_1T_2 + T_2^2\}p^2 + \{(M-1)T_1 + T_2\}T_1^2T_2p^4}$$

$$= \frac{M-1}{M}(T_1-T_2)p\left\{1 - \frac{(1+T_1^2p^2)\{(M-1)T_1 + T_2\}T_2p^2}{M + \{MT_1^2 + (M-1)T_1T_2 + T_2^2\}p^2 + \{(M-1)T_1 + T_2\}T_1^2T_2p^4}\right\} \quad (8)$$

The phase-displacement is consequently zero for either $M=1$ or $T_1=T_2$ as is obvious, and also for $1+T_1^2p^2=0$, which is impossible. If the shunt is non-inductive $T_2=0$ and $\tan\psi=\frac{M-1}{M}T_1p$; while if the instrument is non-inductive and the shunt inductive,

$$\tan\psi=-\frac{M-1}{M+T_2^2p^2}T_2p.$$

Again, if both T_1p and T_2p are small compared with unity, the expression for $\tan\psi$ reduces to $\frac{M-1}{M}(T_1-T_2)p$. This of course means that the tangent of the phase-displacement between the instrument and mains is the fraction $\frac{M-1}{M}$ of that between the instrument and shunt, as is obvious geometrically. Finally, if T_1 and T_2 are nearly equal,

$$\tan\psi=\psi=\frac{M-1}{M(1+T^2p^2)}(T_1-T_2)p \quad . \quad . \quad . \quad (9)$$

where T is the mean time constant as before. When T is small this reduces to the last expression.

As an illustration of the error produced by shunting a wattmeter, we may assume a case where the time constant of the main circuit has a value of ·0025 as in the Kelvin balance above cited, and suppose that it is shunted with a non-inductive shunt of a nominal multiplying power of 10. We then have $M=10$, $T_2=0$, and T_1p at 50~

$$=314\times\cdot0025=\cdot785.$$

Hence

$$M'=M\sqrt{1+\left(\frac{M-1}{M}\right)^2T_1^2p^2}=10\sqrt{1+\cdot81\times\cdot785^2}=12\cdot25.$$

The phase-displacement is

$$\tan\psi_1=\frac{M-1}{M}T_1p=\frac{9}{10}\times\cdot785=\cdot705;$$

so that the current in the wattmeter lags about 35° behind that in the main circuit. Hence the instrument would read 22·5 per cent. too low owing to the error in the ratio, while the phase-displacement would cause the power factor to be apparently about ·5 when actually zero.

This strikingly illustrates the enormous errors which may be produced by shunts in wattmeters. For any given small displacement ψ we have from (9)

$$\frac{T_1 - T_2}{T} = \frac{\Delta T}{T} = \frac{M}{M-1}\left(\frac{1-T^2p^2}{Tp}\right)\psi \quad . \quad . \quad . \quad (10)$$

and ψ will then be the maximum error in the power factor for loads nearly in quadrature. Consequently, in the above case, if the error in the power factor is not to exceed ·01,

$$\frac{\Delta T}{T} = \frac{10}{9}\frac{(1-\cdot 785^2)}{\cdot 785} \times \cdot 01 = \cdot 00542 \quad \text{or} \quad \cdot 54 \text{ per cent.};$$

and we saw before that for the ratio to be correct to 1 per cent.

$$\frac{\Delta T}{T} = 1 \cdot 53 \text{ per cent.}$$

Transformers.

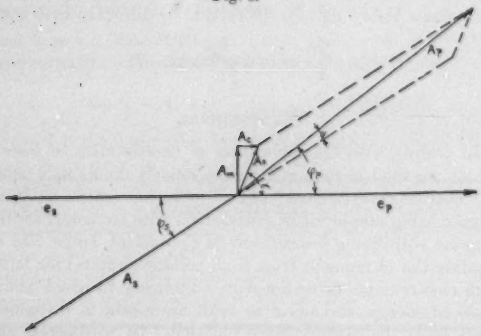
In dealing with the application of transformers to instruments, we shall have to consider separately their employment as "current transformers" or as "voltage transformers"; the former being employed in conjunction with ammeters for the purpose of utilizing instruments of convenient range, and to insulate the instrument from high voltage circuits; the latter with electromagnetic or hot-wire voltmeters to avoid undue waste of energy, and again as with ammeters to disconnect them from the high-pressure circuits. In wattmeters and energy-meters of the induction type, transformers are frequently employed in both the main and shunt circuits simultaneously.

The relations between the primary and secondary currents and voltages have been obtained by Steinmetz and others from the well-known vector diagram, but it will be well to treat the problem *ab initio* as we require the results to be exhibited in the most convenient form.

Current Transformers.—It is most convenient to deal at first with the magnitudes and phase relations of the currents, and we can afterwards apply the results to the investigation of the voltage relations.

Let A_p = primary ampere turns $= C_p n_p$.
A_s = secondary " " $= C n_s$.
A_m = magnetizing " " $= C_m n_p$.
A_c = core loss " " $= C_c n_p$.
A_0 = no load " " $= C_0 n_p$.
ϕ_p = lag of primary current behind core E.M.F.
ϕ_s = lag of secondary " " " "
α = lag of no load " " " "
ψ = phase displacement between primary and secondary currents.

Fig. 2.



Then we have

$$\hat{A}_p + \hat{A}_s = \hat{A}_0 \text{ vectorially;}$$

or writing the currents in the Steinmetz notation,

$$\hat{A}_s = -A_s \cos\phi_s - jA_s \sin\phi_s \quad . \quad . \quad . \quad . \quad (11)$$

$$\hat{A}_0 = A_c + jA_m.$$

Hence

$$\hat{A}_p = A_0 - A_s = A_s \cos\phi_s + A_c + (A_s \sin\phi_s + A_m) \quad . \quad (12)$$

and

$$\overline{A_p} = \sqrt{(A_s \cos\phi_s + A_c)^2 + (A_s \sin\phi_s + A_m)^2}$$

$$= \sqrt{A_s^2 + 2A_s(A_c \cos\phi_s + A_m \sin\phi_s) + A_0^2}.$$

If A_0 is small in comparison with A as it should always be in practice, we may write

$$\frac{A_p}{A_s} = 1 + \frac{A_c}{A_s}\cos\phi_s + \frac{A_m}{A_s}\sin\phi_s. \quad . \quad . \quad (13)$$

If R_c is the ratio of the primary to secondary currents, and R_t the ratio of secondary to primary turns, we have

$$R_c = R\left(1 + \frac{A_c}{A_s}\cos\phi_s + \frac{A_m}{A_s}\sin\phi_s\right). \quad . \quad . \quad (14)$$

Hence the ratio of the currents can only be equal to the transformation ratio if A_c and A_m are both zero or

$$\tan\phi_s = -\frac{A_c}{A_m} = -\cot\alpha.$$

This implies that the secondary current should be in quadrature with the no load current, as is geometrically obvious.

To test the constancy of transformation we have

$$\Delta R_c = R_t\left\{\Delta\left(\frac{A_c}{A_s}\right)\cos\phi_s + \Delta\left(\frac{A_m}{A_s}\right)\sin\phi_s\right\}, \quad . \quad (15)$$

which shows that for constant ratio of transformation both A_c and A_m must be proportional to A_s or that

$$\tan\phi_s = -\frac{\Delta\left(\frac{A_c}{A_s}\right)}{\Delta\left(\frac{A_m}{A_s}\right)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

For constant secondary impedance A_s is proportional to the core E.M.F. and hence to the induction density in the core, B. Hence the current vector should be perpendicular to the curve of which the ordinate is $\frac{A_m}{A_s}$, representing the reluctance of the circuit, and the abscissa is $\frac{A_c}{A_s}$ or the ratio of the core-loss current to B. It is obvious that this is when the secondary load is a leading one, and this result is contrary to the frequently quoted experiments of Mr. Campbell *, who

* Phil. Mag. vol. xlii. p. 271; Journal I. E. E. vol. xxxiii. p. 654.

is claimed to have found experimentally that better constancy of ratio is obtained with an inductive instrument. In the case, however, where there is an air-gap in the transformer, the curve connecting $\frac{A_m}{A_s}$ with $\frac{A_e}{A_s}$ may be practically horizontal, and in this case the regulation will be worst on a non-inductive instrument, and better the greater is either the lag or lead of the secondary current. In estimating the value of Mr. Campbell's experiments it must not be forgotten that the two Kelvin balances employed by him must have been of very different impedance and consequently the core inductions would have been decidedly different. This of itself would probably have been sufficient to produce the effect found. Experimental evidence will be found below on this point.

Effect of variation of frequency.—Increase of frequency increases the impedance of the instrument if inductive, but at the same time increases the E.M.F. for the same core induction. With a fully inductive instrument, therefore, and if eddy currents are absent, variations in frequency should have no effect upon the ratio. This agrees with Mr. Campbell's theoretical and experimental conclusions.

Phase Difference.—Reverting to equations (11) and (12), we have

$$\tan \phi_p = \frac{A_s \sin \phi_s + A_m}{A_s \cos \phi_s + A_e},$$

Hence

$$\tan \beta = \tan (\phi_p - \phi_s) = \frac{\frac{A_m}{A_s} \cos \phi_s - \frac{A_e}{A_s} \sin \phi_s}{1 + \frac{A_e}{A_s} \cos \phi_s + \frac{A_m}{A_s} \sin \phi_s} \quad . \quad (17)$$

which is zero for A_m and A_e both zero or for $\tan \phi_s = \frac{A_m}{A_e}$ as is obvious.

Hence for zero phase-displacement the instrument should be so inductive that the ratio of its reactance to resistance equals the ratio of the magnetizing to the core-loss currents of the transformer.

As the relation between $\frac{A}{A_s}$ and $\frac{A_e}{A_s}$ is not in general linear it follows that the phase-displacement cannot be constant, but

it is fairly evident that the best constancy would be approximately obtained when the actual displacement is smallest, *i. e.* with an inductive instrument.

With variation of frequency it is again obvious that with a fully inductive instrument, without eddy currents in the transformer, the phase-displacement should be unaffected by frequency. The less inductive the instrument the more rapidly should the phase-displacement decrease with frequency, owing to the drop in the induction density.

P.D. Relations.—Writing the secondary current in the form $A_s(\cos\phi_s + j\sin\phi_s)$ and the primary

$$A_s\cos\phi_s + A_c + j(A_s\sin\phi_s + A_m),$$

let r_p and r_s be the primary and secondary resistances and x_p and x_s the primary and secondary reactances for coils of a single turn each.

Then

$$V_p = E + \{A_s\cos\phi_s + A_c + j(A_s\sin\phi_s + A_m)\}(r_p - jx_p)$$
$$-V_s = E - A_s(\cos\phi_s - j\sin\phi_s)(r_s - jx_s),$$

from which

$$V_p = E + A_s\left\{r_p\left(\cos\phi_s + \frac{A_c}{A_s}\right) + x_p\left(\sin\phi_s + \frac{A_m}{A_s}\right)\right.$$
$$\left. -j\left[x_p\left(\cos\phi_s + \frac{A_c}{A_s}\right) - r_p\left(\sin\phi_s + \frac{A_m}{A_s}\right)\right]\right\} \quad (18)$$

and

$$-V_s = E - A_s\{r_s\cos\phi_s + x_s\sin\phi_s - j(x_s\cos\phi_s - r_s\sin\phi_s)\} \quad (19)$$

Voltage Ratio.—Taking the ratio of $\frac{V_s}{V_p}$, simplifying, and neglecting squares of small quantities, we have

$$R_v = R_t\left\{1 - \frac{A_s}{E}[(r_p + r_s)\cos\phi_s + (x_p + x_s)\sin\phi_s] - \frac{A_c}{E}r_p - \frac{A_m}{E}x_p\right\},$$

which will be seen to be correct from the vector diagram.

Denoting $r_p + r_s$ by r the equivalent resistance,

$x_p + x_s$ by x „ reactance,

and $\sqrt{r^2 + x^2}$ by I „ impedance,

$$R_v = R_t\left\{1 - \frac{A_s}{E}(r\cos\phi_s + x\sin\phi_s) - \frac{A_c}{E}r_p - \frac{A_m}{E}x_p\right\}. \quad (20)$$

For an electrostatic or high-resistance voltmeter, $A_s=0$ and we have

$$R_v=R_t\left\{1-\frac{A_c r_p+A_m x_p}{E}\right\}. \quad . \quad . \quad . \quad . \quad (21)$$

For a hot-wire instrument $\phi=0$ and

$$R_v=R_t\left\{1-\frac{A_s r+A_c r_p+A_m x_p}{E}\right\}. \quad . \quad . \quad (22)$$

Returning to formula (20) we have

$$\frac{\Delta R_v}{R_v}=r_p\Delta\left(\frac{A_c}{E}\right)+x_p\Delta\left(\frac{A_m}{E}\right)$$

if the instrument and internal impedances are constant. This is independent of the phase of the secondary current and can only be zero if A_c and A_m are proportional to E, which is only the case for an air-core transformer. For an iron-core transformer the best constancy of ratio is thus obtained by making A_c and A_m and also r_p and x_p as low as possible, the core being of the best quality iron, and the coils subdivided and intercalated.

P.D. Phase Relations.—From equations (18) and (19) we obtain

$$\tan\psi_p=-\frac{x_p\left(\cos\phi_s+\frac{A_c}{A_s}\right)-r_p\left(\sin\phi_s+\frac{A_m}{A_s}\right)}{E+A_s\left\{r_p\left(\cos\phi_s+\frac{A_c}{A_s}\right)+x_p\left(\sin\phi_s+\frac{A_m}{A_s}\right)\right\}}A_s$$

$$\tan\psi_s=\frac{x_s\cos\phi_s-r_s\sin\phi_s}{E-A_s\{r_s\cos\phi_s+x_s\sin\phi_s\}}A_s.$$

If, as should be the case with voltage transformers, the resistance and inductive drop in the transformer are small compared with E, we have

$$\psi_p-\psi_s=\beta=-\frac{A_s(x\cos\phi_s-r\sin\phi_s)+A_c x_p-A_m r_p}{E} \quad . \quad (23)$$

as is geometrically obvious.

For the phase-difference to be zero it follows that either the resistance and inductance must be zero, or that

$$\tan\phi_s=\frac{A_s^2 rx\pm(A_c x_p-A_m r_p)\sqrt{A_s^2(x^2+r^2)-(A_c x_p-A_m r_p)^2}}{A_s r^2-(A_c x_p-A_m r_p)^2}. \quad (24)$$

In this expression if A_s is large compared with A_c or A_m $\tan \phi_s = \frac{x}{r}$, or the ratio of the reactance to the resistance of the instrument should be the same as for the transformer, and the instrument should therefore be fairly inductive.

With a non-inductive instrument

$$\beta = \frac{A_s x + A_c x_p - A_m r_p}{E}, \quad . \quad . \quad . \quad . \quad (25)$$

and if the instrument takes no appreciable current compared with A_c or A_m,

$$\beta = \frac{A_c x_p - A_m r_p}{E} \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

To find the variation of phase-displacement with voltage we have evidently from (23)

$$\Delta\beta = x_p \Delta\left(\frac{A_c}{E}\right) - r_p \Delta\left(\frac{A_m}{E}\right),$$

and this is evidently zero for an air-core transformer.

Relation between the Secondary Current and the Magnetizing and Core-Loss Currents.—If l is the length of the magnetic path, a its area of cross section, and I_s the total secondary impedance (reduced to 1 turn), we have

Effective core E.M.F. $\bar{e} = \frac{a\bar{B}p}{10^8} = \frac{a\hat{B}p}{\sqrt{2} \times 10^8}$ volts per turn;

Secondary ampere turns $A_s = \frac{\bar{e}}{I_s} = \frac{a\hat{B}p}{\sqrt{2} \times 10^8 I_s}$,

from which

$$\hat{B} = \frac{\sqrt{2} \times 10^8 I_s A_s}{ap} \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Assuming the core loss to be $\eta \hat{B}^e$ ergs per c.c. per cycle:—

Total core loss

$$w_m = \frac{\eta \hat{B}^e V_s}{10^7} \text{ watts},$$

and the core-loss ampere turns

$$A_c = \frac{w_m}{\bar{e}} = \frac{5\sqrt{2}}{\pi} \eta l \left\{ \frac{\sqrt{2} \times 10^8 I_s A_s}{ap} \right\}^{(e-1)} \quad . \quad . \quad (28)$$

For the magnetizing current

$$A_m = \frac{10}{4\pi} \frac{Bl}{\mu}$$

or

$$A_m = \frac{10^9}{4\pi} \frac{l}{\mu a p} I_s A_s. \quad . \quad . \quad . \quad . \quad . \quad (29)$$

These relations are sufficient to enable the magnetizing and core-loss currents to be calculated when the total secondary impedance is known. The resistances are of course readily calculable, while the equivalent inductance of the whole transformer for a single primary and secondary winding may be calculated from the following formula, deduced from that given by Prof. Kapp :

$$L = C \frac{\sqrt{2}}{10^{10}} \frac{P}{LK^2} (3KS + T) \quad . \quad . \quad . \quad . \quad (30)$$

P being the mean perimeter of the coils in inches,
L the length of gap from iron to iron, parallel to insulation between coils,
S the thickness of the insulation in inches,
K the number of sections per coil,
T the length of the winding space,
and C a constant given as 44 for core and 55 for shell type transformers.

From personal experiment, however, the author has found that the value 44 is more nearly correct for shell type transformers.

Experimental.—As some of the conclusions arrived at are of importance, and are somewhat at variance with existing ideas, some experimental evidence will be of value. A brief reference only will be necessary here as some of these tests have already been published *, together with a description of the methods by which the results were obtained.

The principal transformer tested was made for the tests, and was of the shell type ; the core having a magnetic path of 35·3 cm. and a sectional area of 71 sq. cm. Four independent coils, each of 150 turns of No. 18 D.C.C. copper wire, were wound side by side, the resistance of each coil being 1·2 ohms. Fig. 3 shows the core, and fig. 4 the

* "Phase Displacements in Resistances and Transformers," Electrician, No. lviii. p. 160.

Fig. 3.—Isometric view of Transformer Core.

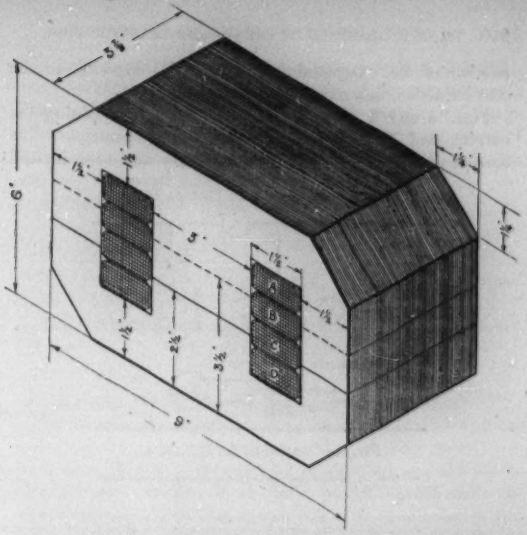
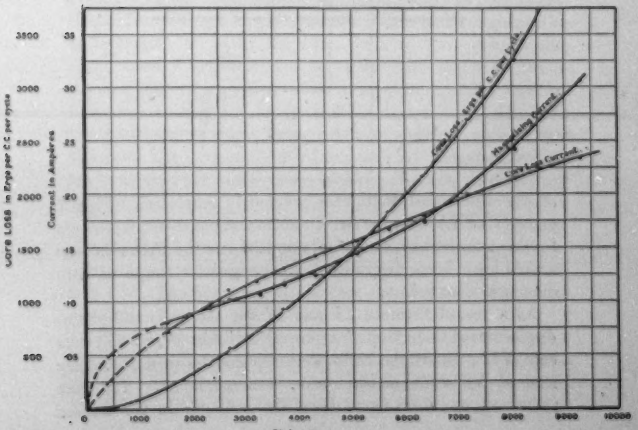


Fig. 4.—Relation of Magnetizing and Core-loss Currents to Induction.



B in Gausses.

relation of the magnetizing and core-loss currents to the core induction, determined experimentally.

Figs. 5 and 6 show the connexions for determining the current and P.D. ratios and phase displacements. The current ratio is determined by an ordinary wattmeter instead

Fig. 5.—Connexions for Current Test.

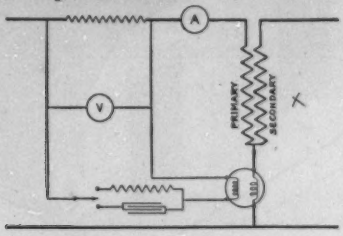
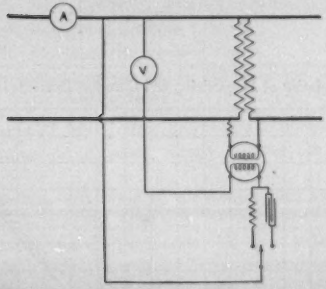


Fig. 6.—Connexions for P.D. Test.



of the differential wattmeter previously employed. Figs. 7 and 8 (Pl. IX.) show the current ratio and phase-displacements for this transformer, while figs. 9 and 10 give the P.D. relations which have been determined over a greater range of core induction than before.

As a test of formulæ 14 and 17 for the ratio and phase-displacement when used as a current transformer, we will take case VII., where the coils were separated and a secondary load of 5·9 ohms non-inductive was employed. The core induction was then found from the voltages in the unused

coils and was taken at 1565 for a secondary load of 5 amps. On measuring the equivalent reactance of the transformer it came out at 5·12 ω or 2·56 ω for the secondary leakage reactance. From this we get a total secondary resistance of 7·1 ω, reactance of 2·56 ω, and impedance of 7·5 ω approx. $\cos\phi_s = \cdot 94$ and $\sin\phi_s = \cdot 34$.

Formula 14 gives us as the loss of ratio per cent.

$$100\frac{R_t - R_e}{R_t} = 100\left\{\frac{A_c}{A_s}\cos\phi_s + \frac{A_m}{A_s}\sin\phi_s\right\},$$

which works out at 1·92 per cent., while the experimental value is 1·8 per cent.

The phase-displacement in the same case may be taken as

$$\frac{A_m}{A_s}\cos\phi_s - \frac{A_c}{A_s}\sin\phi_s,$$

which works out to ·58°, the experimental value being ·75°.

In the case of a lagging load as in curve VI. but with the coils close together, the calculated loss of ratio is 1·87 per cent. as against 2·7 per cent. by experiment, and the phase-displacement works out at ·57°* which agrees with the experimental value. The agreement between calculation and experiment is not always therefore very close, but this is probably due to the difficulty of determining the core induction. In a current transformer the magnetic leakage may be quite comparable with the main working flux, owing to the small value of the latter, and hence the distribution of flux in the core is very irregular. The mean value of the induction, however, as calculated by formula 27, agrees very fairly with the experimental value, being 1575 as against 1565 in case VII., and 1345 as against 1385 in case VI. Calculations have not been made in other cases, as the induction densities are too low.

The statement on p. 243 that a non-inductive secondary circuit should give a better ratio than an inductive one of the same impedance, is amply borne out by comparing curves V. and VI. The advantage of a low induction density is also manifest.

Calculations of P.D. ratio and phase-displacement show an

* The curves for current ratio and phase-displacement appeared in the 'Electrician' (*loc. cit.*), but in case VI. the phase-displacement was unfortunately given with reversed sign.

agreement within ·1 per cent. in the ratio and ·01° in the displacement, which is sufficient for most purposes. The theory is in each case confirmed.

Tests made with instrument transformers of various makers have in no case shown anything like such good results, and the writer is of opinion that these results represent nearly the best that can be obtained without using some special alloy of low core-loss, and also of high permeability. In view of the importance which magnetic leakage assumes in current transformers, the secondary current nearly always lags 20° to 30°, and the magnetizing current is therefore nearly as important as the core loss-current.

Fig. 11 (Pl. IX.) gives the relation between A_c and A_m and is consequently the locus of the end of the no load vector A_0 for various inductions. This vector is therefore over a considerable part of the range, at about 45°, and consequently the best ratio should be obtained with a secondary circuit leading by about 45°. In fig. 12 a curve is given for the relation of $\frac{A_c}{B}$ to $\frac{A_m}{B}$ which is of use in working out the P.D. ratio and phase displacement.

Figs. 13 and 14 show curves of ratio and phase-displacement for three current transformers made by Messrs. Everett & Edgcumbe and Messrs. Nalder Bros. & Thompson. In the former, the primary and secondary windings were close together, giving good ratio but large phase displacement; and it is noticeable that throughout lower phase-displacement is got by greater loss of ratio, as indicated in the tests on the experimental transformer. In the Nalder transformer, the primary and secondary coils are wound on thick porcelain bobbins side by side. The leakage is therefore somewhat large, producing a variation of ratio of about 8 per cent. between 2 and 18 amperes; but this is accompanied by a phase-displacement which is less than ·2° over the range from 6 to 18 amperes or the upper two-thirds of the range. Such a transformer therefore, if used with a wattmeter, *and calibrated with it*, should do fairly well for commercial testing.

In conclusion it should be noted that the tests on all four transformers give results which are not very widely different and the results may therefore be taken as fairly representative of the behaviour of instrument transformers.

The writer wishes to express his thanks to his senior Demonstrator, Mr. A. C. Jolley, for great assistance in the experimental work, and to Mr. A. F. Burgess, B.Sc., for checking the calculations.

Discussion.

Mr A. Campbell expressed his interest in the thorough manner in which Dr Drysdale had gone into the question of the use of transformers with measuring instruments. With regard to the Author's criticism of his (Mr Campbell's) work on the subject (which was carried out twelve years ago) he should like to mention several points. The main object of the first quoted paper was to show that with air-core transformers the transformation ratio becomes more and more constant (for various frequencies) the higher we make the time-constant of the secondary circuit; thus high inductance and low resistance are wanted. Mr Campbell stated that in his paper he also gave several experiments to show that iron-ring transformers "may in many cases be used in a similar way," care being taken to have the resistance of the secondary circuit small enough. In a later paper he stated that to make the ratio sufficiently constant and independent of frequency we require relatively low resistance and high inductance in the secondary circuit. Dr Drysdale showed that for the special case of constant frequency, relatively high inductance does not give the most constant ratio. His (Mr Campbell's) experiments were not complete enough to settle this point and he was careful not to dogmatize on the matter. He was glad that Dr Drysdale had elucidated it.

Dr A. Russell expressed his interest in the paper. He suggested that the Author should take the mutual inductance between the shunt and the instrument into account. If L_1, R_1 be the constants of the instrument and L_2, R_2, those of the shunt, and if M be the mutual inductance between them, the multiplying factor for the reading will be the same whatever the frequency provided that $\frac{L_1-M}{R_1}=\frac{L_2-M}{R_2}$.

Mr Kenelm Edgcumbe referred to the fact that the Author recommended small core-losses, and pointed out that he had seen it stated that in special cases it was possible to improve the ratio and phase errors by increasing the core-losses.

Dr Drysdale, in reply to Mr Campbell, said his praise of the air-core transformer was justified, but in practice instrument-makers were forced to use iron for commercial reasons. Similarly, the ring form of transformer was generally impossible as in practice instrument transformers were used to insulate the observing instrument from high pressure mains, and considerable insulation between the windings was necessary. The reason why an iron-cored transformer behaved differently to one with an air core chiefly resided in the core-loss current which was not proportional to the magnetizing current. The method of testing the transformers would be shown in operation in the laboratories.

XV. *Experiments on Artificial Fulgurites.*

By DOROTHY DEANE BUTCHER*.

[Plates X.–XIII.]

Introduction.

FULGURITES are tubes formed by lightning striking sand or rock and fusing it. The rock fulgurites somewhat resemble worm-borings lined with a glassy substance. It is with the sand fulgurites that we are concerned.

These latter have been found in many different parts of the world—Drigg in Cumberland, Senner in Lippe, Dibla near Lake Chad, the Libyan Desert, South Carolina, Florida, and many other places. They have been found in the sand-dunes at Bondi, near Sydney, New South Wales, by Mr. J. W. Grimshaw, who has kindly given us the opportunity of examining his collection.

The fact that the tubes found are of two sorts—strong, rocky, thick-walled tubes, and smooth, cylindrical tubes with glassy walls—was pointed out by Dr. Fiedler in 1817 (1), and he suggested the possibility that one sort was formed by positive and the other by negative electricity. This idea seemed to be supported by the fact that the cross section of a thick tube shows a radial structure bearing a marked resemblance to the Lichtenberg figures made by a spark from a positively charged conductor.

It was with the view of testing this hypothesis that attempts were made to form fulgurites artificially.

Natural Fulgurites.

The thin-walled tubes are found much more frequently than the others. The former are very often squeezed together and flattened as if the tubes had been subjected to pressure when in a molten condition. This is probably due to the pressure exerted on the thin walls by the surrounding sand and the simultaneous decrease of pressure within the tube as the hot gas cools (Pl. X. fig. 5).

Of the reports of fulgurites examined only 8 localities

* Read April 10, 1908. Communicated by Sidney Skinner, M.A.

produced thick tubes, whereas there were 18 localities where thin tubes were found, and in five cases the thin tubes had retained their cylindrical form (figs. 1 2, 3). Amongst the Bondi fulgurites both thin and thick walled specimens occur.

Some of the tubes are branched whilst others appear to be unbranched, but as very few are found unbroken it is difficult to make sure of this. The tubes appear to be more or less tapering as they go down, but very few have been excavated to their ends. The fulgurite found near Dresden by Dr. Fiedler and now in the British Museum (2) is 17 feet long, and consists of 532 pieces; that found at Drigg in Cumberland was traced for over 30 feet without any sign of the end being found (3).

Amongst the tubes found near Macclesfield in a bed of sand 100 feet thick, one was removed complete, and it was found to taper almost to a point. These tubes started 12 feet below the surface and were 9 or 10 feet long, with an internal diameter 0·7 to 0·1 inch (4).

As a rule the tubes are vertical, but they are sometimes inclined at an angle of 45° to 60° either for the whole or for part of their course, and when there are branches these are occasionally horizontal or even inclined upwards. Good tubes are found only in sand. It appears, therefore, that the chemical nature of the soil determines the completeness of the tube.

In one case at Zankendorf near Vienna (5), where there was a bed of moist clay under the sand, the tube terminated in a small hollow lump with several openings in it. The course of the lightning could be clearly seen under these as long red threads penetrating the clay for about 8 inches, and radiating in all directions like fine roots. There was no fusion after the lightning left the sand, but the fact that these threads were caused by the lightning was shown experimentally by the clay turning red when heated artificially.

Microscopical and chemical analysis show that the tubes are composed of silica, being often purer in composition than the sand in which they are formed (6). The melting-point of silica is very high, but it is possible that the salts present in

the soil act as a flux and render a somewhat lower temperature sufficient (7), but even then the temperature developed by the electricity must be very high.

Artificial Fulgurites.

Since the means at our disposal make it useless to attempt to produce a large quantity of heat by a spark, some material with a lower melting-point than silica had to be used. The substance must be capable of being reduced to a fine powder, must fuse easily, and should be a poor conductor of electricity. Several substances were tried such as salt, sugar, borax, resin, and sealing-wax, but the best results were obtained with powdered glass.

This material mixed with common salt had already been used in attempts to manufacture artificial fulgurites in Paris in 1828 (8). The "Batterie du cabinet de Charles," the largest Leyden battery in Paris at that date, was used and tubes were made 25 mm. long and 1½ to 3 mm. across. Unfortunately no drawings of these tubes are available, and the only description given is that they are "just like the natural ones, but more delicate than those found by M. Fiedler."

In our experiments we used a battery of 12 Leyden jars made of thin glass beakers 8 in. high covered with tinfoil to about 2½ in. from the top. These were charged by a small Wimshurst machine worked by a motor.

The powdered glass or resin was placed in a hole in a block of paraffin wax, and the spark passed from a point above to a metal plate below the hole.

When the spark was passed through a mass of finely powdered resin about 8 mm. thick, a circular hole with slightly melted walls was formed, but the walls fell in when the powder was shaken. A similar result was obtained with powdered borax crystals.

The chief difficulty lay in overcoming the explosive effect of the oscillatory spark. Unless the powder was tightly packed into the wax and covered securely, it was blown about the room. With the idea of obviating this, the oscillations of the spark were damped by introducing resistance in the circuit in the form of a piece of wet string. The

result was most successful, the tubes obtained being larger and sufficiently thick to bear handling.

The resistance used was about 80 cm. of wet half-inch tape. With too long a string there was no sharp discharge but a gradual oozing of the electricity, and the precise length required seemed to vary on different days—probably because there was more evaporation from the string on dry days.

Some of the tubes were branched, but there was no definite difference between the two ends which could be ascribed to difference of polarity. The branches were always towards the plate end, whichever way the current passed (Pl. XIII. fig. 16).

The early attempts with powdered glass were unsuccessful, but this was found to be due to the powder being so coarse that the spark passed through the air between the particles without melting the powder together. With powder which had been sifted by repeated washing and allowing the larger particles to settle, fine smooth tubes were obtained—the finer the powder the smoother were the walls of the tube.

The largest tube thus obtained was 9·5 mm. long and ·5 mm. in cross section. The length could not be increased as the jars would not give a spark of greater length.

Experiments with the powder under pressure greater than the atmospheric up to 30 in. excess gave very similar results.

When the wet string was removed from the circuit no tubes could be made in the powdered glass under the ordinary conditions.

To explain this action of the wet string, experiments were made with a form of Kinnersley's electric thermometer. This apparatus consisted of a T-piece of narrow glass tubing having the tail of the T turned up again and ending at a somewhat higher level than the crosspiece. Platinum wires were sealed into the two arms of the crosspiece, and the junction of this with the vertical tube was blown out into a small bulb. The vertical tube was half filled with water and acted as a manometer tube, being open to the air at the free end.

When a spark passed between the two platinum wires, the air in the bulb was headed and expanded driving the water out of the U. The level of the water served to measure the

expansion of the air, and thus, indirectly, the heat developed by the spark. It was found that the oscillatory spark threw the water violently up in the tube, but the water immediately returned to a point which served to measure, by its distance from the original level, the heat developed. As was expected, the heat developed without the string was some three times greater than with the string. It was evident, therefore, that the repulsions between the elements of the oscillatory current prevented the formation of a tube with that form of discharge.

Once or twice when using the undamped spark, when the powder was very tightly packed into the wax cylinder and the two terminals sealed on the ends by a thick coating of wax to prevent any possibility of the powder escaping, very fine lacework tubes were obtained of much greater size than those got with the damped spark. They were about 1 cm. long and 2 to 4 mm. in diameter, but fell to pieces on being touched, the walls being a mere network of fine threads and looking more as if the particles had been stuck together, than like the result of a fusion of the particles themselves These tubes were only obtained once or twice, most attempts only resulting in blowing away the powder.

The small size of the tubes formed was apparently merely due to the want of power of the electrical apparatus; the length of the tubes which it was possible to make being approximately equal to the length of the spark which could be obtained from the jars.

We might therefore hope to get much larger tubes if we could enlarge the potential and capacity of the Leyden jars used, but those obtained are quite of sufficient size to prove the question which was under consideration.

The results of these experiments may be summarized as follows:—

1. The tubes are formed by fusion of the powder which surrounds the column of air in which the spark passes. The length and thickness of the tube depend on the energy of the spark, and also on the character of the spark, *i. e.* whether it is unidirectional or oscillatory. This latter is shown by the fact that the same quantity of energy stored in the Leyden jars does not form a tube when the discharge is abrupt and

oscillatory, but forms one if the discharge is rendered less oscillatory by the intervention of a wet string resistance, although a large portion of the energy in the latter arrangement is absorbed in overcoming the resistance of the string.

2. There is no appreciable difference in the two ends of a tube provided that the two electrodes are alike. When one electrode is a point and the other a flat plate, any branching that may occur will be towards the plate whichever electrode is made positive.

In nature, the flat plate would be represented by the moist lower strata of the soil. Therefore we cannot say from the character of the tube whether the lightning discharge was from a positive or negative cloud.

3. The difference between thick and thin tubes is due probably to a difference in the sharpness of the flash and the resulting explosive effect. When the explosive effect is great and the quantity of material melted is small, the result will be a large-bored thin-walled tube. Whether this remains circular or becomes pressed together and distorted, depends merely on whether the fused matter has time to cool before the outward pressure of the blowing has been overcome by the inward pressure on the surrounding sand or not. In Nature, the damp sand or soil probably acts as the damp string in these experiments, and consequently causes many lightning discharges to be unidirectional. In the experimental tubes the outward pressure was so great and the quantity of fused material so small, that the walls were broken through and left as a mere network.

Such network tubes much larger and stronger have also been found naturally (9), although it is more frequent to find only one or two isolated holes in the walls (figs. 7 & 9).

Bibliography.

(1) Karl Gustav Fiedler : "Ueber die Blitzröhren und ihrer Entstehung." Gilbert's Annalen, vol. lv. pp. 121–164.

(2) Do. Gilbert's Annalen, vol. lxxiv. p. 213.

(3) Transactions of Geological Society, vol. v. p. 617.

(4) Geological Survey of Great Britain : Memoir on Sheets 81 N.W. and 81 S.W. by A. H. Green, p. 76.

(5) Gilbert's Annalen, vol. lxxiv. p. 213.
(6) Proceedings of U.S. National Museum, vol. ix. p. 83.
(7) Taschenbuch der gesammt Mineralogie, vol. i. p. 313.
(8) Beudant Hachette & Savart: Annales Chimique et de Physique, t. xxxvii. p. 319.
(9) American Journal of Science, vol. xxviii. p. 252.

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Discussion.

Dr Alexander Russell congratulated the Authoress on her interesting paper. Although he had never found a fulgurite he had noticed several phenomena which seemed connected with their formation. Several years ago, during a thunder-shower on the sea, he noticed several flashes striking its surface. Where they struck puffs of steam apparently shot upwards. The appearance was not unlike the splash made by a gannet when diving. It was probably caused by the evaporation of the water in the path of the discharge. Now, a striking peculiarity about fulgurites is that they have a hollow core. The material originally filling it has probably been vaporised. It is therefore highly probable that when a lightning flash strikes sand, a puff of smoke will be shot out from the end of the fulgurite. When testing insulating materials with high voltages, a puff of smoke from the perforation is often the first indication of the breakdown. Electricians frequently notice that when a fuse consisting of a thin wire embedded in sand "blows," a tube of siliceous material is formed. When there is considerable power involved, the tube is raised to a very high temperature and, like the glower of a Nernst lamp, it allows quite a considerable current to pass through it. It would be of interest if the Authoress would investigate the conducting power of fulgurites when heated. The difference between thick-walled and thin-walled fulgurites appears to be only one of degree. If we assume that the energy to be got rid of is the same in the two cases, then, when a lightning discharge has a long path to traverse before reaching a good conducting stratum, the fulgurite would in general be thin-walled, but when the path is short it would be thick-walled. It is obvious that a very appreciable amount of energy must have been expended in making some of the fulgurites, parts of which were exhibited.

Dr C. H. Lees asked if fulgurites, when found, were hollow or filled with sand &c.

The Chairman asked if there were any changes in the magnetic properties of a material when it was formed into a fulgurite.

Mr Skinner expressed his interest in the remarks of Dr Russell on the electrical conductivity of fulgurites, and pointed out that if the sand contained salt the fulgurites would probably be conducting. The suggestion that there might be changes in the magnetic properties was interesting and might be tested by experiment.

XVI. *An Experimental Investigation of the Nature of the γ Rays.* *By* W. H. Bragg, *M.A., F.R.S., Elder Professor of Mathematics and Physics in the University of Adelaide, and* J. P. V. Madsen, *D.Sc., Lecturer on Electrical Engineering* *.

In papers recently published in the Proceedings of the Royal Society of South Australia (May and June 1907) and in the Philosophical Magazine (October 1907) an attempt was made to show that the æther-pulse theory of γ and X rays might prove to be incorrect after all, and that most of the known properties of these rays could be explained more simply and directly on the supposition that they were material and consisted of neutral pairs. The arguments were based on a comparison of known phenomena with deductions from each of the two opposing hypotheses. At that time there did not seem to be any opportunity of appeal to a decisive experiment.

The object of this paper is to give a preliminary account of an investigation which appears to us to give the final answer as regards the γ rays, and to show that they are material in nature.

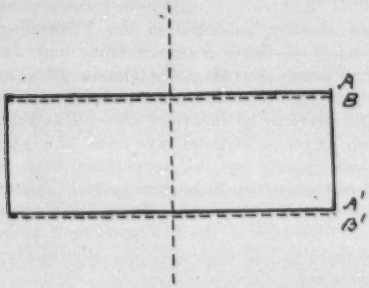
The argument is as follows:—

Secondary radiation which is excited in an atom by a passing wave or pulse must be distributed symmetrically with regard to a plane passing through the atom perpendicular to the direction of motion of the pulse. If we speak of the primary pulse as going forwards, the secondary radiation is just as likely to go backwards as forwards. This is a well-recognized principle. For example, J. J. Thomson divides the secondary radiation due to γ rays into two equal parts which he supposes to move away symmetrically in opposite directions, and, for convenience of calculation, parallel to the direction of the primary rays ('Conduction of Electricity through Gases,' p. 406). Suppose, therefore, a pencil of γ rays to pass normally through a plate so thin that its absorption may be neglected, the secondary radiation

* Read April 10, 1908.

should be exactly the same on the two sides of the plate in amount, in quality, and in distribution; and it ought not to be possible to discover, by any comparison of the secondary radiations on the two sides, which is the face of entry and which of emergence.

Fig. 1.



Consider now the ionization-chamber represented in fig. 1. The two ends are closed by plates, of which A and A′ are alike; so also are B and B′. The material of A and A′ is different to that of B and B′. The nature of the side walls is of no consequence. A pencil of γ rays passes along the axis of the chamber, which is represented by a dotted line. The ionization current within the chamber is measured as usual by inserting a high-potential electrode connected to an electroscope.

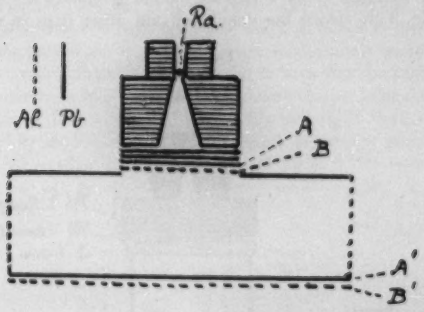
When the plates A and B are inverted there is a change in the amount of the current: so also when A′ and B′ are inverted. By an extension of the principle already stated it ought not to be possible, on the æther-pulse theory, to discover which way the rays are going (up or down in the figure) by comparing the consequence of inverting A and B with that of inverting A′ and B′.

As a matter of fact the direction can be discovered with ease; the more easily the greater the difference between the atomic weights of A and B.

For example, in one experiment of ours the chamber was

of cylindrical form, 7·5 cms. high and 25 cms. diameter. The plates used were aluminium and lead; the thickness of each plate was a little less than 2 mm. Inversion of the top plates A and B made a difference in favour of Al of less

Fig. 2.



than 1 per cent.; *i. e.*, the current was slightly larger when the Al was next the chamber. On the other hand, inversion of the bottom plates made a difference of 44 per cent. in favour of Pb; *i. e.*, the current was 44 per cent. larger when the Pb was on top. The details are shown in the figure. Allowance was made for all radiation other than that which proceeded down the conical opening in the lead block.

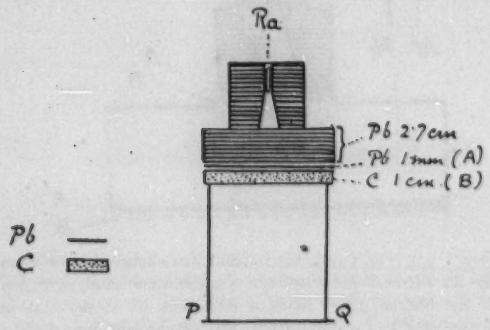
It may be well to point out that this effect cannot be ascribed to any complication due to secondary or tertiary rays. No doubt the radiation in the chamber is very complex; but the fact is immaterial. Provided that the chamber is symmetrical in the first place, then the secondaries must be symmetrical also if the æther-pulse theory is correct, and therefore the tertiaries and so on. Nor is it necessary to consider whether the secondary radiations are β rays or scattered γ rays.

Also it must be remembered that the secondary radiations which enter the chamber have their origin almost entirely in a very few millimetres of material bordering on the chamber. Therefore the γ rays are in almost exactly the same condition,

both as to quality and as to quantity, when they excite secondary radiations from the top plate as they enter the chamber, and secondary radiations from the bottom plate as they leave.

The details of the experiment may be varied greatly; but in all the cases we have tried the want of symmetry is obvious. In fig. 3 are shown the details of one other case, in which

Fig. 3.

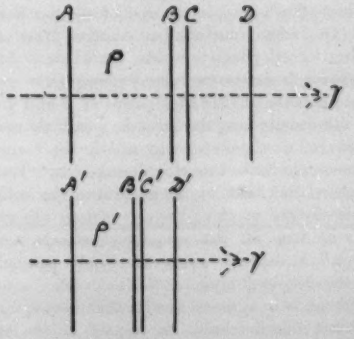


(1) Current with plates arranged as above .... 50·8
(2) „ „ A and B inverted ..... 54·4
(3) As in (1), but base PQ changed to carbon . 50·5

carbon and lead were the materials used, and the form of the chamber was different. It seems unnecessary to give more, because in the first place the experiments are easy to repeat: and in the second place, the complete quantitative analysis of the figures depends on several factors, the influence of which is imperfectly understood, such as the previous screening of the rays, the form of the chamber, and the respective parts played by the original γ rays, cathode rays, and secondary γ rays if any such exist. The experiments as they stand show how far away is that symmetry which the æther-pulse theory demands. It seems to us that there is no escape from the conclusion that the γ rays are not æther pulses.

Let us therefore proceed to consider the hypothesis that the γ rays are material. In the paper already mentioned it was argued that they might well consist of neutral pairs, liable to be broken up on encountering atoms or parts of atoms; and that the secondary cathode radiations might be the negative particles thus set free. Let us suppose, provisionally, that the particles when set free move at first in the direction of the γ stream, but are subsequently scattered in the usual manner of β rays. [It is here that the absence of symmetry arises. On the pulse theory the particles should go equally backwards and forwards; indeed, if they were ejected by atomic explosions, the result of energy accumulated by passing pulses as suggested by J. J. Thomson in the case of X rays, they would move equally in all directions.]

Fig. 4.



Wigger gives a table (*Jahrbuch der Radioaktivität*, Bd. ii. p. 431) showing that the γ rays are absorbed according to a density law pretty strictly, except for the smaller thicknesses in the case of the substances of larger atomic weight. Assume this law to hold good: and also assume for the present that the absorption of β rays follows the density law. The latter is only roughly true, of course; but we

may deal with quantities in a broad fashion first, and make the proper amendments afterwards.

We can now compare the quantities of cathode radiation which should emerge from the far sides of two plates of different densities, ρ and ρ'. Let these be represented by AD and A′D′ in fig. 4 (p. 265); and let BC and B′C′ be corresponding strata of equal weight, in fact let $AB/A'B' = BC/B'C' = CD/C'D' = \rho'/\rho$. Let the plates be crossed by equal pencils of γ rays, as shown in the figure. A certain quantity of γ radiation is absorbed in crossing BC: in the language of our present hypothesis we should say that a certain number of γ particles are stripped of their positives, and the negative remainders go on. An equal number of negatives are set free in B′C′ because the two strata are of equal weight. Of those set free in BC only a certain number emerge from the face D because of the absorption of the plate CD. Since CD and C′D′ are of equal weight a similar absorption occurs in the case of the particles set free in B′C′. Thus the same number emerges from each plate. Integrating for all effective strata, the whole cathode radiations emerging from the two plates are equal.

We thus find that if the absorptions of β and γ rays both followed the density law, the secondary cathode radiation on the far side of a plate—we may call it the "emergence" radiation—would be the same for all materials. There should be no such relation between the amount of the radiation and the atomic weight of the plate as various observers have shown to be true for the secondary cathode radiation of "incidence," a relation which is closely parallel to that found in the case of β rays.

Experiment is in agreement with this theory, for it shows that no such relation exists in respect to the emergence radiations: in marked contrast to what happens in the case of the radiation from the front sides of the plates of various materials—the incidence radiations.

It is true that the emergence radiations are not all equal, but this is to be expected, because (1) the amount of secondary cathode radiation depends, as Kleeman has shown, on the previous screening of the γ rays, (2) the β rays are not absorbed strictly according to a density law, (3) the γ rays

also depart from this law. We have made no serious attempt as yet to disentangle the effects of these various disturbing factors. In fact the task promises to be long and intricate, for it will be necessary to find out how much of the ionization in the chamber is due to each class of rays: to discover the law of distribution of the radiations in space so that the form of the chamber may be allowed for, if necessary: to find out the nature of the departures from the density law of those β and γ rays which are in question, and so on. Nevertheless the results are satisfactory, so far as we have gone. The amount of emergence radiation is found to depend on the previous screening of the rays. In one case the inversion of a C, Pb pair of plates from $\longrightarrow$ C, Pb to $\longrightarrow$ Pb, C altered the current in the ratio 1 : 1·11 when the rays had been previously screened by Pb; but in the ratio 1 : ·96 when the screen was changed to C. Again, when the rays had previously passed through an iron screen, the inversion $\longrightarrow$ Pb Fe to $\longrightarrow$ Fe Pb changed the current in the proportion 1 : 1·12, but when a lead screen was substituted for an iron one the change was 1 : 1·04. In illustration of the effect of the second disturbing factor mentioned above, we have found that, other things being equal, the substances of small atomic weight give the most secondary radiation, in a general way; and it may be no coincidence that in some cases we have found Sn and Fe to give surprisingly small amounts. This is in agreement with what is to be expected; for it is clear, on consideration of the argument already given, that the greater the β-ray absorption of a substance in proportion to its density, the less "emergence" radiation should issue from it. Some observers have found Sn and Fe to possess exceptional absorbing powers. We do not wish, however, to lay any stress upon these last observations, some of which we may not have interpreted correctly; but we mention them in order to show that the inequalities that are found to exist between the emergence radiation of various substances promise to be reducible to order as soon as the difficulties of interpretation have been surmounted.

Let us now consider the cathode radiations on the front sides of the plates. Of the cathode particles set free in BC

and moving at first in the direction of the γ rays, a certain proportion, say p, is returned by what is beyond. These move towards the face A, and a certain number of them succeed in reaching it and emerging therefrom. In the case of the other plate the proportion returned is p'; the absorption in B′A′ is the same as in BA because the weights are the same. Comparing the two plates stratum by stratum, we find that the "incidence" radiation of one plate is to the incidence radiation of the other plate as p to p'. Now p and p' are the well-known constants of the β rays.

When a stream of γ rays is allowed to fall upon a plate, the cathode radiation which issues from the place of incidence must be divisible into two parts. One consists of scattered β particles derived from the stream of such particles which was travelling with the γ rays before incidence, and which was formed during the previous transit of the screens employed—solid, liquid, or gaseous. This part is scattered to an extent which depends on the atomic weight of the plate, according to the usual (McClelland's) law of β particles. The other part is originated in the plate itself in the manner just described, and the amount of it is also regulated according to the β-ray law. When, therefore, observers have measured the secondary radiation due to γ rays, and have found a law corresponding to that for β rays, the reason of the correspondence has been that they really were measuring the secondary radiation due to β rays. Properly speaking, the secondary radiation produced by γ rays, or rather *from* γ rays, is proportional to the density of the substance traversed (*cf.* Wigger's table); and this is only another form of the law of absorption of γ rays.

The relative importance of the two parts of the incidence radiation just mentioned must depend on the circumstances of the experiment*. The researches of Kleeman (Phil. Mag. Nov. 1907) show very well how the second part, which is influenced by previous screening, modifies the effect of the first part, which is not so influenced, but which follows the law of β rays strictly.

* In a recent letter addressed by one of us to 'Nature,' too much stress was laid on the part played by the first part under all circumstances.

It is easy to show, by comparing corresponding strata at the front and back of one plate, that the incidence radiation should be somewhat less than p times the transmitted radiation,—somewhat less, because the cathode radiation which is turned back is scattered and softened in the process.

To sum up :—

On the æther-pulse theory we ought to find perfect symmetry in the secondary radiations from the two sides of a plate; but experiment shows nothing of the kind.

On the material, or neutral pair theory, the "incidence" radiations should follow the β-ray law. This is known to be the case. The "emergence" radiations should not follow the β-ray law; and experiment shows that they do not. If the density law held for both β and γ rays, and if the γ rays were homogeneous, the emergence radiations should all be equal. As already explained, experiment shows that the observed inequalities give promise of ready explanation on the ground that no one of these suppositions is quite true.

It is perhaps better not to extend the preliminary account of these experiments by any lengthy discussion of the issues arising from them. Many points that invite consideration have been discussed already in the papers first referred to. Moreover, our own further experiments are incomplete; and their full interpretation is not yet certain. We will therefore confine ourselves to one or two questions which seem to be of special interest.

The X rays resemble the γ rays so closely that it is practically inconceivable that the two radiations should be essentially different. The secondary cathode radiations which are set free when X rays impinge on any material must therefore have been part of the X-ray stream, and must start their independent existence by moving on in the line of the X-ray motion. Their velocity is much smaller than that of the secondary cathode rays due to γ rays, and they are much more readily scattered. It may still remain an open question whether or no the X-ray stream contains æther pulses. Perhaps their existence must be supposed in order to explain

the velocity experiment of Marx, and the diffraction experiment of Haga and Windt. Possibly they are also required in order to explain Barkla's polarization experiments; but we do not think that the experiment described by Barkla in 'Nature' (Oct. 31, 1907) is in any way decisive.

It seems proper to consider a possibility that the negative particle, when it moves on in the original line of motion of the pair from which it came, retains also its original velocity. It is a striking fact that the cathode particle due to the γ rays has the same speed, very nearly, as the β particle issuing from the original radioactive material. And it looks quite unlike a coincidence that similar comparisons can be made in the case of the X rays. The secondary cathode radiations due to these rays have velocities which, at the least, are of the same order as the velocities of the cathode particles in the X-ray bulb. If we examine the table given by Innes (Proc. Roy. Soc. Aug. 2, 1907, p. 461), and if we may be allowed to adopt an interpretation differing somewhat from the author's, but more natural, it seems to us, in view of the conclusions of this paper, we find that the velocities of the electrons emitted by all the metals are practically the same, zinc being an exception because it is unable to break up the hardest rays. We find that the velocities range from about 6×10^9 to $7{\cdot}5 \times 10^9$ for soft rays, and 6×10^9 to 8×10^9 for hard rays. Remembering that bundles of X rays are very heterogeneous, the natural conclusion seems to be that the softest rays give the slowest speeds, and that the velocity of the secondary rays increases with the hardness of the X rays from which they are derived. Now the hardness of the rays grows with the speed of the cathode particles in the bulb. Is it then possible that the cathode particle is first set in motion by the electromotive force in the bulb, strikes the anticathode and picks up a positive there, becomes neutral and is now called an X ray, is subsequently stripped of the positive and becomes a secondary cathode particle, the identity of the negative remaining the same throughout and its speed invariable or nearly so? The difficulty comes in when we try to consider the part played by the mass of the positive.

And, again, may not the β and γ forms be interchangeable at times? A γ particle which had been stripped of its

positive and become a secondary cathode or β ray would be lost to measurement as a γ ray ; and we should thus have an explanation of how the γ rays are "absorbed," and why the absorption follows an exponential law. And in the same way, if a β particle picked up a positive it would disappear from view as a β particle ; it would be "absorbed."

Although we have made a few experiments with magnetic fields, we have not yet come to any conclusion as to whether or no there are γ pairs which have become loosened in the attachment of positive to negative, forming a softer and more ionizing radiation. Their existence might be suspected, since there is an analogous effect in the case of X rays ; and probably they would be found more at the back of the penetrated plate than in front of it.

A few further experimental illustrations are shown diagrammatically in figs. 5 and 6 with the explanations attached.

Fig. 5.—The upper figure shows the general arrangement. The lower figures are diagrammatic, and show the currents for different arrangements of the Pb and Al at the bottom of the chamber, and at the top with the exception of the plate through which the γ rays enter. Inverting the top plates makes little difference when the upper of the two plates at the bottom is Al ; but an appreciable difference when it is Pb, because in the latter case a good deal of secondary radiation is thrown up by the Pb, and there is a tertiary from the top plate.

The same, when the conical opening is completely filled by a Pb stopper :—

12·6	12·6	17·7	18·3

The differences show the effects of those rays only which are stopped by the Pb stopper :—

27·7	28·7	39·9	44·7

These show the effect of inverting that portion only of the

Fig. 5.

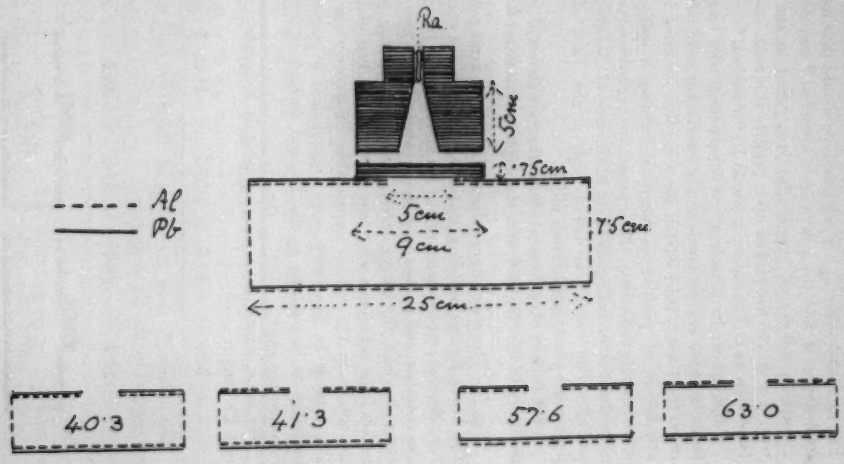


Fig. 6.

top plate where the γ rays enter. Three Pb plates = ·55 cm.; Al plate = ·16 cm.

The same, with Pb stopper inserted :—

14·1 14·7.

Fig. 6.—The upper figure shows the general arrangement. The wall of the cylindrical vessel was of brass : a Pb or an Al lining could be inserted as shown. The lower figures are diagrammatic, and show the currents for different arrangements of Pb and Al at top, bottom, and sides. Inversion of the plates *through* which the γ rays pass into the chamber makes little difference ; but there is a great change if the material is changed on which the γ rays fall, or the "emergence" radiations from the top plate. The base is of less importance than in fig. 5 ; but the sides of more importance. This should clearly be so, for geometrical reasons. When the conical opening was filled by a Pb stopper the currents were all reduced considerably, but retained the same proportions pretty nearly.

When a small pencil of β rays was admitted through a hole in the centre of the top plate a change of the material of the bottom became more effective, and of the sides less effective than before; but this difference became smaller when thin Al sheets were so placed as to scatter the β rays on their entry into the chamber.

In conclusion we should like to add that Wigger was the first, so far as we know, to show clearly that the secondary radiation of Al, on the far side of the plate, was greater than that of Pb. A comparison of the emergence radiations of different metals was made by Dawes (Phys. Rev. xx. p. 182), who showed that they did not follow the law of the incidence radiations. The same effect was indicated in the experiments of Eve (Phil. Mag. Dec. 1904). We have little doubt that the interesting experiments of Mackenzie (Phil. Mag. July 1907) are to be explained on the lines indicated in this paper. In fact it is clear that this is the case in a broad sense ; but it is difficult to give a complete explanation until the laws are so completely worked out that they can be

applied to the interpretation of experiments which are really very complicated, although at first sight they may seem to be simple.

Discussion.

Dr R. S. Willows criticised the paper and expressed the opinion that the experiments described could all be accounted for by an extension of the æther-pulse theory. On the other hand, he instanced phenomena which would be difficult to explain on the neutral pair theory.

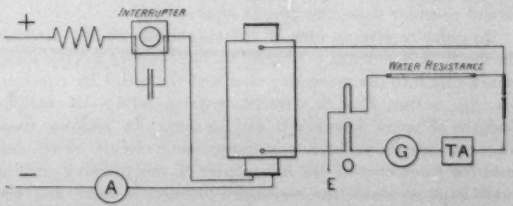
XVII. *Short Spark Phenomena.* *By* W. Duddell, *F.R.S.**

[Plates XIV.–XVI.]

In connexion with some measurements of the current in the secondary circuit of an induction-coil, I have noticed two curious effects, which are probably well known but which I do not remember having seen described anywhere.

The apparatus in use consisted of a 12-inch Newton induction-coil which was supplied from the 200 volt direct-current mains. A large resistance was placed in series with the primary of the coil to limit the current, and the current was interrupted by means of a mercury-jet interrupter; the connexions are shown in fig. 1. The secondary circuit

Fig. 1.



contained a galvanometer G to measure the mean current and thermo-ammeter T. A. to measure the root-mean-squared current.

The galvanometer was specially constructed for the purpose

* Read April 10, 1908.

so as to have a sufficiently low sensibility without using a shunt. It was of the moving-coil type and was well insulated from earth by means of porcelain insulators. The sensibility was such that 1 milliampere gave a scale-deflexion of 200 divisions (1 division equals 1/40 in.). The thermo-ammeter had a resistance of about 101·5 ohms and gave its full scale-deflexion for about 70 milliamperes R.M.S. value.

By breaking the current through the primary by means of a switch, the direction of the deflexion of the galvanometer corresponding to breaking the primary current was determined. A deflexion, in this direction I will call, in what follows, a positive deflexion, and a deflexion in the opposite direction, that is corresponding to the make of the primary current, I will call a negative deflexion.

When there was no spark-gap in the secondary circuit and the coil was in action, the mean current, as read by the galvanometer, was zero, as it should be, and the root-mean-squared current had a value of about 3·8 milliamperes.

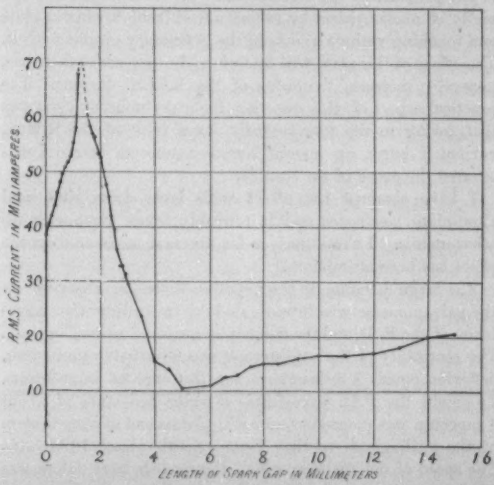
If now a microscopic spark-gap, say between two aluminium points, is introduced into the secondary circuit two curious effects take place. Firstly, the R.M.S. current enormously increases in value; and secondly, a very large deflexion is produced on the galvanometer, reading the mean current, and this deflexion is in the *negative* direction, that is to say, in the direction corresponding to making the primary circuit. I will consider these two effects separately.

In order to give an idea of the magnitude of the increase in the R.M.S. current produced by introducing a very small spark-gap into the secondary circuit of the coil, I have plotted in fig. 2 the R.M.S. current corresponding to various lengths of spark between 0 and 15 mm. In making these observations the resistance in the primary circuit of the coil and the frequency of the interrupter of the primary current were kept constant, the resistance being 137 ohms and the frequency 75 interruptions per second. The current through the primary of the coil was about 1/4 ampere.

A large number of points were taken and are plotted on the curve. They do not give very consistent results, so it is difficult to draw a smooth curve through them; but the general appearance of the phenomenon can be seen. On

introducing a spark-gap 1/10 mm. long the root-mean-squared current instantly rose from 3·8 to 38·5 milliamperes, and continued to increase with increasing spark-length until it reached a maximum at a spark-length somewhere in the neighbourhood of 1·4 mm. The exact point is slightly uncertain owing to the R.M.S. current just exceeding the range of the instrument that was in use. From 1·6 to 5 mm. spark-length the R.M.S. current gradually fell in value and

Fig. 2.



attained a minimum value of 10½ milliamperes. Further increase in spark-length produced a gradual increase in the R.M.S. current up to the maximum length of 15 mm. that was used in the experiments.

There is no doubt in my mind as to the cause of this effect. It is due to very high frequency oscillations being set up in the wires connected to the secondary circuit of the coil when a spark-gap is introduced. The magnitude of the oscillations

will depend on the voltage between the terminals of the spark-gap just before the spark passes and on the resistance that the spark-gap offers. Now the P.D. between the terminals of the spark-gap will increase with increasing length and so will the resistance, so that on increasing the spark-length we have two conflicting agencies at work, one tending to increase the magnitude of the oscillatory current, and the second tending to decrease the magnitude. I think that it is due to this differential action that the curve is such a curious shape.

The presence of the oscillations in the secondary circuit can easily be made evident by taking a well insulated metal plate and touching various points in the secondary circuit with it. The effect of this plate will be to largely increase or decrease, generally increase, the value of the R.M.S. current. The practical aspect of this question from my point of view was that, owing to the unexpectedly large value of the R.M.S. current, I burnt up several thermo-ammeters before I discovered the cause of the trouble.

I have obtained the effect with brass, iron, zinc, and aluminium electrodes and it probably takes place with all other metals. I think that, so far, the best metal to show the effect has been aluminium.

The large deflexion in the negative direction observed on the galvanometer was investigated by recording the wave-forms of the P. D. and the current by means of an oscillograph. The sensibility of the oscillograph was adjusted so that 1 mm. deflexion equals 1 milliampere for the current wave-forms. To obtain the P.D. wave-forms a water resistance of about 1 megohm was placed in series with the second moving system of the oscillograph so that 1 mm. equals about 1000 volts. The speed of the plate on which the records were taken was 1500 mm. per second. For this series of tests, the frequency of interruption was 75 per second and the resistance in series with primary of the induction-coil was 37 ohms.

Records were made for a series of spark-lengths between aluminium-point electrodes. I have selected from these some typical results which are shown in Plates XIV.–XVI. figs. 3, 4, 5, 6, 7, 8.

Fig. 3 is the current wave-form when the spark-gap is short-circuited, length 0. The straight line across the centre

of the figure is the true zero line. Deflexions above this zero line represent current in the positive direction and below the zero line represent current in the negative direction. It will be noticed that the maximum current in the positive direction (14·5 milliamperes) is much less than the maximum current in the negative direction (35 milliamperes), but the length of time that the negative current lasts is much shorter than that which the positive current lasts; so that the areas of the two sides of the zero line are equal and the mean current zero.

The smallest gap that I could make in this circuit between the aluminium electrodes, at once changed the wave-form to the type shown in fig. 4, which is for spark-length 1 mm. In this figure, the straight parts of the curve along the centre of the figure are in the position of zero current. This was carefully checked by taking records with a fixed datum-line at the zero of the curve. The line was afterwards moved to the lower part of the figure in order not to hide small details of the curve near the zero. The effect of introducing the spark-gap of 1 mm. is, while leaving the maximum current on the two sides of the zero at practically the same value, to reduce the area of the curve on the positive side of the zero line nearly to zero, so that instead of the areas on the two sides being equal there is a large excess of area on the negative side causing a large mean current in the negative direction.

On increasing the length of the gap a small area on the opposite side of the zero line again begins to form, which increases with increasing length of gap, until the condition shown in fig. 6 is reached. In this figure, the spark is sometimes rectifying or stopping the current flowing round the circuit in the negative direction altogether. We have in this figure the large triangular current wave form which corresponds to the current flowing in only one direction round the circuit, and the smaller triangular current pulses accompanied by a large current in the negative direction at make.

Further lengthening the spark-gap brings it into the normal condition of long sparks; the current wave of this is shown in fig. 7. The datum-line across the centre of the plate, in this case, is true zero line.

It was necessary in order to obtain the wave-form of these

longer lengths to disconnect the circuit for recording the P.D. wave-form, as the leak which it formed, having a resistance of only 1 megohm, prevented the sparks passing across the gap.

Fig. 8 shows the P.D. wave-form with so long a spark-gap that the spark could not jump across it. This is the normal wave-form given on the secondary of the induction-coil when supplied from a high voltage direct-current circuit.

It will be noted that the voltage induced in the circuit at the break is about 23,000 volts and at make about 13,500, so that the make voltage exceeds one half of the break voltage. Also, the voltage induced at the make dies away less rapidly than that induced at the break. It may be enquired how it is possible, if the make voltage is less than the break voltage, for the current to be larger at make than at break. I think that this question must be answered by noting that during the make period the primary of the induction-coil is connected to the supply mains so that the energy may be directly transferred from the primary to the secondary circuit by magnetic induction; that is to say, as long as we maintain a steadily increasing flow of current into the primary, we can continue to take energy from the secondary. During the break period, however, things are very different. The whole of the energy that we can get out of the secondary is that stored up in the magnetic field which is linked with the secondary winding. The greater part of this magnetic field will pass through the core. The magnetization of the core will depend upon the resultant magnetizing ampere-turns which is equal to the primary ampere-turns less the secondary ampere-turns. At the moment of break the current in the secondary is at a value of say 35 milliamperes, and the current in the primary cannot have exceeded $\frac{200 \text{ v.}}{37 \text{ ohms}}$ or 5·4 amperes.

I do not know the exact ratio between the numbers of turns on the primary and on the secondary of this induction-coil, but it is probably of the order of 100, so that the 35 milliamperes flowing in the negative direction in the secondary would correspond to a demagnetizing current of about $3\frac{1}{2}$ amperes in the primary, which would leave a comparatively small margin of resultant magnetizing ampere-turns.

If this is the case, the energy that can be got out of the secondary on break is very limited, which would account for the rapid dying away of the current to zero when even a very small spark-gap is introduced in the circuit. Directly the length of the spark-gap is sufficient to prevent the current flowing in the negative direction round the secondary circuit, the whole of the demagnetizing effect of the secondary current is done away with, and under the conditions of the experiment the magnetizing current in the primary is mainly limited by the large resistance in the circuit; hence, we get a very much larger amount of energy available directly the secondary current is prevented from flowing in the negative direction. This I think accounts for the large difference in the size of the current waves in fig. 6, and shows the great importance of preventing any current from flowing round the secondary circuit in the negative direction at make when using the induction-coil on a high voltage supply.

I made certain that the phenomenon was not due to any want of symmetry in the points of the spark-gap by reversing the electrodes of the gap and also by interchanging the connexions to it. The material of the electrodes did not seem to appreciably affect the results, but the shape of the electrodes was important in so far that the spark-length at which the galvanometer deflexion changed sign depended on the shape. Thus with two spheres the galvanometer deflexion changed sign at a shorter length than with points.

I have brought these two observations forward in the hope that in the discussion either my views as to the explanations may be confirmed or that better ones may be suggested.

Discussion.

Dr Alexander Russell thanked Mr Duddell for his valuable paper and for the exceedingly interesting and very successful demonstration of short-spark phenomena. The Author's complete analysis of the phenomena he has discovered would be a great help in elucidating the difficult theory of the action of the induction-coil and would throw light on many still unexplained effects. He believed that the change of sign of the direct current component in the secondary circuit of an induction-coil was first discovered by Mr Duddell, and the Physical Society was to be congratulated on having been the first Society to which this phenomenon was clearly described and demonstrated. He considered

that the rise in the value of the effective current when the air-gap was widened was a resonance phenomenon. The capacity *between the electrodes* was in series with the inductance of the secondary, and hence, as the Author pointed out, Kelvin oscillatory discharges were continually taking place. The shape of the curve showing the connection between the effective current and the distance between the electrodes was strong evidence in support of this view. The explanation of the direct current component when the electrodes are at considerable distances apart, may be made by remembering that the wave pulses of E.M.F. round the secondary at "make" and "break" are of very different shapes although their integral values are the same. The maximum value of the induced E.M.F. at break is much greater and the consequent rush of electricity is more "impulsive" than at make. Hence, when the electrodes are sufficiently far apart the air-gap is only broken down by the break E.M.F., and so we get a unidirectional pulsating current of triangular shape. When the electrodes are closer together we get an alternating current, the shape of the positive and negative waves being quite different owing to the differently shaped waves of E.M.F. at make and break. When they are very close together, the Author's oscillograms show that we get a pulsating unidirectional current in the direction due to the "make" E.M.F. The reason for this strange phenomenon is by no means obvious.

Mr S. Skinner expressed his interest in the paper, and referred to the experiments on the electrolysis of steam carried out by J. J. Thomson by a modification of Perrot's method. He found that with short sparks the hydrogen appeared at the positive electrode instead of at the negative as in ordinary electrolysis. Perhaps this phenomenon was due to the same causes as the effects shown by the Author.

Dr R. S. Willows asked if the maximum current obtained was connected in any way with the minimum sparking voltage.

Prof. Cassie pointed out that the minimum sparking voltage occurred at distances very much less than those used in Mr Duddell's experiments.

Mr C. C. Patterson asked if the Author had tried experiments with spark-gaps of different capacities.

Mr A. Campbell asked if it was possible that the rectifying action of the spark-gap might be similar to that of an ordinary aluminium electrolytic rectifier.

Mr Duddell said he did not think the effect was in any way connected with the minimum sparking voltage. He had tried varying the materials of the electrodes and using electrodes of various capacities and the effect was the same.

XVIII. *The Production of Small Variable Frequency Alternating Currents suitable for Telephonic and other Measurements.* *By* B. S. COHEN, *A.M.I.E.E.**

[Plates XVII.–XX.]

INTRODUCTION.—The devices described in the latter part of this paper are the outcome of experiments carried out in the Investigation Branch of the Engineering Department of the National Telephone Company, in order to obtain suitable alternating currents both of simple and complex wave form to act as substitutes for the voice in telephonic measurements. Although primarily designed for this purpose, it is considered possible that the apparatus to be described is capable of more extended use.

A short summary of the methods known to the author of obtaining small alternating currents of the kind suitable for telephonic measurements may prove of interest as an introduction.

Statement of the problem.—The ordinary telephonic current is a few milliamperes at a potential of about 2 to 10 volts and is of complex wave form. The frequency of the fundamental harmonic generally lies between 100 and 300 complete periods per second, *i. e.* it varies from 100~ to 300~, the highest harmonic having a frequency of 4000~ to 5000~, although all harmonics above 1500~ are comparatively unimportant. The average frequency of the whole wave is about 800~. From consideration of these data it follows that the ideal device for supplying such alternating current is one which will give currents of any frequency lying between 100~ to 500~, singly or in combination. The output of this apparatus should be about 1 watt, and it is necessary that it should be capable of working uniformly over a fairly considerable period of time.

Simplicity and portability are also highly desirable.

Alternators.—A sine wave alternator coupled to a motor, the combination being capable of running at a constant speed for a considerable period, is a very useful form of apparatus

* Read May 22, 1908.

for investigations in connexion with telephony. A suitable machine is, however, difficult to obtain.

The Western Electric Company build a machine with an output of about 30 watts at frequencies varying from 800~ to 1800~, and the wave form is stated to resemble a sine curve closely at all loads.

Messrs. Siemens and Halske also make a machine with an output of 3 or 4 watts at about the same frequencies. Both these machines are of the inductor type, the purity of the wave form being secured by the shape of the teeth and pole faces.

Mr. Duddell's work with high frequency alternators is well known. He has also made several medium frequency machines, one of which, with a wound rotor, gives an output of about 20 watts, but only runs up to about 500~.

In the Investigation Department of the National Telephone Company we have built a small alternator of the inductor type on the lines of the Siemens machine. This is illustrated in Pl. XVII. fig. 1.

With a 30 volt battery connected to the field windings, the output is only about ·3 watt, but the wave form is fairly good, *i. e.* it approximates to a sine wave. By decreasing the air gap, the output can be considerably increased at the expense of the purity of the wave form.

With the best air gap (about ·015 mm.) the P.D., which is independent of the frequency, is 10·5 volts.

Fig. 2 (Pl. XVII.) shows the wave form of this machine when the frequency is 800~. Measurements of the capacity of small mica condensers made with the current from this alternator on the assumption that it is a pure sine wave, are accurate to within about 3 per cent., and this forms a very good test.

For more accurate measurements this machine has been used with a wave filter, consisting of series inductances of low effective resistance and parallel capacities, similar to that described by G. A. Campbell (Phil. Mag. March 1903). The wave form when at a frequency of 800~ this filter is used is shown in fig. 3 (Pl. XVII.).

Humming Telephone.—The interaction between a receiver and a transmitter setting up an alternating current whose frequency depends on the free period of the apparatus

can be utilized. The method, however, is untrustworthy, and the output small.

The wave form is also far from sinusoidal, as might be expected when it is considered that it is influenced by the action of the polarized electromagnet in the receiver. Fig. 4 (Pl. XVII.) shows the wave form using the local battery type of telephone instrument.

Mr. A. Campbell's modification of the humming telephone consists of a steel bar with a free period of the frequency desired, and this is set in motion by a polarized electromagnet, which has attached to it a light microphone, which maintains the interaction and supplies current to the external circuit through a transformer. This apparatus is more reliable than the humming telephone. The output, however, is limited by the transmitter, and the wave form is similar to that of the ordinary humming telephone *.

Organ Pipes etc.—Organ pipes, other wind instruments, and tuning forks sounding in transmitters, were experimented on for a short time, and might under certain conditions give satisfactory results. It is difficult to avoid trouble, however, with all such arrangements, owing to irregularities introduced by the transmitters and to their limited output.

Vibrating Wire Interrupters.—The vibrating wire interrupter described by Wien, Orlich, Campbell, and others, has when suitably modified given very satisfactory results, and I propose to describe a special form of this apparatus †. It has been found very difficult to construct a vibrating wire which will run reliably at a frequency much above 300~, and I have been unable to find references to wires which could be maintained in vibration at a frequency higher than 500~. The output is also very limited, and the frequency difficult to determine and vary.

* For further particulars of these devices see Note by R. Appleyard, Elect. Rev. pp. 57 & 656, vol. xxvi. 1890; Paper on Humming Telephones, F. Gill, Journal I. E. E. vol. xxxi. 1901; A. Campbell, Proc. Royal Soc. p. 208, June 12th, 1906; J. E. Taylor, Journal I. E. E. p. 396, vol. xxxi. 1901; F. Dolezalek, *Zeitschrift für Instrumentenkunde*, p. 240, Aug. 1903.

† (1) Max Wien, Wied. *Ann.* xlii. p. 593 (1891).
do. do. xliv. p. 681 & p. 689 (1891).
(2) Orlich, *Electrotechn. Zeitsch.* vol. xxvi. (1903).

In order to overcome these drawbacks the instrument illustrated in fig. 5 (Pl. XVIII.) has been designed. The upper figure shows details of mercury cups and electromagnet, and the lower figure gives a general view of the whole instrument. The wire is steel, 1·06 mm. in diameter, and supports a soft iron armature which is maintained in vibration by an electromagnet with a laminated core of stalloy, which is an alloy of iron much used in electrical work, a mercury cup and a platinum wire contact. The wire is provided with a tension adjuster, and will run steadily at frequencies varying from 100~ to 250~.

The frequency of the alternating output is quite independent of the wire frequency, and is obtained by means of a separate circuit, mercury cup, and contact. The mercury cups are made from glass tube 4 mm. in internal diameter, and these are enclosed in brass tubes to which the mercury is connected by means of a platinum wire fused into the glass.

The cups thus made slip into brass adjustable carriers from which they can readily be removed for cleaning purposes, and the carriers admit of a fine vertical adjustment.

Experiments have been made with cups of varying diameter. Those with the smallest diameters gave the best results.

It would appear that the mercury in the small cups remains much steadier under the action of the vibrating contact than in those of larger diameter, owing possibly to the greater surface-tension effect.

Single Action Circuit.—Several forms of oscillating circuit have been used of which the two principal ones may be described as "single action" and "double action" respectively.

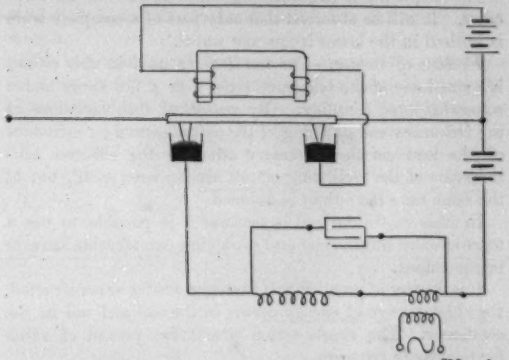
Fig. 6 shows the single action circuit and also the wire driving circuit. The latter requires no explanation. The former consists of a platinum contact and mercury cup which in the make contact position connects a battery to a capacity in parallel with an inductance and transformer primary.

On the break between the cup and contact an oscillation occurs in the circuit.

By this method a series of damped trains of oscillations of

any frequency can be produced, the trains following each other with the frequency of the wire vibrations.

Fig. 6.—Single Action Circuit.



Theory of Action.—Oscillations will occur when $\frac{R^2}{4L} < \frac{1}{K}$, their frequency being given by $\frac{1}{2\pi}\sqrt{\frac{1}{LK} - \frac{R^2}{4L^2}}$. With the values of L, K and R used, the last term can be neglected and formula becomes $\frac{1}{2\pi}\sqrt{\frac{1}{LK}}$.

The total output in watts, excluding all losses, will be the energy stored, divided by the interval of time between each succeeding train of waves.

The equation to the line joining the peaks of the damped waves is $Y = e^{-\frac{aR}{2L}}$, so that for small damping L must be large and R small, which also gives the maximum output.

In one of the circuits used the inductance is ·037 henry, and its effective resistance which is practically invariable over the range of frequency used is 10 ohms. (These figures include the resistance and inductance of the primary of transformer.)

Fig. 7 (Pl. XVIII.) shows an oscillogram of damped wave trains produced in the manner just described.

The three frequencies are 490~, 959~, and 2880~ produced by using the three capacities, 2·88 mfd, ·8 mfd, and ·1 mfd respectively.

The frequency of the vibrating wire was 90~ in all three cases. It will be observed that only part of a complete train is utilized in the lower frequency waves.

Problem of Output.—The transformer used in this circuit is a small one of the telephone type, with a few turns and a somewhat loose coupling. By reason of this, variations in the frequency and damping of the output caused by variations of the load on the secondary affecting the effective self-induction of the oscillating circuit are rendered small, but at the same time the output is reduced.

In cases where the load is constant it is possible to use a more efficient transformer and so obtain a considerable increase in the output.

It is worthy of mention that in the apparatus as constructed, the chief storage of energy occurs in the coil and not in the condenser. The single action circuit has proved of value for telephonic purposes.

Double Action Circuit.—The double action circuit previously referred to is illustrated in fig. 8.

This circuit, unlike the single action one, gives a pronounced oscillation on the make contact, as well as on the break.

On referring back to fig. 6 it will be seen that no oscillation can occur on the make contact as the battery short-circuits the oscillating circuit during that period.

By introducing resistance into the battery circuit, however, an oscillation on the make contact can be obtained in addition to the break oscillation. This oscillation is of the frequency of the main oscillating circuit.

Although the results might have been predicted theoretically, the following series of oscillograms (Pl. XIX. fig. 9) are inserted as they show in a rather interesting manner the effect of insertion of resistance.

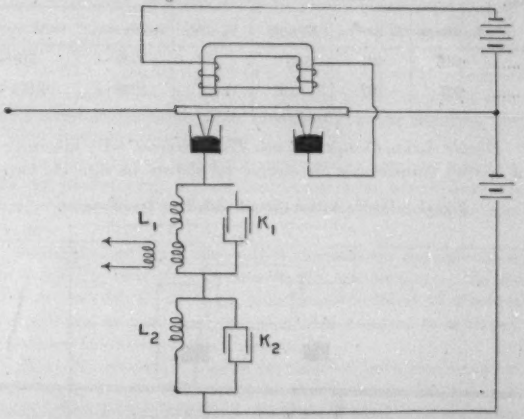
This series shows the variations obtained in the make contact oscillations in a circuit as shown in fig. 6, when the resistance in series with the battery (which is of negligible internal resistance) is varied from 4000 to 0 ohms.

The make oscillation is most evident when the resistance is reduced to 1000 ohms.

At 300 ohms the oscillation begins to appear during the current rise on the make contact, whilst at 100 ohms the oscillation is entirely confined to the rise during the make contact.

The resistance is still large enough to cause the current to rise very quickly to its maximum value, thus giving a nearly square shouldered wave.

Fig. 8.—Double Action Circuit.



As the resistance is still further reduced the current takes longer to rise to its maximum value and at the same time the oscillation is reduced, until ultimately at 0 ohms the familiar current rise curve with no superimposed oscillation is produced.

In the double action circuit as illustrated (fig. 8) a capacity shunted by an inductance is introduced in place of the simple resistance. This somewhat modifies the action of the circuit. If the inductance L_2 is inserted as shown in the figure and the capacity K_2 omitted, the effect is to give

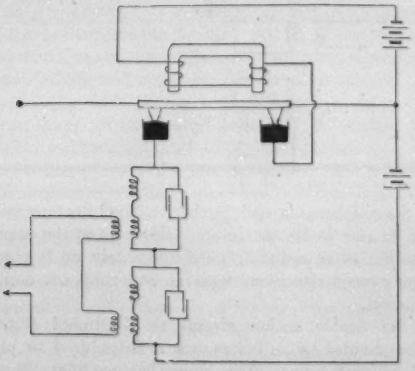
an oscillation on the make contact, the frequency of which is determined by the combined inductance of L_1 and L_2 and the capacity K_1. When the capacity K_2 is added a similar effect is produced. The resulting make frequency being due to the combined effect of both inductances and capacities. The break oscillation will be that due to L_1 and K_1 only.

Fig. 10 (Pl. XIX.) shows two waves produced by this circuit. The following small table gives the necessary data:—

Wave.	Values of				Frequency of oscillation :	
	L_1 henrys.	L_2 henrys.	K_1 mfds.	K_2 mfds.	make contact.	break contact.
a	·037	·037	1·67	·167	1020~	715~
b	·037	·037	·167	1·67	1020~	2160~

Double Action Circuit with two Transformers.—By inserting a second transformer in circuit as shown in fig. 11, two

Fig. 11.—Double Action Circuit with Two Transformers.



separate and distinct oscillations can be obtained. If the circuits are similar two separate waves will be produced the shape of which will be exactly similar.

By joining up the two secondaries so that these two waves either assist or oppose each other, some interesting results are obtained.

Fig. 12 (Pl. XIX.) shows two 900~ break oscillations assisting one another. The make oscillations oppose each other and are therefore wiped out, the result being to give a wave exactly similar to that obtained with the single action circuit, but with about double the amplitude.

When the secondaries are joined up so that the break oscillations oppose each other, whilst the make oscillations assist, the result, if these oscillations are equal in frequency but unequal in amplitude, is to give a continuous and more or less uniform wave.

The first wave in fig. 13 (Pl. XIX.) shows the effect of opposing two 850~ oscillations, and the second the effect with two 490~ oscillations. The continuous waves resulting are 850~ and 490~ respectively.

The regularity of the continuous wave can be best modified by slightly varying one of the inductances, which have air cores by inserting in the cores one or two strands of fine iron wire.

It is obvious since the output depends on the difference between the two circuits that it will not be large. To increase the output, different inductances for each of the two circuits can be used, and the capacities adjusted so as to get the same frequency in each circuit.

It is of course impossible to obtain a pure sine wave by this method, and the extent of the impurity has not yet been examined analytically. Some tests have, however, been made by using these waves for measurement purposes. These are given further on.

Complex Wave Form.—It is worth noticing that a number of double action circuits can be operated by means of a single mercury-cup and contact. From fig. 11, it will be seen that any number of oscillating circuits can be joined in series. The oscillations which ensue on break are quite independent, each having the frequency of its particular circuit.

The make oscillations will depend on the total capacity and inductance of all the circuit.

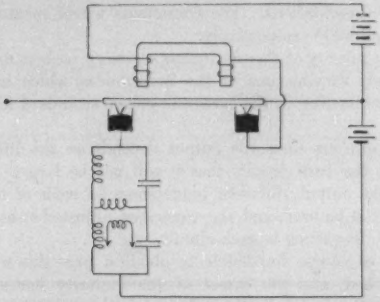
The secondaries of the transformers can be joined up in

series or in parallel to suit the external circuit conditions, and can also be joined to oppose or assist each other. By this means complex waves of definite formation can be built up.

Fig. 14 (*a*) Pl. XX. shows only the make oscillations when three oscillating circuits were used, and (*b*) gives the resulting complex wave, including both make and break oscillations. The three break frequencies are 415~, 590~, and 1442~ respectively.

Modified Double Action Circuit.—Fig. 15 shows a modified form of double action circuit which has been found to give a greater output when obtaining continuous oscillations by opposition than the circuit previously described.

Fig. 15.—Modified Double Action Circuit.



The two transformers are replaced by a single one put in such a position that both oscillations pass through it.

Fig. 16 (Pl. XX.) shows continuous waves of 1020~ and 3700~ produced by this method.

The variation in the three 1020~ waves shown are produced by the insertion of iron in the core of one of the inductances.

(*a*) No iron.
(*b*) One strand, } No. 22 S.W.G. Soft Iron Wire.
(*c*) Two strands, }

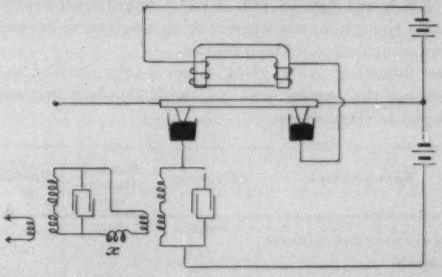
The effect of superimposing an Oscillation on an Oscillatory Circuit of similar period.—If the oscillations produced by

one circuit are superimposed on a second similar circuit, as shown in fig. 17, a continuous oscillation results.

Fig. 18 (Pl. XVII.) shows the resulting wave forms when each of the two circuits is tuned to 830~, (*a*) being the applied oscillation, and (*b*) the resultant. The *modus operandi* appears to be as follows :—

The damped oscillation applied to the second circuit causes resonance in the latter, and the oscillation set up will be in the reverse direction of the original oscillation inasmuch as it will tend to rise to a maximum whilst the latter is falling.

Fig. 17.—Continuous Oscillation Circuit.



By suitable adjustment of a series resistance, or preferably an inductance placed at x, therefore, it is possible to obtain in the second transformer a wave which is the resultant of two equal and opposite trains of waves. This resultant wave is practically continuous.

Mercury Contact effect.—A phenomenon, in connexion with a vibrating-point contact in a mercury cup, has been noticed to which I have not been able to find any reference. I refer to a series of sparkless points which occur at regular intervals as the distance between the contact and the mercury is varied. With the vibrator described in this paper the contact point has an amplitude of motion of about 4 mm.

If the cup is adjusted whilst the contact is vibrating until the point just makes contact on the mercury surface, a spark is observed.

If then the distance between the mercury and the contact

is slowly decreased, a series of sparkless points will be reached which are very sharply defined, as when such a point is reached, a variation either way of about ·07 mm. gives the sparking condition again.

The writer has not so far made any investigation into the reason for this action.

It is, however, most probable that this action is partly mechanical as at the sparkless points the vibrating surface of the mercury seems to be set in a rigid condition.

Use of continuous Waves for Absolute Measurements.—The wave forms illustrated in fig. 13 (Pl. XIX.), fig. 16 (Pl. XX.), and fig. 18 (Pl. XVII.) might conceivably be accurate enough for use when making absolute, as opposed to merely comparative, measurements.

The following table gives some results arrived at by measuring the current and p.d. with standard condensers subjected to these waves.

Wave form used.	Frequency.	Standard Capacity tested.	Percentage error.
Wave similar to that illustrated in fig. 13	1442	·167	−1·8
	854	·33	−1·7
	523	1·0	−1·0
Wave similar to that illustrated in fig. 16 (*a*)	833	·167	+3
	1442	·167	+4·2
Wave similar to that illustrated in fig. 18 (*b*)	833	·33	+2·4
	833	·167	+1·8
	1440	·167	+ ·5

These results were determined by inserting a non-inductive resistance of known value in series with the condenser, taking the potential-difference across both by means of a sensitive electrostatic voltmeter, and then calculating the capacity by the formula $A \cdot 10^{16}/pV$, where A and V are the effective values of the charging current and potential-difference respectively. This is very satisfactory as very little time was spent in adjusting the waves.

So far as these tests go it would appear that the amount of

the departure from the sine shape in these continuous waves is comparatively inconsiderable.

Telephonic Measurements.—The special form of vibrating wire described was primarily designed for telephonic measurements; and although the tests carried out in this direction are far too extensive to describe in any detail in this paper, a brief outline may perhaps be given.

Telephonic measurements are now mainly carried out by expressing the attenuation of speech-waves in any line circuit or apparatus in terms of the attenuation over a given length of a certain type of cable line settled on as a standard.

It has been found that the words representing the numbers 1, 2, 3, 4, 5, embody all the frequencies of telephonic importance, and these numbers are invariably used for testing purposes.

Inspection of the oscillograms of these five words as spoken by a number of persons shows that they may be approximately represented by fig. 19 (Pl. XVII.), which is given by the equation

$$y = \cdot 29 \sin pt + \cdot 32 \sin 2pt + \cdot 39 \sin 3pt + \cdot 55 \sin 4pt + 1 \cdot 06 \sin 5pt + 6 \cdot 5 \sin\left(6pt - \frac{\pi}{2}\right) - 1 \cdot 06 \sin 7pt - \cdot 56 \sin 8pt - \cdot 39 \sin 9pt - \cdot 32 \sin 10pt - \cdot 29 \sin 11pt,$$

where $p = 2\pi \times 145$.

The general resemblance of this wave to some of those produced artificially, see figs. 7 and 12, is obvious on inspection.

It has been found that by replacing the human voice and transmitter by the vibrating wire producing these waves, and by using the correct frequencies, similar results to those produced by the voice can be obtained.

Having obtained a satisfactory substitute for the human voice, it is next necessary to replace the ear and receiver by some measuring instrument.

Simple measurements of the comparative current attenuation over the standard line and the line or apparatus under test do not give accurate results, and it was suggested by my assistant Mr. A. J. Aldridge, to whom I take this opportunity

for expressing my indebtedness for considerable help given in carrying out the investigations embodied in this paper, that direct measurements of the comparative volumes of sound issuing from the telephone receiver could be made by allowing the receiver to sound into a transmitter connected up in the usual manner to a battery and induction-coil, and to measure the current in the secondary of the induction-coil.

This method has been found to solve most of the difficulties, and with the vibrating wire at the sending end of the line and at the other end of the line a receiver sounding into a transmitter, the latter being connected to a barretter used as an alternating current milliammeter, it has been found possible to carry out comparative telephonic measurements with greater accuracy than with the voice and ear, owing to the elimination of the personal equation and the greater sensibility to volume variation of the apparatus used.

This method can be used for such diversified tests as comparison of loaded with unloaded lines, comparison of standard and non-standard apparatus, such as receivers, transmitters, induction-coils.

Conclusion.—Apart from telephonic measurements, the modified vibrating-wire interrupter may possibly have a field for both comparative and absolute electrical measurements, and especially for use with the vibration galvanometer. The ease with which the frequency can be varied together with the steadiness of the resulting wave and the simplicity of the apparatus are certainly great recommendations. The author must express his indebtedness to the National Telephone Co. for the facilities afforded to him, to Dr. A. Russell for great help in the revision of this paper, to Mr. A. Campbell for references to humming and vibrating wires, and a tribute must also be paid to the oscillograph which has been of such great assistance in these investigations.

Discussion.

Mr A. Campbell mentioned that with his vibrating-bar microphone hummer he had found it easy to obtain more than one frequency from the same bar by altering the supports to suitable nodal points and tuning the magnet circuit with a condenser. Harmonics of the fundamental

frequency are thus produced. He asked if any of the composite wave forms shown gave vowel sounds in the telephone.

Dr Russell congratulated the Author on his interesting and valuable paper, and thanked him for the instructive demonstration he had given of methods of producing high-frequency currents suitable for telephonic and other measurements. He understood that the Author's main object was not the obtaining of currents of excessively high frequency, but obtaining currents having frequencies varying between 100 and 5000 which followed the harmonic law. Some of the methods described, in particular the Author's "double action circuit" method, seemed well adapted for this end. In some of these circuits there are two free periods of vibration, and it was exceedingly interesting to see the accuracy with which the oscillograph gave the resultant of two slightly damped trains of high-frequency waves of different periods. Tuning for resonance the Author obtains a very perfect sine wave. He thought that the thanks of the Meeting were also due to the National Telephone Company for the facilities they had afforded Mr. Cohen.

Mr. Cohen expressed his interest in Mr. Campbell's statement that it was easy to get more than one frequency from his vibrating bar hummer. The currents shown, when passed through a telephone receiver, did not produce vowel sounds.

XIX. *A Vacuum-Tube Model for Demonstrating the Propagation of Alternate Currents in Cables.* By Dr. C. V. Drysdale *.

[Plates XXI. & XXII.]

As is well known, the propagation of electric waves along conductors can be conveniently shown on the small scale by means of a helix, fed from a suitable source of high frequency, high potential oscillations. In this case the helix is found to glow with brush discharges at the antinodes of potential, remaining dark at the nodes. Experiments of this kind have been shown by Prof. J. A. Fleming and by G. Seibt †. The glow is, however, very faint, and may be rendered much more visible by the use of a vacuum-tube, as Prof. Fleming himself suggests.

When experimenting lately in this direction, the present writer noticed that if a long vacuum-tube was laid parallel to

* Read March 27, 1908.

† See 'The Principles of Electric Wave Telegraphy,' by Dr. J. A. Fleming, pp. 251-261.

the helix, light and dark portions were observed, but that these did not coincide with the points of maximum and minimum potential respectively; and it was of course evident that such a tube is influenced not by the actual potential, but by its slope or $\frac{\partial V}{\partial x}$ along the tube. As the intrinsic equation of propagation is $\frac{\partial V}{\partial x} = IC$, it is clear that the brightness of the tube is proportional to the current in the helix, and that therefore the whole distribution of current and potential can be exhibited by vacuum-tubes fixed parallel to and perpendicular to the helix respectively. A few photographs of the effects observed were published in a recent paper*, but these were obtained by successive exposure of the same tube in different positions.

In order to show these effects with the greatest convenience the model shown in fig. 1 (Pl. XXI.) has been made. The helix is 182 cms. long and 2·65 cms. in external diameter. It is wound on a glass tube with 25·5 turns per cm. of 13 mil. D.S.C. wire; and has a resistance of ·43 ω, an inductance of $\frac{\pi^2 d^2 n^2}{1000} = 44$ microhenrys per cm. The capacity from experiments with a metal tube of similar diameter appears to be $9{\cdot}2 \times 10^{-6}$ mfd. per cm.

This helix is supported in a wooden frame, and has above it a carbon-dioxide vacuum-tube 1 inch diameter. Immediately below the helix and about 2 mm. from it are the spherical bulbs of 18 vertical carbon-dioxide tubes, which are mounted in brass sockets on a metal strip fixed to a shelf at the bottom.

The helix is excited from a 10 in.-spark induction-coil with an oscillation-circuit consisting of six leyden-jars of about ·002 microfarad each, which can be connected in series or parallel; and an adjustable helix having a maximum inductance of ·000365 henry.

In order to exhibit the effect of different receivers at the end of the line a water resistance, consisting of a tube about 5 mms. diameter and 30 cms. long, was made up, as well as

* "The Theory of Alternate Current Transmission in Cables," Electrician, vol. lx, No. 10, p. 358.

an adjustable parallel-plate condenser about ·7 sq. ft. in area, and an inductance-coil of 110 yards of No. 16 G.P. covered wire in a hank about 10 in. diameter. By means of mercury cups and paraffin blocks, various arrangements of leyden-jars and loads could be easily made.

The photographs, figs. 2 to 9 (Pl. XXII.), are of the apparatus, and of the effects observed when currents of different frequency were employed, and different terminal conditions. The sending end was in each case on the left. In figs. 3, 4, and 5 the frequencies were about 600,000, 1,140,000, and 1,800,000~ per sec. respectively, giving one, two, and three half-waves. In each case the current and potential waves are out of phase; and at the higher frequency, where resistance is of less effect, they are almost exactly in quadrature, as required by theory.

The remaining figures are all with the low frequency and should be compared with fig. 3. Fig. 6 shows the effect of earthing the receiving end through the water resistance; and that, as theory indicates, the current and P.D. become much more uniform in magnitude. In fig. 7 the receiving end is earthed, resulting in a shifting of the waves to the right, so that a second current-loop appears.

Fig. 8 illustrates the effect of increasing the capacity of the helix by introducing an earthed wire inside it, and shows that with the same frequency the wave-length is decreased, so that there are two half-waves instead of one.

Finally, fig. 9 shows the effect of earthing the receiving end through a condenser. The current-loops are shifted forward as when the receiving end was earthed, and at the same time they are much more definite. With inductance the same result is observed, but the shift is backward, as the theory indicates.

No attempt has as yet been made to check the results by theory with great accuracy, nor does it seem probable that this will be of much value, as the presence of such a large number of vacuum-tubes alters the capacity and introduces leakance into the helix. But the general effects observed are of value in illustrating the theory of wave-propagation.

The author's thanks are tendered to Mr. C. M. Dowse who has taken considerable trouble in arranging the apparatus and

experiments, and to Mr. A. C. Cossor for the construction of the vacuum-tubes, which were only obtained sufficiently uniform after a considerable amount of trouble.

Discussion.

Mr W. Duddell expressed his interest in the experiments, and asked why the points of zero potential were not more clearly marked. Was it due to absorption of energy by the tubes or to the superposition of oscillations with higher frequencies? Had the Author used a single long tube with a strip of tinfoil on the outside? He also asked what gas the Author used in the vertical tubes, and whether he had tried helium.

Dr Drysdale in reply said that the tubes were filled with CO_2. It was somewhat difficult to ensure their equality, and no attempt had therefore been made to use rare gases. Doubtless still better results could be obtained, but the CO_2 tubes were sufficiently bright for most purposes. He believed that the energy taken by so many tubes was the cause of the absence of complete darkness at the nodes, as with a single tube sharper results had been obtained. For this reason the single tube with a strip of tinfoil on the side had not been used.

XX. *An Elementary Treatment of the Motion of a Charged Particle in a Combined Electric and Magnetic Field.* *By* W. B. Morton, *M.A., Professor of Natural Philosophy, Queen's College, Belfast* *.

One of the methods employed by Prof. J. J. Thomson in his fundamental work on the ratio of charge to mass of a corpuscle † depends on the theorem that a charged particle, liberated without appreciable velocity at a metal surface, and moving under the combined influence of a constant electric force, urging it at right angles to the surface, and a uniform magnetic field parallel to the surface, will describe a cycloid. The result comes at once by integration of the equations of motion of the particle, but in view of the importance of the matter it may be worth while to point out that it can be established in a geometrical way without requiring anything more advanced than a knowledge of the velocity and acceleration of a point on a

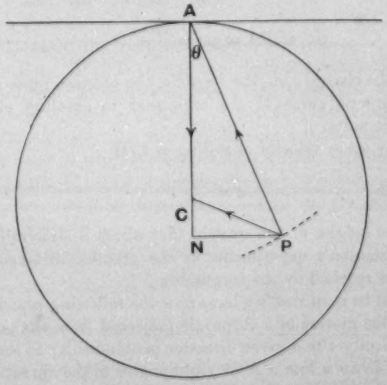
* Read June 12, 1908.

† J. J. Thomson, Phil. Mag. [5] xlviii. p. 547 (1899).

wheel which is rolling steadily along a straight line. It is only necessary to show that the acceleration of such a point can be resolved into two components, one of constant magnitude and direction, perpendicular to the line along which the circle rolls, and the other perpendicular to the instantaneous direction of motion of the point and proportional to its velocity. These correspond to the accelerations given to the corpuscle by the electric and magnetic fields respectively. If the point is on the circumference of the circle its velocity will vanish when it is the point of contact with the fixed line, so its initial circumstances coincide with those of the corpuscle liberated at the plate. Therefore their paths are identical.

By taking the tracing point anywhere in the plane of the circle, we get the more general result that a corpuscle projected in any manner in a plane perpendicular to the magnetic lines, will describe a trochoid.

Fig. 1.



Let P (fig. 1) be any point in the plane of the circle which is rolling along the horizontal line with constant angular velocity ω.

Then A is the point of zero velocity,

C the point of zero acceleration.

Therefore the velocity of P = its velocity relative to A = ω.AP, in direction perpendicular to AP, and the acceleration of P = its acceleration relative to C = ω^2.PC along PC.

Replace this by

ω^2.PA along PA and

ω^2.AC parallel to AC.

This is evidently the resolution required, for the latter component is constant in magnitude and direction, and the former is $\omega \times$ (velocity of P) and is perpendicular to this velocity.

On identifying the accelerations with the electric values we obtain at once the size of the circle and its rate of motion.

Using a for the radius and v for the velocity of P at the instant

$$\omega^2.a = \frac{Ee}{m}$$

$$\omega^2.PA = \omega v = \frac{Hev}{m},$$

where the charge e on the particle, the electric force E, and the magnetic force H are expressed in absolute electromagnetic units.

Hence ωa = velocity of rolling = E/H

and

$$a = \frac{Em}{H^2e}.$$

It is the height of the cycloid ($2a$) which is determined in Prof. Thomson's experiments as the greatest distance from the plate reached by the corpuscles.

It will be seen that we have now the following representation of the motion of the corpuscle projected from the point P with velocity v in a given direction perpendicular to the lines of H. Draw a line PA at right angles to the direction of projection (towards the side to which H exerts its force), measure off a length $v/\omega = vm/He$ to the point A. Describe a circle of radius Em/H^2e with A as its highest point, and let this circle roll, along the horizontal tangent at A, with

velocity E/H, carrying the point P along with it. Thus the size and motion of the circle are independent of the manner of projection.

In further confirmation of the agreement between the motion of the charged particle and this geometrical representation, it may be noticed that the mechanical force arising from H, being always perpendicular to the motion, does no work on the particle. Thus the kinetic energy of the latter is derived wholly from the work of the electric force, and we have

$$\tfrac{1}{2}mv^2 = Eey + \text{constant};$$

where y is the vertical distance below the horizontal plane.

In the rolling circle we have

$$\begin{aligned} v^2 = \omega^2 . PA^2 &= \omega^2 \{AC^2 + CP^2 + 2AC(AN - AC)\} \\ &= 2\omega^2 ay \quad + \text{constant} \\ &= \frac{2Ee}{m} y \quad + \text{constant} \end{aligned}$$

in agreement with the last.

When P is on the circumference of the circle we get the cycloidal path, the corpuscle coming to rest on the fixed straight line. If P is outside the circle the trochoidal path is looped, and the corpuscle crosses the fixed line when its direction of motion is vertical. When P is inside the circle the path is a wavy one, at mean distance a below the line.

It is easy to express the condition which determines the form of path, in terms of the magnitude and direction of the velocity v. Let θ be the angle between v and the horizontal $= PAC$ in figure 1.

Then we have a looped path if

$$PA > 2a \cos\theta,$$

i. e.

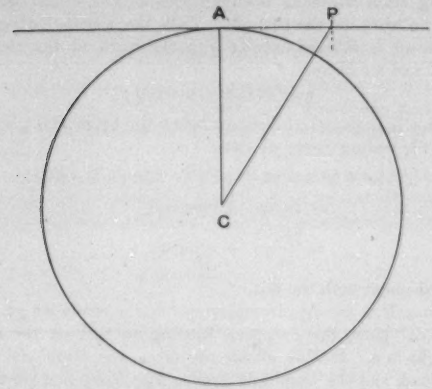
$$v > 2\frac{E}{H}\cos\theta.$$

So the path is looped or waved according as the velocity of projection is greater or less than the component of the velocity of the lowest point of the circle, resolved along the direction of projection.

It seems of some interest to find the greatest distance from

the metal plate which would be reached by a corpuscle projected normally with velocity u, say, instead of being liberated without velocity. In this case P is outside the circle, and the plate takes the place of the line along which the circle rolls.

Fig. 2.



Taking P on this line we have (fig. 2)

$$u = \omega \,.\, \mathrm{AP},$$

$$\therefore \qquad \mathrm{CP} = \sqrt{a^2 + \frac{u^2}{\omega^2}}.$$

The maximum distance reached is

$$a + \mathrm{CP} = a\left\{1 + \sqrt{1 + \frac{u^2}{a^2\omega^2}}\right\} \text{ instead of } 2a$$

$$= \frac{m}{e\mathrm{H}^2}\{\mathrm{E} + \sqrt{\mathrm{E}^2 + u^2\mathrm{H}^2}\}$$

Thus the correction is of the second order in the ratio of the velocity of expulsion to the velocity E/H. The latter velocity in the experiments described by Prof. Thomson (*loc. cit.*) was of order 10^8 cm. sec.$^{-1}$.

There is an alternative mechanical way of describing the motion in the cycloidal case. It can be shown without difficulty that it is identical with the motion of a cycloidal pendulum, or of a particle sliding on a smooth cycloidal wire, the range of swing extending from cusp to cusp. Here the acceleration of gravity takes the place of the electric force, and the normal reaction of the wire or string that of the magnetic force. But this analogy cannot be extended to the trochoidal motion; the velocity of the moving corpuscle does not alter in the same way as that of a particle sliding under gravity on a smooth trochoid.

23rd May 1908.

XXI. *A Directive System of Wireless Telegraphy.* *By* E. Bellini *and* A. Tosi*.

Part I.—Bilateral Directive Wireless Telegraphy using Closed Oscillatory Circuits: (i.) for Transmission; (ii.) for Reception; (iii.) for both Transmission and Reception.

Part II.—New Unilateral Directive Method: Simultaneous Use of Closed Oscillatory Aerial Circuits and the Open Oscillator or Vertical Antenna.

The authors, having lately obtained some interesting and very remarkable results in the course of their work upon a further development of their directive system, propose to give an account of the arrangements employed and the results attained.

Since the complete understanding of these experiments depends upon the properties shown by the closed oscillatory circuits used by the authors, it is considered desirable, in the first part of the paper, to recapitulate, as briefly as possible, the main features of the original method, thus enabling a more comprehensive view to be taken of the later arrangements, as well as supplying the necessary information to those who are not familiar with the earlier work, which has been fully dealt with elsewhere†.

* Read June 12, 1908.

† Electrical Engineering, ii. pp. 771–775 (1907), and iii. pp. 348–351 (1908). The blocks referring to this part of the paper have been kindly lent by the Kilowatt Publishing Company.

Part I.

The aerial of the bilateral system is composed of vertical closed oscillatory circuits.

The closed oscillatory circuit is, as is known, an oscillatory circuit, the metallic portion of which forms a geometrically "nearly closed" figure, the electric field of which is concentrated in a very narrow region, and the conditions of vibration of which are independent of the position of the spark.

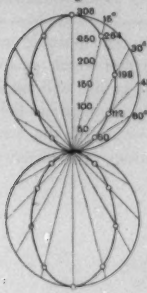
Although in such circuits the electric field is situate almost exclusively between the armatures of the condenser, none the less there always exists an electric field of dispersion the lines of force of which go from different points of the circuit to neighbouring conductors and to earth. Consequently, if one of these circuits be vertically placed in proximity to the ground, and with the condenser at the upper part, a number of lines of force will reach the earth, giving rise to an electromagnetic field which propagates itself in space. Since one of the armatures of the condenser has at the same instant an equal potential of opposite sign to that of the other, the electromagnetic field of one side will be opposed in phase relative to the field of the other side. As a result it is evident that in the direction perpendicular to the plane of the circuit there is no radiation whatever; that is to say, the said type of circuit radiates principally in its own plane.

An oscillatory circuit placed in an electromagnetic field will become the seat of an E.M.F. due to the variation of magnetic flux across the surface bounded by the circuit. This E.M.F. will vary with the cosine of the angle which the plane of the circuit makes with the direction of propagation of the field.

Using the Duddell thermogalvanometer, the authors have determined the diagrams of the energy emitted or received by such a circuit. The energy diagrams obtained were equal and composed of two equal tangent curves (fig. A) whose equation is $E = E_0 \cos^2 \alpha$, where α is the angle which the direction of propagation makes with the plane of the

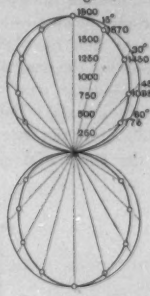
circuit. The diagram of the electromagnetic field is composed of two equal tangent circles (fig. B).

Fig. A.



Energy Transmission Diagram (Microwatts).

Fig. B.



Transmission Diagram of Electromagnetic Field Intensity (Microamperes).

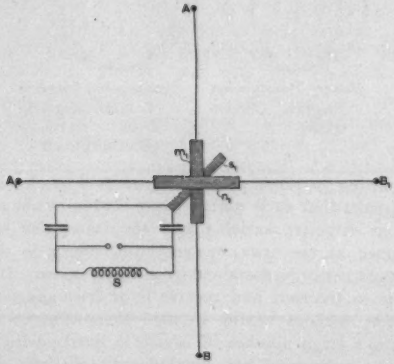
It is owing to these properties possessed by closed oscillatory circuits that such circuits were chosen by the authors for use as directive aerials; and the triangular form of circuit, open at the apex, was employed, owing to the fact that it could easily be sustained by a single mast. In order to be able to transmit and receive to or from any direction whatsoever, without having to turn the aerial, or having recourse to a large number of aerials in fixed positions, the aerials were built up of two equal closed oscillatory circuits, vertically placed and mutually perpendicular. This aerial system is connected up to special instruments for transmission and reception which the authors have called "Radiogoniometers."

The radiogoniometer for transmission comprises two fixed windings, m_1, n_1, mutually perpendicular, one of which is inserted in each of the two aerial circuits AB, A_1B_1, as shown diagrammatically in horizontal projection in fig. C. A third winding s, enclosed by the two first, is arranged so

as to be movable on its vertical axis, which axis is that of the intersection of the planes of the two fixed windings. This winding s is included in the oscillatory exciting circuit, by means of contact brushes. S is the secondary of the spark coil or transformer.

It is evident that on placing the movable winding in different positions the magnetic flux which passes through the two fixed windings will vary, as also will the oscillatory currents in the aerials. The electromagnetic field in space will be produced by the superposition of the partial fields created by the two aerial circuits. Its intensity will be

Fig. C.

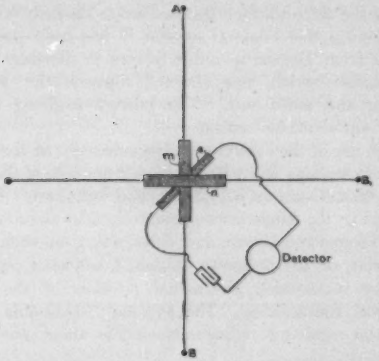


independent of its direction, and this latter will coincide with the direction of the central plane of the movable winding. The law of angular variation of the energy and of the magnetic field is the same as for a single circuit.

The apparatus is provided with a fixed dial over which moves a pointer rigidly attached to the movable winding. On setting the pointer in the direction of the station to which the signals are to be sent, the emission will be a maximum in the vertical plane which contains the transmitting and receiving stations—that is to say, both towards the receiving

station, and at 180° from this. The receiving radiogoniometer is similar in construction to that for the transmission. The fixed windings m, n (fig. D) are connected to their respective aerial circuits. The movable winding s_1 is suitably connected up to the detector, through two brushes.

Fig. D.



The oscillatory currents set up in the two aerials, when these latter are subjected to the influence of an electromagnetic field, create, in the interior of the fixed windings, a resultant magnetic field of an intensity which is independent of the direction of the transmitting station. When, therefore, the movable winding is perpendicular to this resultant field, that is, its central plane is oriented towards the transmitting station, the effect on the detector will be a maximum. In this position the reception is practically limited to the plane containing the transmitting station, and the pointer indicates on the dial the direction of this plane. One is, however, not able to determine *which* side of the receiving station the transmitting station is located.

The receiving radiogoniometer is provided with special contacts for tuning the fixed windings to the lengths of the waves to be received, which is effected by varying the number

of active turns equally and simultaneously on the two windings.

The experiments with the system have been carried out at three stations on the French side of the English Channel, at Dieppe, Havre, and Barfleur respectively. The distance Dieppe-Barfleur is 107 miles, all over sea; that between Dieppe and Havre 54 miles, entirely over land; the angle made by the three stations Havre-Dieppe-Barfleur is 23°.

Employing this bilateral system it has been possible to transmit from Dieppe to either Havre or Barfleur without the emission which was directed towards the one post troubling the other one. The energy sufficing for this purpose was about 500 watts.

By the use of the receiving radiogoniometer at the Dieppe and Havre stations the authors have been able to determine the direction of stations both known and unknown. Further, by receiving the same transmission (signals) simultaneously at both Dieppe and Havre, and determining at each station the bearing of the unknown station, it has been possible to locate, by intersection, the actual position of the station which was transmitting. The accuracy obtainable by the use of the receiving radiogoniometer is about one degree of arc.

The triangular closed oscillatory circuits comprising the aerial system were 45 metres high, with a base of 55 metres.

Part II.

The system of wireless telegraphy reviewed in the first Part allows of transmission and reception in the plane in which the station with which one is communicating is situated. That is to say, in transmitting, one is sending the signals to the corresponding station and at the same time in the opposite direction also. As regards the reception, either of two diametrically opposite transmitting stations will affect the receiver in the same way, and it will not be possible to decide as to which side of the receiving station the transmitting station is actually located.

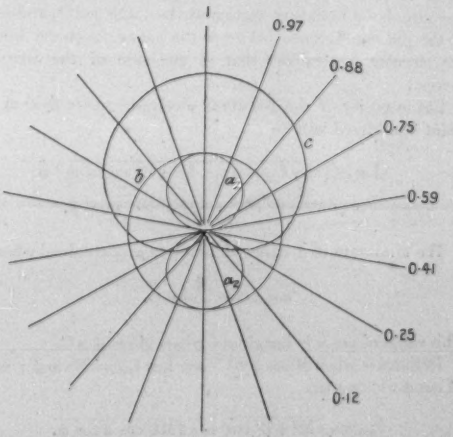
Since the sending of the waves in a single useful direction

is often of great importance, and it is desirable to be able to decide not merely the direction but the azimuth, the authors have worked out a method of achieving this object, and have solved the problem in the following manner.

Transmission.—The diagram of the electromagnetic field of the bilateral directive system already described is composed of two equal tangent circles, and the one half-field, represented by one of the circles, is opposite in phase to the other half-field.

If one operates simultaneously, at the same station, both the directive system and an ordinary system giving a circular emission, whose vertical antenna is symmetrically placed with reference to the directive aerial, the resultant electromagnetic field of the simultaneous emissions of the two systems, can, in general, be obtained by vectorially adding the partial diagrams of the two systems.

Fig. 1.



As an example, let a_1 and a_2 (fig. 1) be the two tangent

circles forming the diagram of the electromagnetic field of the directive system; the circumference b represents the diagram of the system with circular emission, of amplitude equal to the maximum amplitude of the directive system; let the phase-difference between b and a_1 be zero, and that between b and a_2 180°. The resultant diagram c is obtained in this case by adding algebraically the *radii vectores* of the curve a_1 to those of the diagram b and by subtracting the *radii vectores* of the curve b from those of curve a_2. The resultant diagram is represented by a "cardioid."

It is evident from the figure, that the resultant maximum electromagnetic field is directed to one side only; that it has a value double that of the component electromagnetic fields; and that the field on the other side is zero.

The intensity of the electromagnetic field set up by the directive system at a point whose direction makes an angle α with the maximum radiation, is expressed by $C \cos \alpha$. Let M be the intensity of the electromagnetic field produced by the circularly radiating system at the same point, and let ϕ be the phase-difference between the electromagnetic field of the circular system and that of one side of the directive system.

The intensity of the resultant electromagnetic field at the point considered will be

$$I = \sqrt{(M + C \cos \alpha \cos \phi)^2 + C^2 \cos^2 \alpha \sin^2 \phi}$$
$$= \sqrt{M^2 + C^2 \cos^2 \alpha + 2MC \cos \alpha \cos \phi}.$$

The minimum of I with reference to α is obtained when

$$\cos \alpha = -\frac{M}{C} \cos \phi.$$

This value of $\cos \alpha$ is imaginary when $M \cos \phi > C$.

In the case when $M \cos \phi \leqq C$, one has $I_{min} = M \sin \phi$; when $M \cos \phi > C$ one has

$$I_{min} = \sqrt{M^2 + C^2 \cos^2 \alpha - 2MC \cos \alpha \cos \phi}.$$

In the special case when $\phi = 0$ the equation of the resultant

electromagnetic field transforms itself into

$$I = M + C \cos \alpha,$$

which is the equation of a curve that can have three different forms according to the value of the ratio M/C. The curve represented by the condition M = C is the cardioid above mentioned.

But since in wireless telegraphy the action depends chiefly upon the energy, it will be useful to consider this in preference to the intensity of the electromagnetic field.

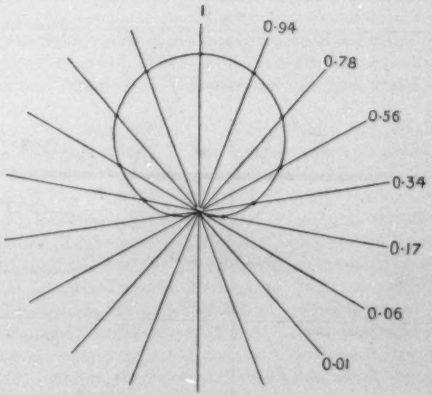
In the general case the energy radiated in the different directions is expressed by the equation—

$$W = M^2 + C^2 \cos^2 \alpha + 2MC \cos \alpha \cos \phi,$$

and in the case of the cardioid by the equation—

$$W = M^2(1 + \cos \alpha)^2,$$

Fig. 2.



The corresponding curve is given in fig. 2.

Fig. 3 represents the energy diagram in the case where $M/C=0{\cdot}72$ and $\phi=53°$.

Fig. 3.

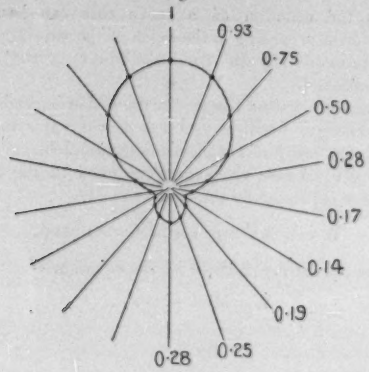


Fig. 4.

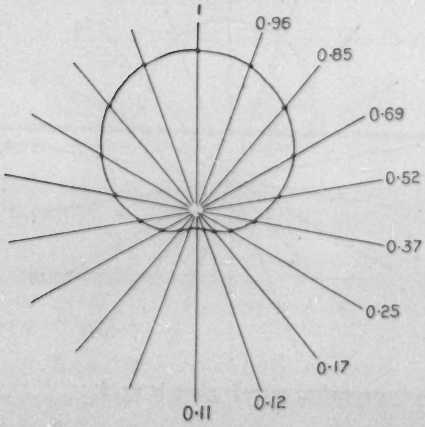
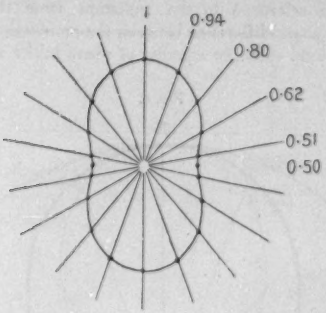


Fig. 4 shows the same diagram for the case where $M=2C$

and $\phi = 0$, and finally fig. 5 the same diagram for $M = C$ and $\phi = 90°$.

Fig. 5.



In consequence, one can conclude that the shape of the diagram of the energy radiated in the different directions depends upon the value of the phase-difference of the component electromagnetic fields, and that the superposition of a circular system on a directive system allows of the concentration of the emitted energy in the desired direction and in the useful sense only.

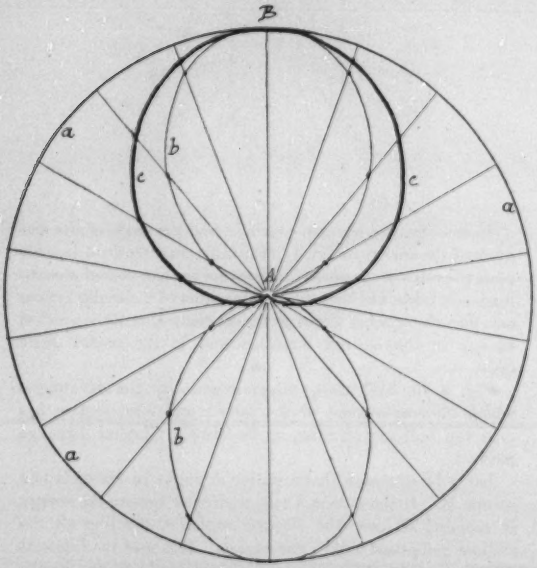
Fig. 6 (p. 316) shows diagrammatically the advantages which the employment of the mixed unilateral system has over the ordinary circular, or the simple bilateral directive method.

Let it be supposed that a station A ought to transmit to a station B. If the station A uses a circular system the energy is radiated all over the circular region a, affecting all the stations comprised within this region. If A uses the bilateral directive system the active area is the region bounded by the two tangent curves b, b. If the station uses the mixed unilateral system adjusted for $M = C$ and $\phi = 0$, the active area is considerably smaller than that with the other methods.

The practical realization of the superposition of the two systems has been effected by employing as the directive

system the bilateral system already dealt with in Part I. It was evident *a priori* that, owing to the different conditions under which the radiation from the directive circuit takes place with reference to the radiation from the vertical antenna, a phase-difference between the emissions of the two systems should exist, for equality of phase in the excitations.

Fig. 6.



As will be seen further on, the experiments have shown that this phase-difference is 90° or near to that value. The aerial of the unilateral system is formed by the aerial of the directive circuit to which has been added a vertical antenna in a position symmetrical with reference to the first. To secure the simultaneous excitation of the closed oscillatory circuits and of the

vertical antenna, various arrangements have been employed which are diagrammatically depicted in figs. 7, 8, 9, 10.

Fig. 7.

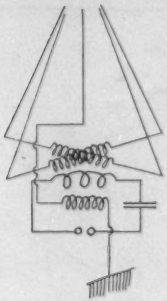


Fig. 8.

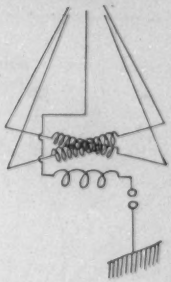


Fig. 9.

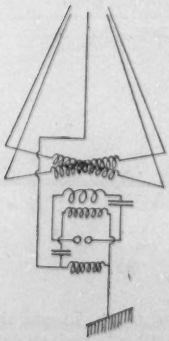
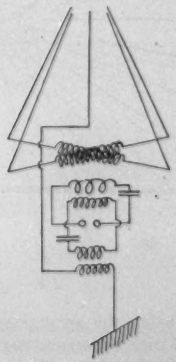


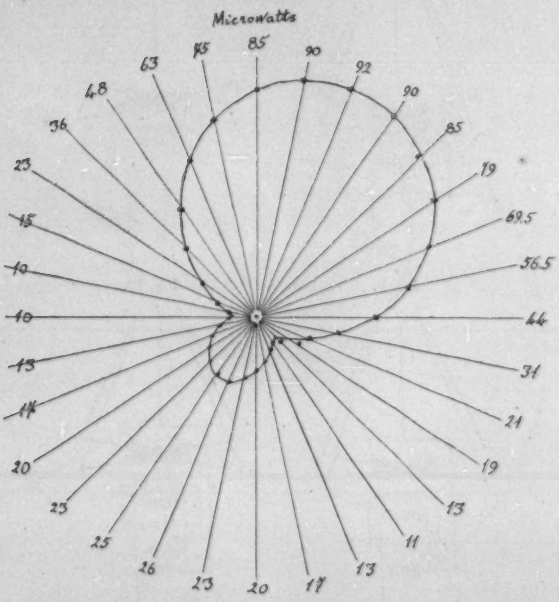
Fig. 10.



In the first of these arrangements (fig. 7) the excitations of the two systems are in phase; in the other three cases they are in quadrature. The diagrams of the energy radiated in the different directions (figs. 11 and 12) are selected from

among the large number which have been determined. They were obtained with the thermo-galvanometer in a similar manner to that employed in the case of the bilateral directive system. In particular the diagram of fig. 11 was obtained

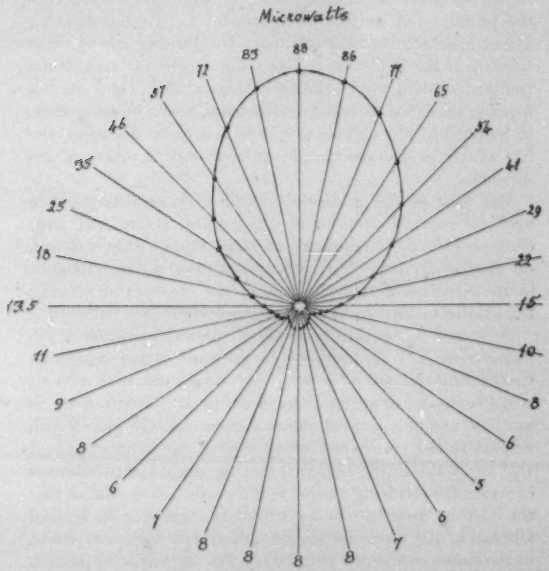
Fig. 11.



when using the arrangement shown in fig. 7; and that of fig. 12 with the arrangement of fig. 10. In the last diagram (fig. 12) the phase-difference between the radiation from the two component systems turns out to be much smaller than in the preceding diagram, and only by means of this last arrangement (fig. 10) have the very small values of the backwardly radiated energy been obtained. This proves the

existence of a phase-difference of 90°, or thereabouts, between the emissions from the two systems. The diagram of fig. 12 shows besides, that the problem of the emission of the energy exclusively on the useful side can be considered as having been practically solved.

Fig. 12.



It is interesting to note that, even if the phase-difference between the radiation from the two systems is as much as a certain number of degrees, and the ratio of the partial fields is slightly different from unity, the backward radiation is always practically zero. The phase-difference has to be over 45° for the backward radiation to become noticeable. In order to see if the energy emitted by each component of the

system and by the resultant system satisfied the theoretical predictions, these have been measured in several cases with the thermo-galvanometer. The results were always in practical agreement with the theory.

The simultaneous excitation of the two systems was effected by adding, internal to the primary of the transmitting radiogoniometer, a winding, fixed in position relatively to the primary and inductively excited by it. To this winding is connected the vertical antenna. On turning the movable winding of the apparatus the excitation of the vertical antenna remains constant whilst the excitations of the closed circuits forming the aerial of the bilateral system, vary. Consequently, to transmit by the unilateral system in a given direction, one has merely to turn the movable pointer until it points in this direction.

The tests of the unilateral system at long distance were made between the stations of Dieppe and Havre, and have confirmed the results obtained at short range—that is to say, on placing the pointer of the transmitting radiogoniometer in the direction of Havre, reception was effected ; on turning the pointer to 180° from this the reception ceased entirely.

Reception.—The same principle of the superposition of the two systems has been applied to the case of the reception. Considering the case of a receiving station provided with an aerial composed of a single closed oscillatory circuit, fixed in position, and of a vertical antenna symmetrically placed with respect to the closed oscillatory circuit, let it be supposed that any transmitting station is moving along a circumference of which the receiving station is the centre ; it is evident that the intensity and phase of the oscillatory current in the vertical antenna of the receiving station will remain constant, whilst in the closed oscillatory circuit they will vary with the position of the transmitting station. The intensity passes through two maxima and two minima, and its phase varies by 180° when the transmitting station passes from one side to the other of the plane perpendicular to the closed oscillatory circuit. Further, although a closed oscillatory circuit utilises the variation of magnetic flux across the surface bounded by such circuit, the E.M.F. generated in it will be in

quadrature with the E.M.F. generated in the vertical antenna, and the same phase-displacement will exist between the currents in the two types of aerials if these latter are both tuned to the length of the waves received.

Let it be supposed, for instance, that the closed oscillatory circuit is oriented in a certain meridian and that the transmitting station in moving about arrives in the plane of the closed oscillatory circuit, on the north side. In such a position, by bringing the actions of the two systems on the detector into phase by any means, the effects of the two will be added. When the transmitting station finds itself, similarly, in the plane of the closed circuit but on the south side, the effects of the two systems counteract one another. If the directive aerial of the receiving station is composed of two closed oscillatory circuits connected to the receiving radiogoniometer, the phase of the E.M.F. induced in the movable winding of the apparatus will vary by 180° when the coil is turned through 180°. So that the action of the directive aerial in one case will add itself to that of the vertical antenna, and in the other case will subtract itself therefrom. It follows that, to the position of maximum reception of the movable coil, there corresponds a position of zero or minimum reception when the coil is turned through 180°, which thus permits of the determination not merely of the direction of the transmitting station, but also of the side which it is on.

The action of the detector depends, as in the case of the transmission, upon the values of the phase-difference between the actions of the two systems and on the ratio M/C. The law of variation of this action is the same as that of the variation of the resultant electromagnetic field at the transmitter and the diagrams are the same.

To obtain quadrature between the actions of the vertical antenna system and that of the directive bilateral system several arrangements have been tried which are shown in figs. 13, 14, 15.

In the case of fig. 13 the phase displacement was obtained by inserting an excess of self-inductance in one or other of the aerial circuits; in the arrangements of figs. 14 and 15

the shift of phase was produced automatically when the circuits were all carefully tuned, but the arrangement of fig. 14

Fig. 13.

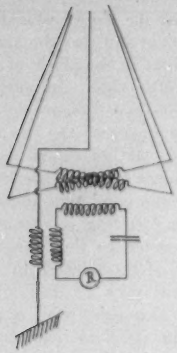


Fig. 14.

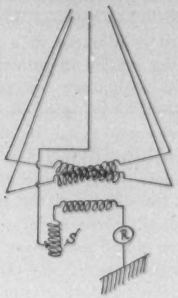
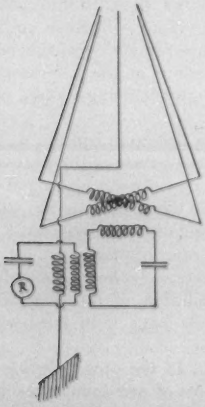


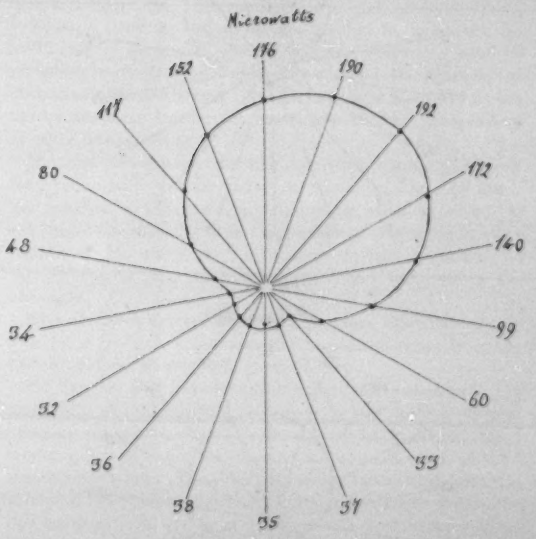
Fig. 15.



is preferable owing to its simplicity, rapidity, and certainty of operation. In this case, as can be seen from the figure,

the vertical antenna is joined to one end of the secondary circuit of the receiving radiogoniometer, through a self-inductance S ; the other end of the secondary is earthed through the detector R. The working of the arrangement is very simple, since it only requires the tuning of the two circuits as in ordinary cases of circular reception. The diagrams of the reception were obtained in the same way as

Fig. 16.



those of the transmission, by means of the thermo-galvanometer in place of the detector, and by turning the movable secondary of the receiving radiogoniometer, the transmitting station being fixed. The curve of fig. 16 in particular was obtained by means of the arrangement shown in fig. 13 ; the curve of fig. 17 (p. 324) by that of fig. 15 ; and the curve of fig. 18 (p. 324) by the arrangement of fig. 14

Fig. 17.

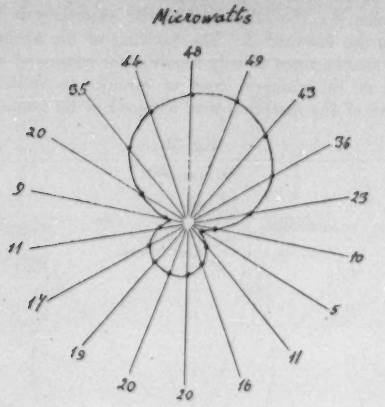
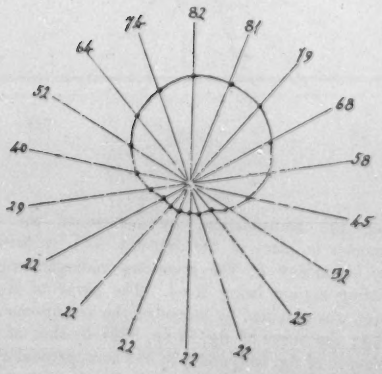

Microwatts
48
49
43
36
23
10
5
11
16
20
20
19
14
11
9
20
35
44

Fig. 18.


Microwatts
82
81
79
68
58
45
32
25
22
22
22
22
22
40
52
64
74

The arrangements have always enabled the sense of the received transmissions to be determined, and in many cases to get no reception when the pointer of the movable coil of the receiving radiogoniometer was turned in the opposite direction, which was precisely the object to be achieved.

The directive system of wireless telegraphy described can be employed in many ways, according to the requirements, owing to the facility with which one can pass from one system to the other. For example, when the operator expects a message from a station whose position is unknown, he could employ the single vertical antenna system; a soon as reception is effected he can, by using either the bilateral or unilateral system, determine the direction and upon which side the transmitting station lies, and make himself independent of other transmissions.

For the transmission the operator could, similarly, employ the system with circular radiation when he does not know the position of the receiving station or when he wishes to send the same message to several stations. He could in other cases use the unilateral or bilateral system according to the position of the stations which should not receive the messages.

The directive system described presents some advantages over other systems, which are very important either from the commercial or the strategic point of view.

So far as the commercial services are concerned, the possibility of transmitting messages to one station without affecting the others, results in the stations having an increased working capacity and consequently in an enhancement of their commercial value. The ability to receive from one determined direction renders the receiving station independent of extraneous transmissions, and even owing to this circumstance alone the working capacity of the stations is raised. Further, the ability to determine the origin of a transmission, apart from its obvious advantages, presents the advantage of enabling the route to be followed by a ship proceeding to the assistance of another in danger, to be indicated.

From the strategic point of view the directive reception enables one to learn of the presence of the enemy in a certain direction and to follow him in his movements. The directive

transmission will allow, when suitably employed, of transmitting to one's friends without the enemy being able to receive the waves—for the simple reason that there are no waves there to be detected in the undesired region.

In conclusion, the following is given as one practical example of the main strategical advantage of the directive system. Supposing it were necessary to send radiotelegraphic messages to the cruiser squadrons on the north east, north, north-west, and west coasts of the United Kingdom; if the messages were sent by an ordinary vertical antenna system they would be perceived in Denmark, in Germany, in Belgium, Holland, and France. Employing, on the other hand, the unilateral system, the messages sent from London to the Fleet would not be received outside the United Kingdom.

The authors desire to thank Mr. L. H. Walter, M.A., for translating this paper from the original.

Discussion.

Dr Fleming said that he had long been acquainted with the interesting experiments of the Authors of this paper, and desired to express his congratulations on the skill and inventiveness with which these investigations had been prosecuted. He was glad to see that the experiments of the Authors confirmed in many ways the theory which he (Dr Fleming) had given of the operation of a bent antenna as employed by Mr Marconi.

When Mr Marconi read his paper at the Royal Society in March 1906 describing his experiments on directive telegraphy, he gave no theory in the matter, but Dr Larmor pointed out in the discussion that an antenna partly vertical and partly horizontal was equivalent to the sum of a magnetic oscillator and an electric oscillator, and shortly afterwards he (Dr Fleming) had gone more carefully into the matter mathematically and showed that the observed effects could be accounted for on this theory.

Both Mr Marconi and he himself had obtained by the same methods as those employed by Messrs. Bellini and Tosi the same type of pear-shaped radiation curves obtained by the Authors of the paper by combining together the effect of closed and open oscillators. Although this theory had been criticised by Dr Mandelstam lately, yet nevertheless there did not seem sufficient grounds for objecting to it. Mr Marconi had, as everyone knew, employed directive antennæ for a long time past in his Power Stations at Poldhu and Clifden, and had also given demonstrations showing that the position of ships out of sight could be located

by means of such receiving directive antennæ. Nevertheless, Messrs. Bellini and Tosi had worked out extremely ingenious arrangements for determining the direction of the radiant point without moving the antennæ themselves. He (Dr Fleming) had also shown that, having the power to locate the radiant point, two stations equipped with such antennæ at a known distance apart could by simultaneous observations determine also the distance of the radiant point, and this might become important in connection with marine work.

It ought also to be noticed that Dr F. Braun, of Strasburg, had by the employment of three open antennæ having oscillations in them of definite phase difference, been able to obtain radiation curves having the form of a cardioid similar to some of those given by the Authors of the paper.

Dr W. H. Eccles warmly congratulated the Authors upon their beautiful and original method. On this method, by merely rotating a small coil of wire on the table, a fixed aerial directive system of any size was made to do what could otherwise only be done by turning the whole system of aerial wires in azimuth. The essence of the system was the piece of apparatus styled the radiogoniometer, which, by causing appropriate component radiation from two fixed wire triangles set at right angles, brought about a resultant radiation in any direction desired—just as if a virtual aerial of the full size of the fixed aerials were being rotated in the air. Someone had compared the result so achieved with that obtained by Marconi's well-known arrangement of a number of fixed bent antennæ with their horizontal portions directed from a centre to various points of the compass. Apart from the fact that it is rarely feasible to fix up a large number of antennæ, the radiogoniometer method has the advantage that the direction of transmission or reception can be altered perfectly smoothly; so that if it can be used, as stated by the Authors, accurately to about one degree of arc, the radiogoniometer is the practical equivalent of 360 bent antennæ.

The looped aerials used by the Authors did not seem to the speaker to be the best kind of radiator for utilising the principle they had developed. The radiogoniometer would prove to be capable of giving excellent results with two fixed antennæ of the bent type set at right angles, one, for example, in the meridian plane and the other in the east-west plane. The two looped aerials of the Authors may each of them be assimilated to a pair of vertical antennæ emitting waves of 180° phase difference, and with the assumption that for a single vertical aerial the inverse square law holds for the propagation of electrical effects from a single aerial, the speaker showed by aid of the ordinary equation of wave-motion that a looped antenna obeying the condition stated emitted two waves of equal period, of phases differing 90° and 180° respectively from the phase of the radiation from either side of the loop, and of amplitudes having a ratio proportional to λ/x. This means that one portion of the resultant radiation obeys an inverse square law and the other portion an inverse cube law; the latter portion is the more important of the two when the distance x is small, the former is the more important when the distance x

is much greater than the wave-length λ. This explains why Dr Fleming's measurements of the radiation from closed oscillators, which were carried out at short distances, gave small promise of powerful propagation to a distance, and why Messrs Bellini and Tosi had found as a fact that good propagation occurred to great distances.

As regards the phases of the two portions of the radiation, the important portion for wireless telegraphy, namely that obeying the inverse square law, was 90° out of phase with the near side of the loop; and thus it is clear that the vertical aerial the Authors had used to cancel the radiation to one side of the sending station, must for that purpose emit radiation 90° out of phase with the near side of the loop. Messrs. Bellini and Tosi had reached this conclusion in the course of their experiments.

After eliciting that the power-supply to the primary of the induction-coil for the Dieppe-Havre experiments was 500 watts, Dr Eccles concluded by remarking that the Authors were worthy of special congratulation because in their method of unilateral transmission by phase and intensity adjustment of several radiators they had eluded the difficulties of managing the phases that had apparently baffled F. Braun when he was using the same principle.

Mr W. DUDDELL complimented the Authors upon their paper and asked for the wave-length of the waves given out by the transmitter. He also asked what would be the effect of opening the closed antennæ at the top.

Mr Tosi, in reply to Dr Fleming, stated that Braun's cardioid diagram was a theoretical diagram only; the method had not succeeded and no actual diagram obtained experimentally had ever been shown.

Referring to the similarity which Dr Fleming said the pear-shaped diagram of Marconi bore to the Authors' diagrams, Mr Tosi could only say that they had purposely shown some bad diagrams as well as some good ones. The good diagram of Marconi's corresponded to the Authors' bad diagram. With the Marconi horizontal aerial it would, moreover, be necessary, as Dr Eccles had pointed out, to employ 360 such wires to obtain an accuracy in locating the bearing of the radiant point equal to that secured with the system now described. Further, if the emitted energy is actually represented for longer distances by Marconi's pear-shaped diagram, it is difficult to understand why the transatlantic messages from Clifden to Glace Bay are readily picked up off the Algerian coast, at right angles to the line of transmission.

In reply to Mr Duddell he said that the wave-length of the waves employed was from 350 to 400 metres.

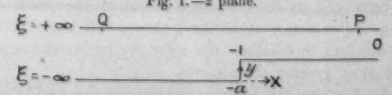
XXII. *On the Resistance of a Conductor of Uniform Thickness whose Breadth suddenly changes, and on the Shapes of the Stream-Lines in the immediate neighbourhood.* By CHARLES H. LEES, *D.Sc., F.R.S., Professor of Physics in the East London College, University of London* *.

A KNOWLEDGE of the resistance of a conductor whose section suddenly changes is of considerable practical importance, but mathematical difficulties have prevented an exact solution of the problem. Lord Rayleigh † has given an approximate solution of the case in which a cylinder of circular section is joined at one end to the plane surface of a large conducting solid, and Professor Hicks ‡ has solved the case of a wire of small diameter ending in the surface of a conducting sphere §.

The mathematical difficulties of the problem disappear if the conductors are of rectangular section and one dimension, *e. g.* the thickness, remains constant, while the other, the breadth, suddenly changes, and the two are joined together either with their axes or with two sides colinear. For a longitudinal flow through the conductors the former case becomes the latter by section through the common axis, which by symmetry is a line of flow.

Let the thickness of the conductor be t, the breadth of the wider part c, that of the narrower b, and let k be the conductivity of the material. Take the plane of the conductor

Fig. 1.—z plane.



as the z plane, fig. 1, and convert the outline of the conductor

* Read June 12, 1908.

† Lord Rayleigh, Trans. Roy. Soc. London, lxi. p. 77, Nov. 1870; and 'Scientific Papers,' vol. i. p. 56. See also Phil. Mag. viii. p. 481 (1904).

‡ Professor Hicks, Messenger of Mathematics, xii. p. 183 (1883).

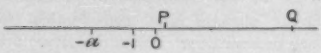
§ A list of cases which have been worked out will be found in Professor Auerbach's articles "Flächenströme" and "Körperliche Ströme" in Winkelmann's *Handbuch der Physik*, 2nd edit. iv. pp. 238, 240.

by means of the Schwarzian transformation

$$\frac{dz}{d\zeta} = A\zeta^{-1}(\zeta+1)^{\frac{1}{2}}(\zeta+a)^{-\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad (1)$$

into the axis of ξ in the ζ plane fig. 2, the corresponding

Fig. 2.—ζ plane.



points being indicated by the lettering in the figures.

In $$z = A\int \frac{1}{\zeta}\sqrt{\frac{\zeta+1}{\zeta+a}}\,d\zeta$$

write $$\frac{\zeta+1}{\zeta+a} = \frac{1}{\theta^2},$$

and the integral becomes

$$z = 2A\int \frac{(a-1)\,d\theta}{(a-\theta^2)(1-\theta^2)}$$

$$= 2A\int\left(\frac{1}{1-\theta^2} - \frac{1}{a-\theta^2}\right)d\theta$$

$$= 2A\left\{\arg\begin{pmatrix}\tanh\\ \coth\end{pmatrix} \pm\sqrt{\frac{\zeta+a}{\zeta+1}} - \frac{1}{\sqrt{a}}\arg\begin{pmatrix}\tanh\\ \coth\end{pmatrix} \pm\sqrt{\frac{\zeta+a}{a(\zeta+1)}}\right\};^{*} \quad (2)$$

where the origin in the z plane is taken at the point corresponding to $\xi=-a$, $\eta=0$ in the ζ plane, and tanh or coth is taken according to whether the quantity under the root sign is less than or greater than unity.

In order to show the connexion between the z and ζ planes graphically, we write

$$\zeta+a = r_a e^{i\theta_a}, \qquad \zeta+1 = r_1 e^{i\theta_1},$$

where r_a, θ_a, r_1, θ_1 are the coordinates of the point $\xi\eta$, with

* The form argtanh has been retained throughout because of the symmetry of the norm and amplitude when this is done.

respect to the points $-a$ and -1 (fig. 2), and

$$r_a^2 = r^2 + 2ar\cos\theta + a^2, \qquad \cot\theta_a = \frac{\cos\theta + \frac{a}{r}}{\sin\theta},$$

$$r_1^2 = r^2 + 2r\cos\theta + 1, \qquad \cot\theta_1 = \frac{\cos\theta + \frac{1}{r}}{\sin\theta}.$$

Resolving each term of the above equation into its norm and amplitude, we obtain

$$z = A\left[\operatorname{argtanh}\left(\pm\frac{\cos\frac{\theta_a-\theta_1}{2}}{\cosh\log\sqrt{\frac{r_a}{r_1}}}\right) - i\arctan\left(\pm\frac{\sin\frac{\theta_a-\theta_1}{2}}{\sinh\log\sqrt{\frac{r_a}{r_1}}}\right) + im\pi\right.$$

$$\left. -\frac{1}{\sqrt{a}}\left\{\operatorname{argtanh}\left(\pm\frac{\cos\frac{\theta_a-\theta_1}{2}}{\cosh\log\sqrt{\frac{r_a}{ar_1}}}\right) - i\arctan\left(\pm\frac{\sin\frac{\theta_a-\theta_1}{2}}{\sinh\log\sqrt{\frac{r_a}{ar_1}}}\right) + in\pi\right\}\right]$$

where m and n are integers. The alternatives tanh and coth of equation (2) both lead to the same result.

On determining the values of the constants A and a so as to make $y=0$ from $\xi=-\infty$ to $\xi=-a$, $y=c-b$ from $\xi=-1$ to $\xi=0$, $y=c$ from $\xi=0$ to $\xi=\infty$, we find that the negative sign must be assigned to the terms in the small brackets, that $A=c/\pi$, $\sqrt{a}=c/b$, and the equations become

$$z = x+iy = \frac{2b}{\pi}\arg\begin{pmatrix}\tanh\\ \coth\end{pmatrix}\frac{c}{b}\sqrt{\frac{\zeta+1}{\zeta+c^2/b^2}} - \frac{2c}{\pi}\arg\begin{pmatrix}\tanh\\ \coth\end{pmatrix}\sqrt{\frac{\zeta+1}{\zeta+c^2/b^2}}$$

$$= \frac{b}{\pi}\left\{\operatorname{argtanh}\frac{\cos\frac{\theta_1-\theta_a}{2}}{\cosh\log\frac{c}{b}\sqrt{\frac{r_1}{r_a}}} - i\arctan\frac{\sin\frac{\theta_1-\theta_a}{2}}{\sinh\log\frac{c}{b}\sqrt{\frac{r_1}{r_a}}}\right\}$$

$$-\frac{c}{\pi}\left\{\operatorname{argtanh}\frac{\cos\frac{\theta_1-\theta_a}{2}}{\cosh\log\sqrt{\frac{r_1}{r_a}}} - i\arctan\frac{\sin\frac{\theta_1-\theta_a}{2}}{\sinh\log\sqrt{\frac{r_1}{r_a}}}\right\}, \quad (3)$$

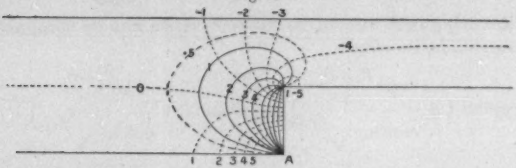
* The term $\operatorname{argtanh}\frac{\cos\theta}{\cosh\log\rho}$ may be more familiar in the forms

$$\tfrac{1}{2}\log\frac{\cosh\log\rho+\cos\theta}{\cosh\log\rho-\cos\theta} = \tfrac{1}{2}\log\frac{\rho^2+2\rho\cos\theta+1}{\rho^2-2\rho\cos\theta+1}.$$

where the arctan terms are to be taken within the limits 0 and π.

The curves in the z plane into which this equation converts the orthogonal system of circles $\theta_1 - \theta_a = \text{constant}$, and $r_1/r_a = \text{constant}$, of the ζ plane, are shown in fig. 3 for the

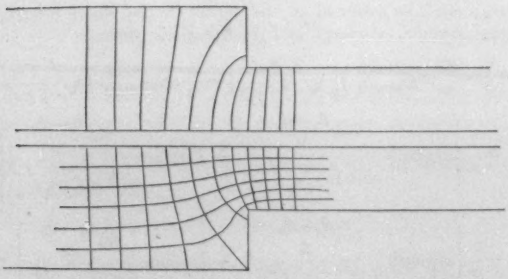
Fig. 3.



Full lines are equipotential lines, dotted lines are stream lines, when the end 1A is kept at potential 9, and the rest of the contour at 0. Ratio of breadths 1 : 2.

case $c/b = 2$. They may be considered either as the potential and stream curves for a source at 1 and an equal sink at A, or for the edge 1A kept at one potential and the whole of the remainder of the edges at another.

Fig. 4.



Lower part equipotential and stream lines calculated from (3); upper part equipotential lines observed. Ratio of breadths 1 : 2.

The curves in the z plane corresponding to $\log \zeta = \text{constant}$ in the ζ plane are shown in fig. 4 lower part, and represent

the equipotential and stream lines for a current flowing along the two strips in series. The upper part shows the observed equipotential lines *.

To determine the resistance of the compound conductor to the flow of current along its length, we find the distribution of the potential in the corresponding problem in the ζ plane. This is the case of a source at the point $\xi=\eta=0$, sending a current C into the upper half of the plane.

The vector function $w=u+iv$ for this case is given by the equation

$$w=\frac{C}{\pi kt}\log\zeta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where t is the thickness of the conductors and k is the conductivity of the material of which they are composed.

If ξ_P, 0, are the coordinates of a point P near O in the ζ plane, and x_P is its X coordinate in the z plane (figs. 1 and 2), then by equation (3)

$$x_P=\frac{2b}{\pi}\operatorname{argcoth}\frac{c}{b}\sqrt{\frac{\xi_P+1}{\xi_P+c^2/b^2}}-\frac{2c}{\pi}\operatorname{argtang}\sqrt{\frac{\xi_P+1}{\xi_P+c^2/b^2}},$$

which since ξ_P is small reduces to

$$x_P=\frac{2b}{\pi}\operatorname{argcoth}\sqrt{1+\frac{c^2-b^2}{c^2}\,\xi_P}-\frac{2c}{\pi}\operatorname{argtanh}\frac{b}{c},$$

$$=\frac{b}{\pi}\left\{\log\frac{4c^2}{c^2-b^2}-\frac{\pi kt}{C}.u_P\right\}-\frac{c}{\pi}\log\frac{c+b}{c-b},$$

in virtue of equation (4).

Hence

$$\frac{\pi kt}{C}.u_P=-\frac{\pi}{b}x_P-\log\frac{c^2-b^2}{4c^2}-\frac{c}{b}\log\frac{c+b}{c-b}.$$

Similarly if ξ_Q, 0, are the coordinates of a point Q near $+\infty$ in the ζ plane, and x_Q is its X coordinate in the z plane,

* I have to thank my colleague Mr. W. H. White for making the observations on a thin strip of german silver transmitting an electric current, from which these curves have been drawn.

then by equation (3)

$$x_Q=\frac{2b}{\pi}\operatorname{argcoth}\frac{c}{b}\sqrt{\frac{\xi_Q+1}{\xi_Q+c^2/b^2}}-\frac{2c}{\pi}\operatorname{argtanh}\sqrt{\frac{\xi_Q+1}{\xi_Q+c^2/b^2}},$$

which, since ξ_Q is large, reduces to

$$x_Q=\frac{2b}{\pi}\operatorname{argcoth}\frac{c}{b}-\frac{2c}{\pi}\operatorname{argtanh}\sqrt{1-\frac{c^2-b^2}{b^2\xi_Q}}$$

$$=\frac{b}{\pi}\log\frac{c+b}{c-b}-\frac{c}{\pi}\left\{\log\frac{4b^2}{c^2-b^2}+\frac{\pi kt}{C}.u_Q\right\}$$

in virtue of equation (4).

Hence

$$\frac{\pi kt}{C}u_Q=-\frac{\pi}{c}x_Q-\log\frac{4b^2}{c^2-b^2}+\frac{b}{c}\log\frac{c+b}{c-b}.$$

Thus

$$\frac{kt}{C}(u_Q-u_P)=\frac{x_P}{b}-\frac{x_Q}{c}+\frac{1}{\pi}\left(2\log\frac{c^2-b^2}{4cb}+\frac{c^2+b^2}{cb}\log\frac{c+b}{c-b}\right)$$

Hence the resistance between two transverse sections of the conductors through the points P and Q situated at considerable distances from the change of section and on opposite sides of it, is equal to the sum of the resistances of the lengths of conductor on the two sides of the change of section, each considered as part of an infinite length, plus the resistance of a length of either conductor equal to

$$\frac{1}{\pi}\left\{2\log_e\frac{c^2-b^2}{4cb}+\frac{c^2+b^2}{cb}\log_e\frac{c+b}{c-b}\right\}$$

times the breadth of that conductor.

When the ratio of the breadths $c/b=n$ becomes large, the correction approximates to

$$\frac{2}{\pi}(\log_e n-\cdot386),$$

and when very large to

$$\frac{2}{\pi}\log_e n.$$

When, on the other hand, the ratio n of the breadths becomes nearly unity the correction approximates to

$$\frac{(n-1)^2}{\pi}\left\{\frac{1}{2}+\frac{1}{n}\log_e\frac{2}{n-1}\right\},$$

and when it is very nearly unity to

$$\frac{(n-1)^2}{\pi}\left\{1{\cdot}19-\log_e(n-1)\right\}.$$

In order to exhibit the character of this correction the following table has been calculated.

CORRECTION to be added to the length of one conductor in terms of the breadth of that conductor, in order to make the total resistance equal to the sum of the resistances of the two conductors, each considered as part of a conductor of infinite length.

Ratio of breadths.	No. of breadths to be added to length.	Ratio of breadths.	No. of breadths to be added to length.
1·0	·0	3·0	·48
1·2	·032	4·0	·65
1·4	·080	6·0	·90
1·7	·17	10·0	1·24
2·0	·25	15·0	1·46
2·5	·37	20·0	1·65

DISCUSSION.

DR RUSSELL congratulated the Author on having obtained the exact solution of an important problem and thanked him for giving it in a form in which it could be utilised readily by electricians. Somewhat similar problems are of frequent occurrence in practice, in particular he instanced the measurement of the resistance of the bonds connecting the rails in electric tramway systems. The difficulty in this case is in knowing where the rail ends and the bond begins.

A small variation in the position of the potential contacts makes a large variation in the reading of the galvanometer. In many cases the only way of attacking the problem is to calculate the resistance from the known resistivity of the metals by the approximate method indicated by Maxwell. An exact solution, therefore, like the one obtained by the Author, would be of great value in checking the accuracy of the approximate method.

FIG. 2.—ORDINARY CAST IRON NO. 1.

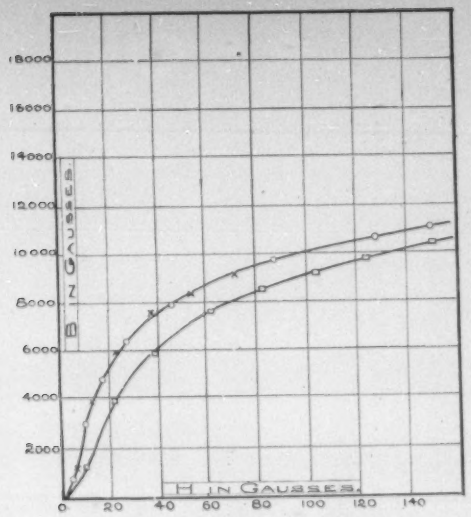


FIG. 3.—ORDINARY CAST IRON NO. 2.

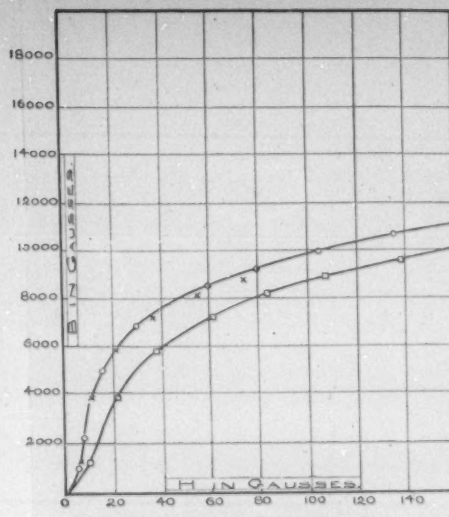


FIG. 6.—WROUGHT IRON NO. 1.

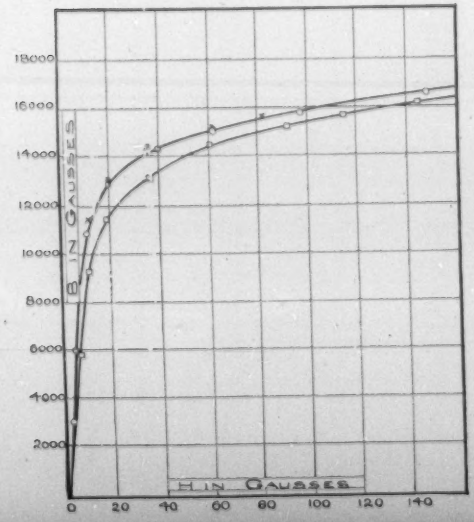


FIG. 7.—WROUGHT IRON NO. 2.

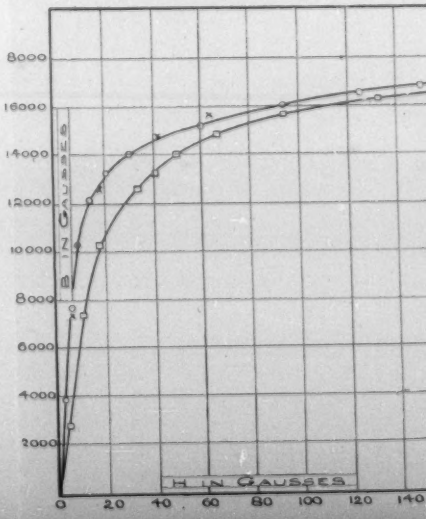


FIG. 4.—SPECIAL CAST IRON No. 1.

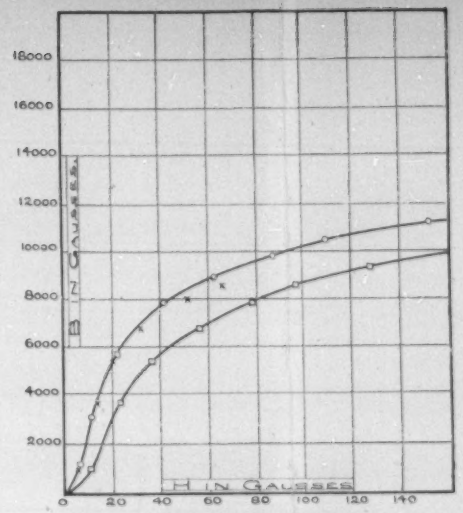


FIG. 5.—SPECIAL CAST IRON No. 2.

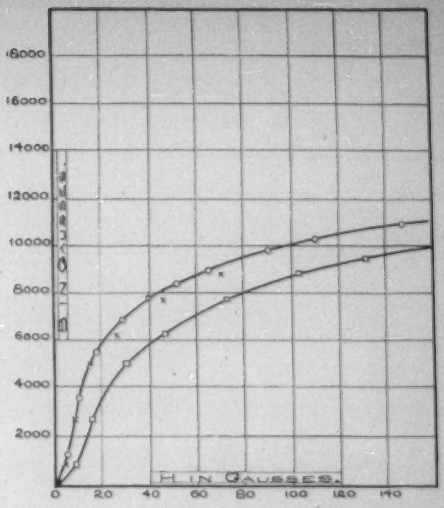


FIG. 8.—MILD STEEL No. 1.

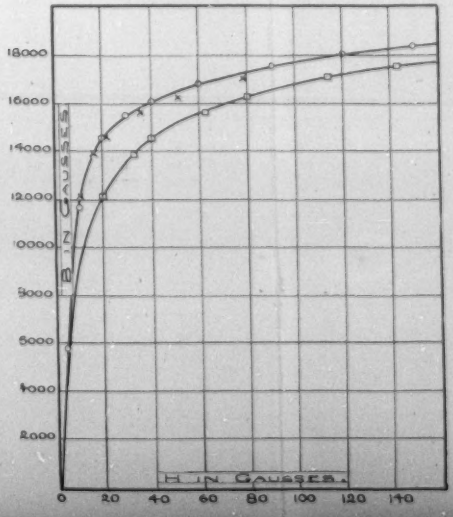


FIG. 9.—MILD STEEL No. 2.

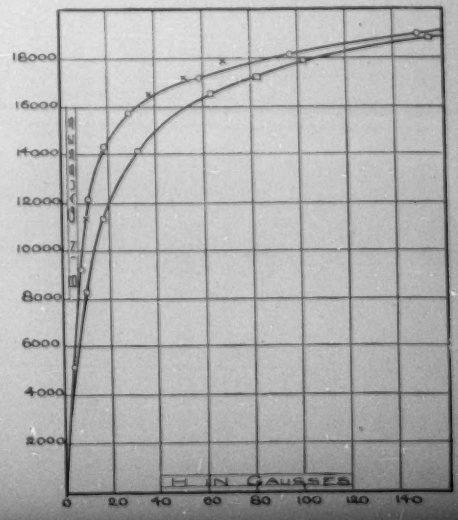
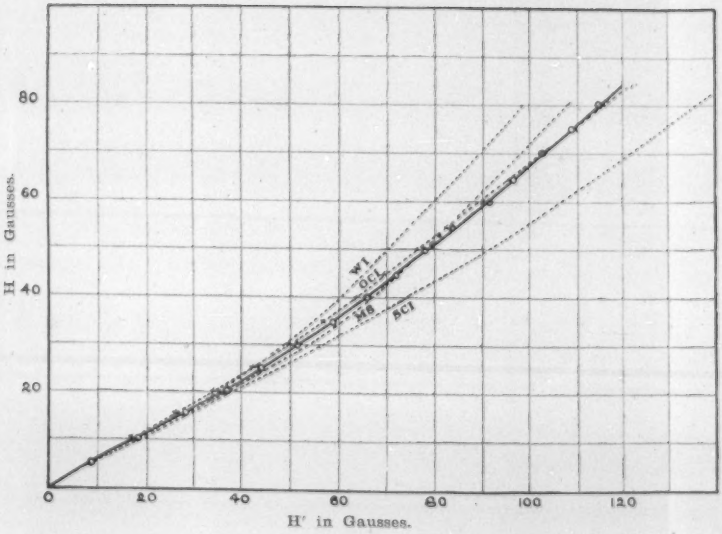


Fig. 10.


80
60
40
20
H in Gausses.
0
20
40
60
80
100
120
H' in Gausses.
WI
OCI
MB
ACI

Fi

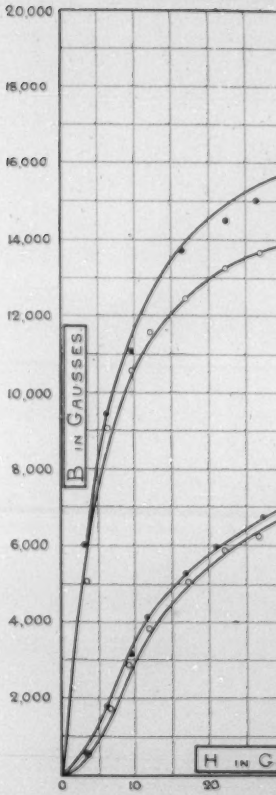

20,000
18,000
16,000
14,000
12,000
10,000
8,000
6,000
4,000
2,000
B in Gausses
0
10
20
H in G

FIG. 11.

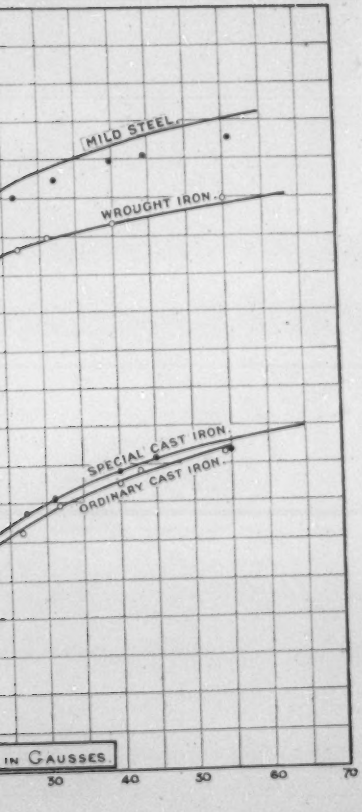

MILD STEEL.
WROUGHT IRON.
SPECIAL CAST IRON.
ORDINARY CAST IRON.
IN GAUSSES.
30
40
50
60
70

FIG. 12.

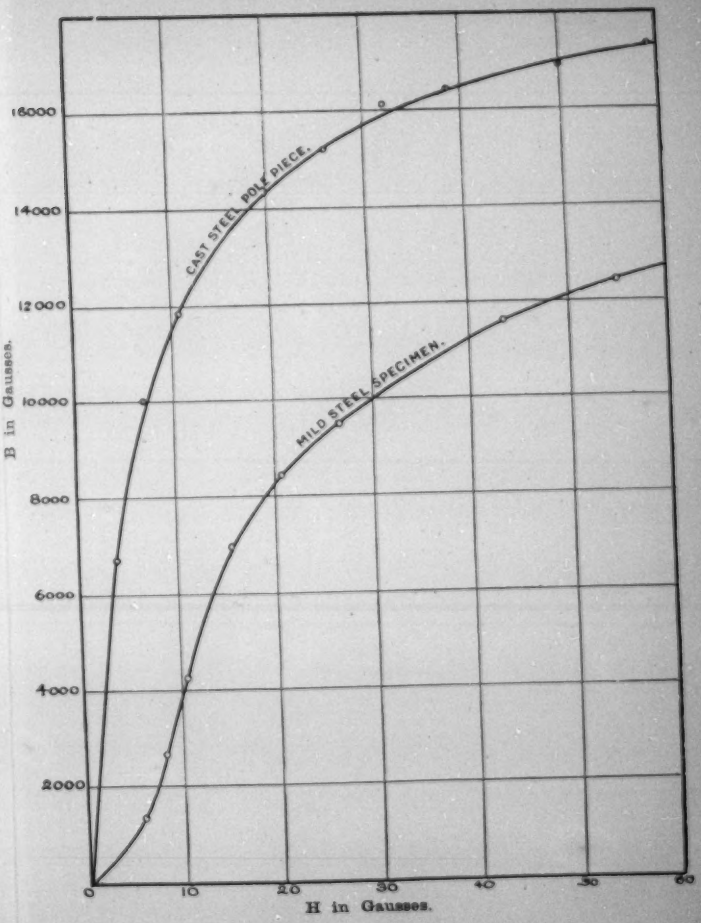

18000
14000
12000
10000
8000
6000
4000
2000
B in Gausses.
CAST STEEL POLE PIECE.
MILD STEEL SPECIMEN.
0
10
20
30
40
50
60
H in Gausses.

FIG. 7.—Ratio Tests on Current Transformer.

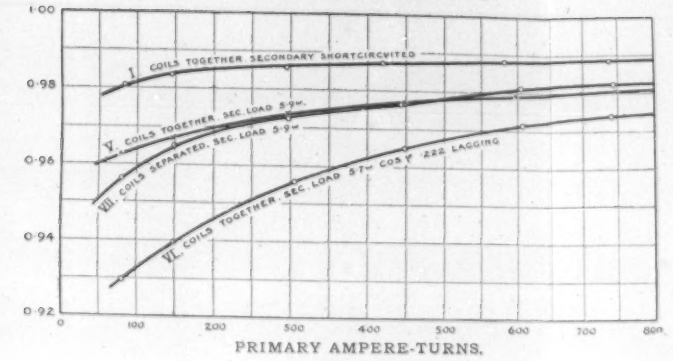


FIG. 8.—Phase Displacement on Current Transformer.

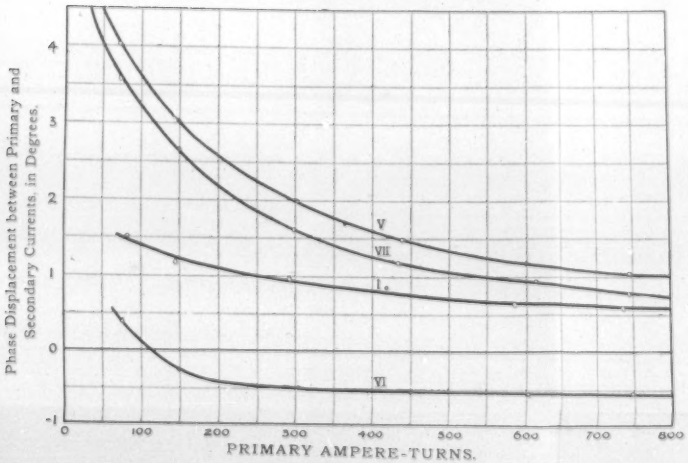


FIG. 9.—Ratio Tests on Voltage Transformer.

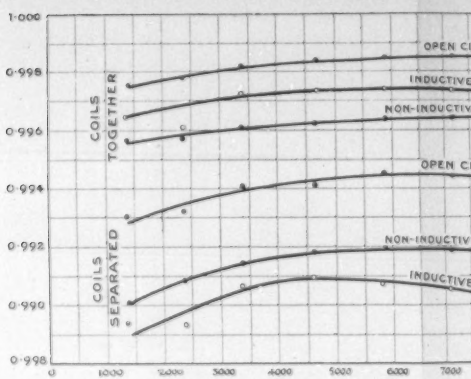
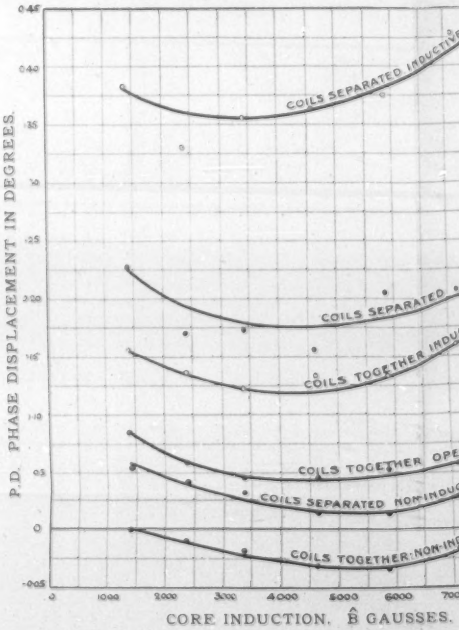
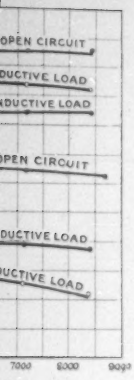


FIG. 10.—Displacement Tests on Voltage Transformer.

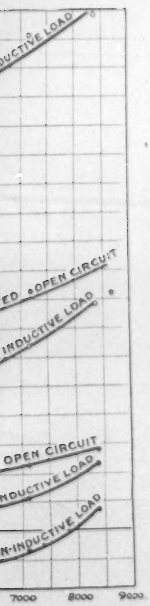


rmer.



ES.

nsformer.



ES.

FIG. 11.—Relation between Core Loss and Magnetizing Current.

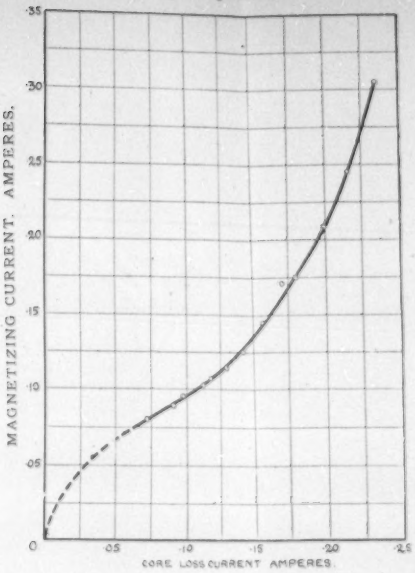


FIG. 12.—Relation between $\frac{A_c}{B}$ and $\frac{A_m}{B}$.

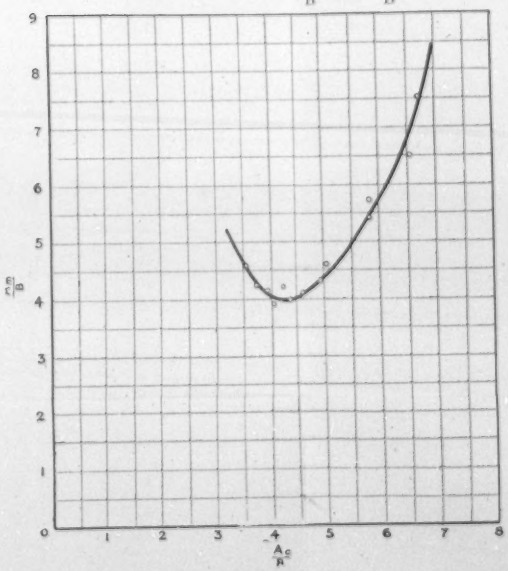


FIG. 13.—Ratio Tests on Current Transformers.

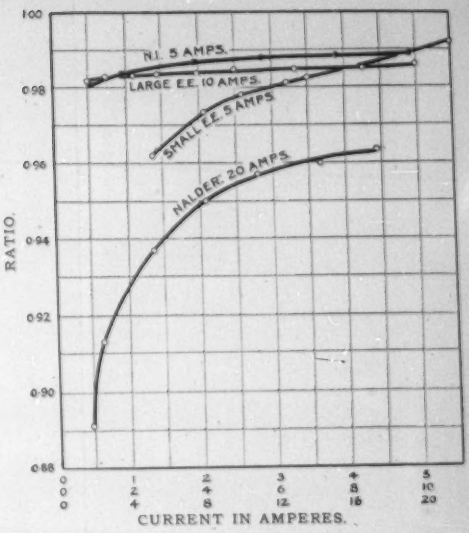
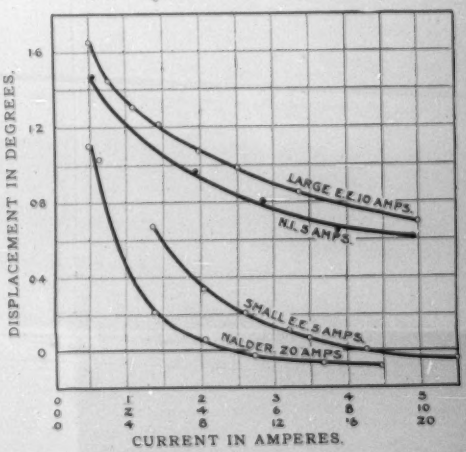
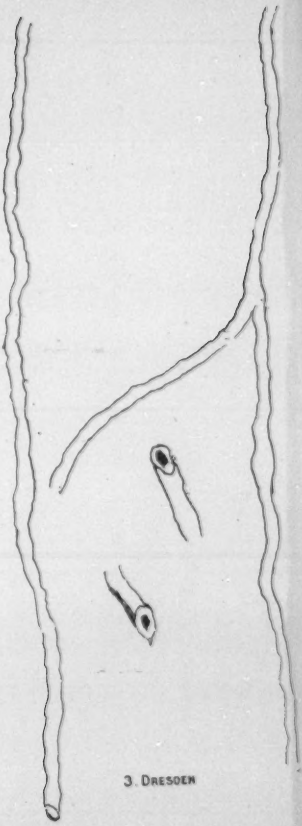


FIG. 14.—Displacement Tests on Current Transformers.



1. DIBLA. LAKE CHAD



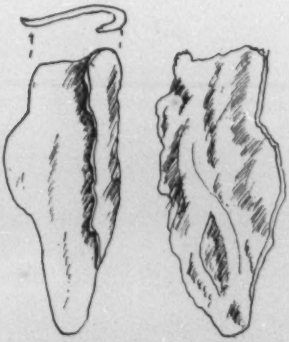
3. DRESDEN

2. ESSEX

ALLED FULGURITES.

4. DIBLA. LAKE CHAD

5. SENNER

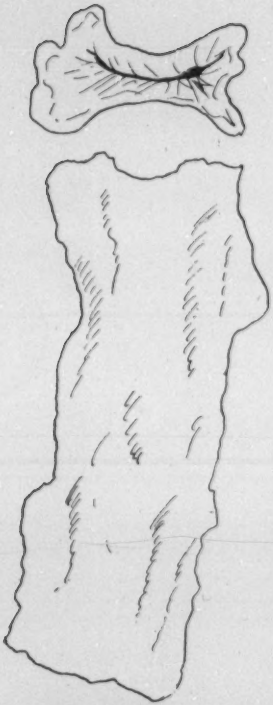


6. DRIGG

7. ESSEX

BUTCHER.

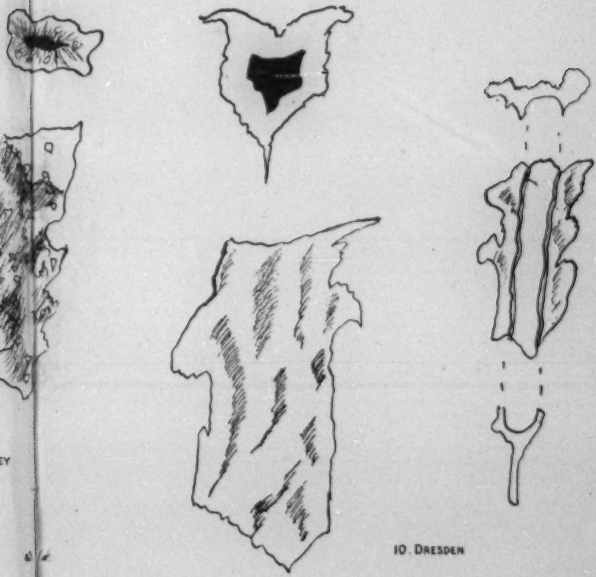
THICK-WALLED



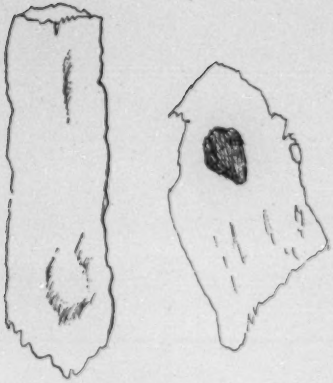
8. SENNER

9. BUTLEY

LED FULGURITES.



10. DRESDEN

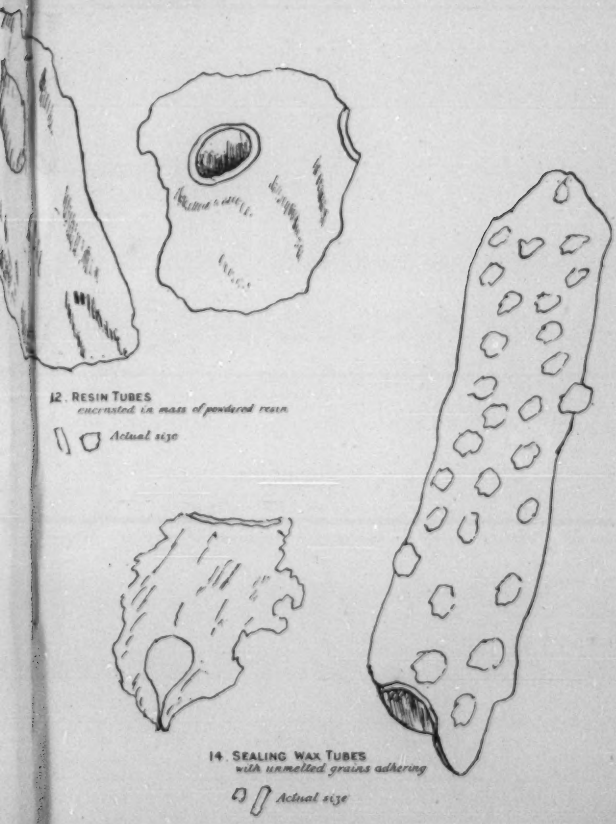


11 BORAX TUBES (magnified)
Actual size

13 TUBES OF CRYSTALLISED BORAX

Actual size

ICIAL FULGURITES.



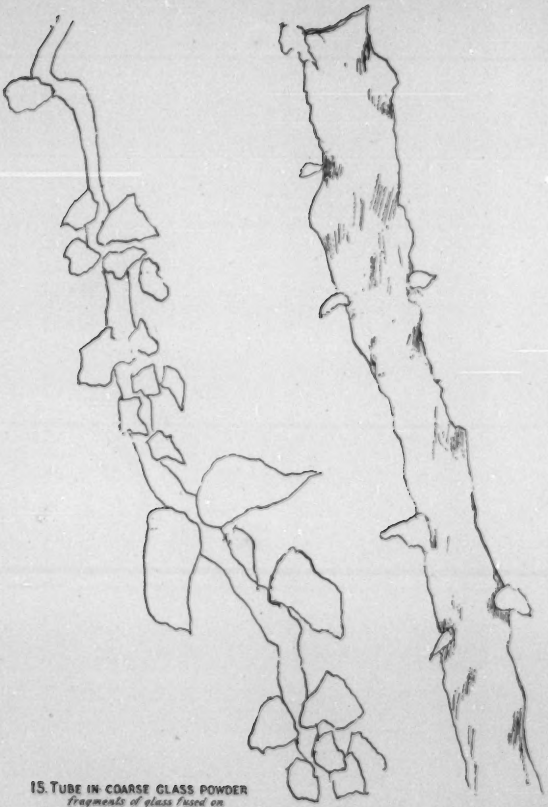
12. RESIN TUBES
encrusted in mass of powdered resin.

Actual size

14. SEALING WAX TUBES
with unmelted grains adhering

Actual size

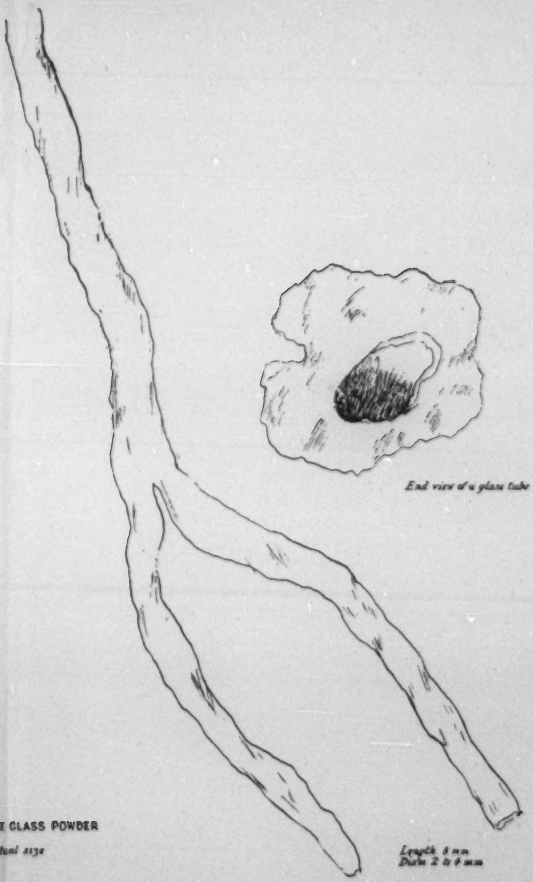
ARTIFICIAL FULGUR



15. TUBE IN COARSE GLASS POWDER
fragments of glass fused on
Actual size 6 mm long 2 mm diam

16 FINE
Actu

LGURITES IN GLASS.



P.D. and Current Wave-Forms of Short Sparks.

FIG. 3.—Spark-length equals zero.

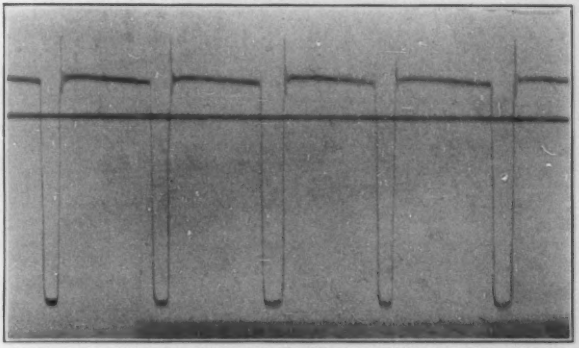
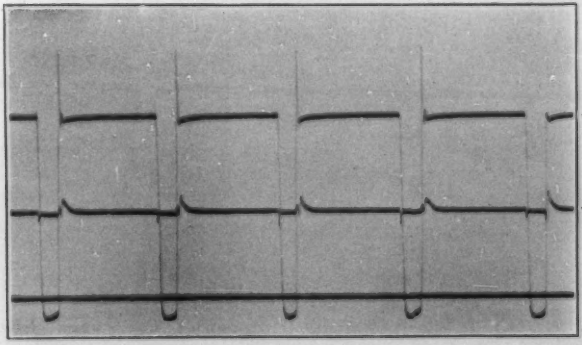


FIG. 4.—Spark-length 1 mm.



P.D. and Current Wave-Forms of Short Sparks.

FIG. 5.—Spark-length 5 mm.

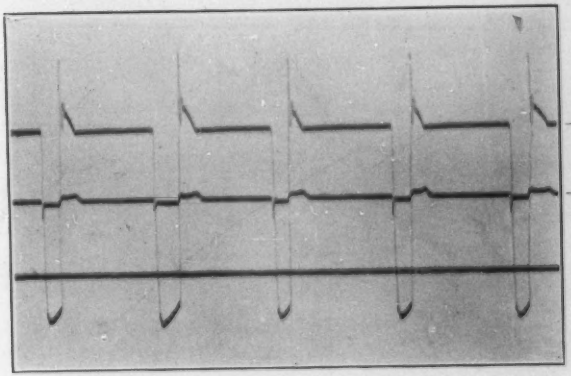
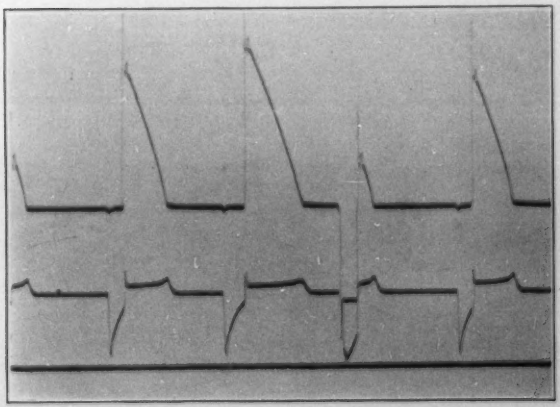
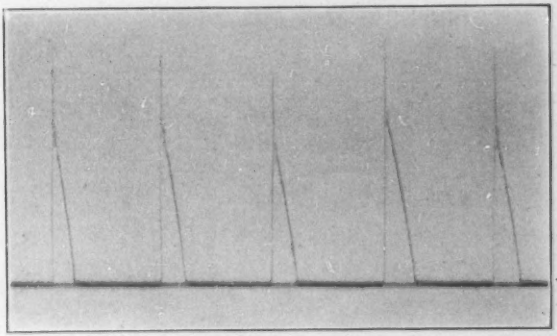


FIG. 6.—Spark-length 15 mm.



P.D. and Current Wave-Forms of Short Sparks.

Fig. 7.—Spark-length 30 mm.



P.D. Wave-Form not recorded.

Fig. 8.—Distance between electrodes too great for sparking to take place.

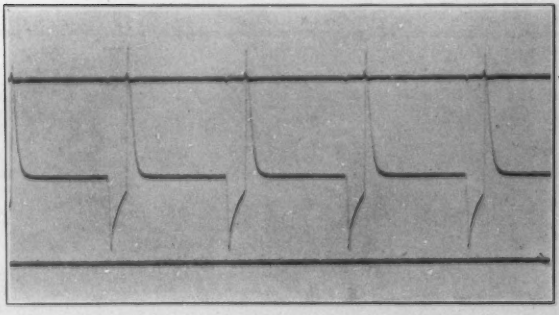
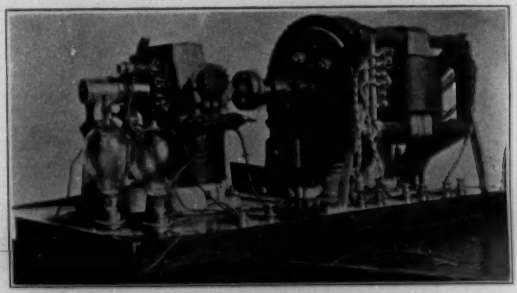


FIG. 1.



N. T. CO. ALTERNATOR.

FIG. 2.

UNFILTERED WAVE.

FIG. 3.

FILTERED WAVE.

FIG. 4.

WAVES in primary and secondary of humming telephone instrument.

FIG. 18.

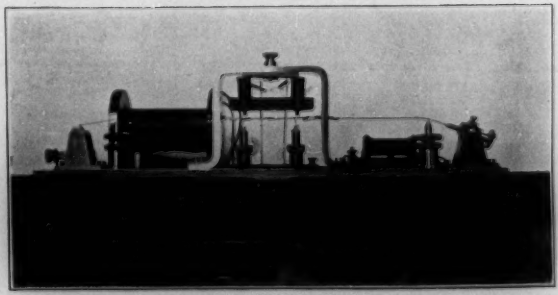
WAVE FORMS. Continuous oscillation circuit.

FIG. 19.

MEAN SPEECH WAVE.

FIG. 5.

VIBRATING WIRE.
Details of mercury cups, etc.



VIBRATING WIRE INTERRUPTER.
General View.

FIG. 7.

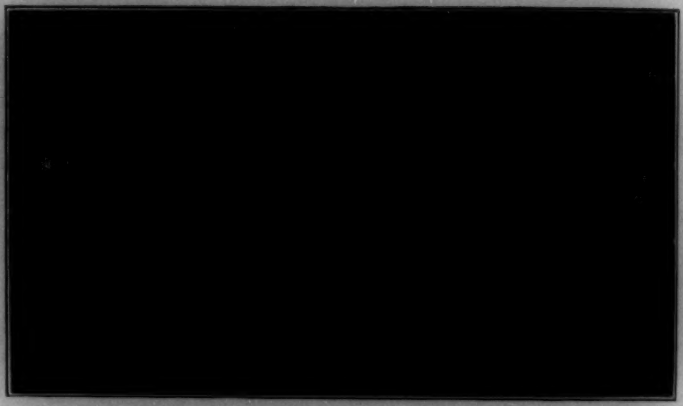
490 ~.

959 ~.

2880 ~.

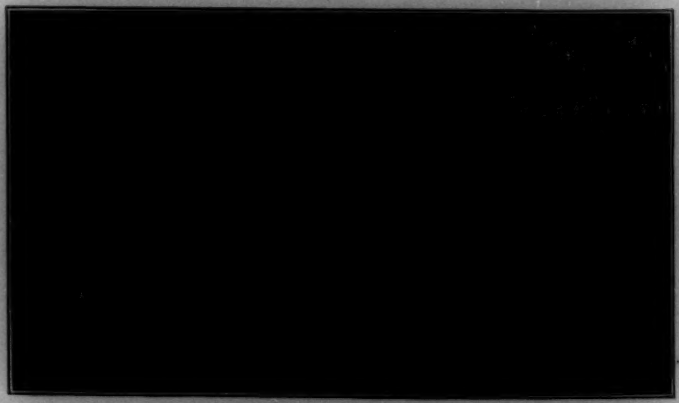
SINGLE ACTION CIRCUIT WAVES.

FIG. 9.



EFFECTS OF RESISTANCE IN SINGLE ACTION CIRCUIT.

FIG. 10.

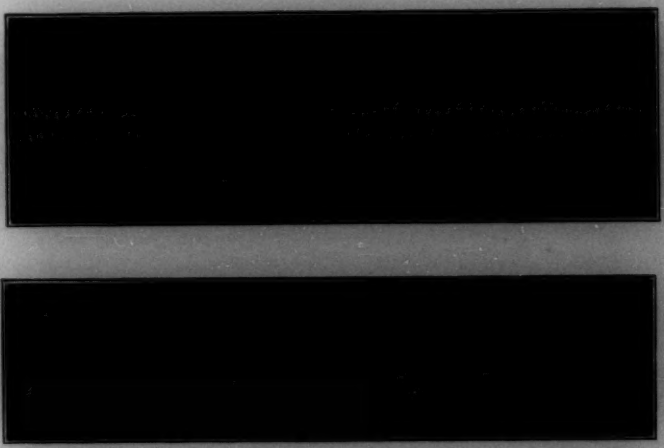


DOUBLE ACTION CIRCUIT WAVES.

FIG. 12.

ASSISTING WAVES.

FIG. 13.

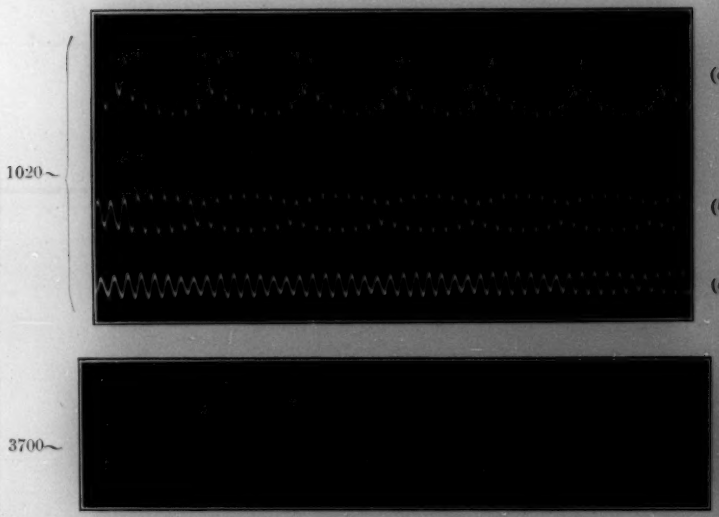


OPPOSING WAVES.

FIG. 14.

COMPLEX WAVE FORM.

FIG. 16.



WAVE FORMS, Modified double action circuit.

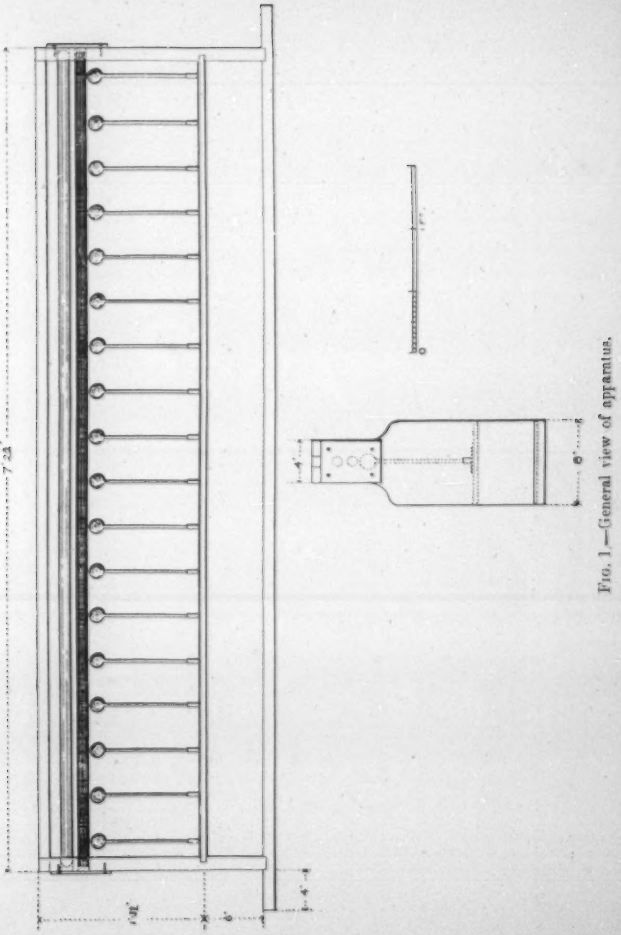


FIG. 1.—General view of apparatus.

FIG. 2.—Apparatus and accessories.

FIG. 3.—Frequency about 600,000. Receiving end open.

FIG. 4.—Frequency about 1,140,000. Receiving end open.

FIG. 5.—Frequency about 1,800,000. Receiving end open.

FIG. 6.—Frequency 600,000. Non-inductive receiver.

FIG. 7.—Frequency 600,000. Receiving end earthed.

FIG. 8.—Frequency 600,000. Capacity increased by internal wire. Receiving end open.

FIG. 9.—Frequency 600,000. Receiving end earthed through condenser.

When, on the other hand, the ratio n of the breadths becomes nearly unity the correction approximates to

$$\frac{(n-1)^2}{\pi}\left\{\frac{1}{2}+\frac{1}{n}\log_e\frac{2}{n-1}\right\},$$

and when it is very nearly unity to

$$\frac{(n-1)^2}{\pi}\left\{1{\cdot}19-\log_e(n-1)\right\}.$$

In order to exhibit the character of this correction the following table has been calculated.

CORRECTION to be added to the length of one conductor in terms of the breadth of that conductor, in order to make the total resistance equal to the sum of the resistances of the two conductors, each considered as part of a conductor of infinite length.

Ratio of breadths.	No. of breadths to be added to length.	Ratio of breadths.	No. of breadths to be added to length.
1·0	·0	3·0	·48
1·2	·032	4·0	·65
1·4	·080	6·0	·90
1·7	·17	10·0	1·24
2·0	·25	15·0	1·46
2·5	·37	20·0	1·65

DISCUSSION.

DR RUSSELL congratulated the Author on having obtained the exact solution of an important problem and thanked him for giving it in a form in which it could be utilised readily by electricians. Somewhat similar problems are of frequent occurrence in practice, in particular he instanced the measurement of the resistance of the bonds connecting the rails in electric tramway systems. The difficulty in this case is in knowing where the rail ends and the bond begins.

A small variation in the position of the potential contacts makes a large variation in the reading of the galvanometer. In many cases the only way of attacking the problem is to calculate the resistance from the known resistivity of the metals by the approximate method indicated by Maxwell. An exact solution, therefore, like the one obtained by the Author, would be of great value in checking the accuracy of the approximate method.

XXIII. *Homogeneous Secondary Röntgen Radiations.* By CHARLES G. BARKLA, *M.A., D.Sc., Lecturer in Advanced Electricity, and* CHARLES A. SADLER, *M.Sc., Demonstrator in Physics, University of Liverpool* *.

THOUGH there are many phenomena of Secondary Röntgen Rays still awaiting investigation, it seems desirable in publishing the results of recent experiments—principally on the homogeneous secondary radiations—that a general survey should be made of the whole subject of "Secondary X-Rays emitted by substances subject to X-Rays," and that a more concise statement of the experimental results and the conclusions based on these should be given. This, indeed, appears a necessity not only in order to make intelligible the results of what would otherwise appear isolated experiments of little significance, but to exhibit the observed limitations or the generality of laws which are continually being tested by further experiments on a variety of substances and under a variety of conditions.

As has been shown in previous papers †, the behaviour of substances subject to X-rays varies enormously with the atomic weight of the substance exposed, and generalizations cannot safely be arrived at except after an extensive series of experiments on a large number of elements.

The results which have so far been found to be perfectly general will be briefly stated ‡ :—

* Read June 12, 1908.

The expenses of this research have been partially covered by a Government Grant through the Royal Society.—C. G. B.

† As frequent references are made to the following papers, they are denoted by the letters *a–g* :—

BARKLA : *a*. Phil. Mag. June 1903, pp. 685–698.
b. Phil. Mag. May 1904, pp. 543–560.
c. Phil. Trans. A, vol. 204, 1905, pp. 467–479.
d. Roy. Soc. Proc. A, vol. 77, 1906, pp. 247–255.
e. Phil. Mag. June 1906, pp. 812–828.
g. Phil. Mag. Feb. 1908, pp. 288–296.

BARKLA & SADLER : *f*. Phil. Mag. Sept. 1907, pp. 408–422.

‡ These results were given in the papers to which reference has already been made ; later experiments have not revealed any exceptions. It appears quite possible, however, that under certain conditions

All substances subject to X-rays are a source of secondary X-rays.

The radiation from a given element is independent of the physical state of the substance and of its mixture or even chemical combination with other elements.

The character of the secondary radiation from an element is independent of the intensity of the primary radiation producing it.

The intensity of secondary radiation from an element is proportional to the intensity of the primary radiation of definite character producing it.

The absorption by a thin sheet of any substance of the secondary rays from various elements subject to the same primary beam is a periodic function of the atomic weight of the radiating substance.

There are, however, groups of elements of neighbouring atomic weight into which substances may conveniently be divided; for when a primary beam of ordinary penetrating power is used, the radiations from the various elements in one group are very similar in properties, while those from elements in different groups differ considerably. But it should be understood that this grouping is somewhat arbitrary, as elements of intermediate atomic weight emit radiations possessing intermediate properties, and the classification depends to a certain extent on the character of the primary radiation. It is, however, convenient for the purpose of description.

H–S Group.

The group of substances of atomic weights from that of hydrogen to that of sulphur appears simplest in behaviour under X-rays of ordinary penetrating power.

Each element, when subject to such a primary beam, emits a secondary radiation which has almost exactly the

exceptions will be found. Crowther, who by careful experiment has further verified some of these results (Phil. Mag. Nov. 1907), finds much less intense radiation from nickel when in combination as nickel carbonyl than we have obtained from the pure element in the solid state. As this cannot be accounted for by variation in the primary beams used, it is perhaps worthy of further investigation.

same penetrating power as the primary producing it. The secondary beam is complex like the primary, and contains the rays of various penetrating powers in approximately the same proportion as the primary (*a* & *b*).

Though it is very difficult, if not impossible, to detect by direct methods a difference in the penetrating powers of primary and secondary beams, when the primary is not more than moderately penetrating, it appears from indirect evidence that the secondary radiation is always slightly more absorbable than the primary (*b*). With more penetrating primary rays the difference is more marked (*b* & *).

The intensity of radiation emitted by these elements is proportional merely to the quantity of matter passed through by a primary beam of definite intensity, if of low to moderate penetrating power: in other words, the intensity of radiation from an atom is proportional to the atomic weight † (*a* & *b*).

The secondary radiation proceeding from one of these substances in a direction perpendicular to that of propagation of the primary is fairly completely polarized, when the rays are of the absorbable type (*d*).

The intensity of secondary radiation from each of these substances varies in different directions perpendicular to that of propagation of a polarized primary beam (*e*).

The amount of polarization in a primary beam, as indicated by the secondary rays, diminishes with an increase in the "hardness" of a given X-ray tube emitting the primary radiation.

The secondary radiation from these substances is approximately twice as intense in the direction opposite to that of propagation of the primary rays as the average in directions at right angles, when the primary beam consists of rays of the easily absorbed type (*g*). (Polarization produces variation in different directions at right angles to the primary beam.)

* Beatty, Phil. Mag. Nov. 1907, pp. 604-614.

† It was considered possible that the discrepancy in the case of hydrogen, as found by one of us, might be explained by the mixture of a small quantity of air. Crowther, however, from more recent experiments has concluded that hydrogen and helium in this group are exceptions to this law of intensity.

This ratio varies somewhat with a variation in the character of the primary rays, but has not been found to exceed 2 : 1.

The results of all the experiments when an easily absorbed primary was used as the exciting beam, may be explained on the theory as given by Professor J. J. Thomson * shortly after the earliest systematic experiments on light gases. The electric displacement in the primary Röntgen pulses when passing over the electrons produces accelerations in these in the direction of that displacement, and thus causes the emission of secondary pulses of equal thickness. The natural deductions from this theory have all been strikingly verified by experiments on substances of low atomic weight when subject to an easily absorbed primary beam.

Before the phenomena of secondary X-rays from these light atoms may be said to be fully understood, we must explain the effects produced when the primary rays change to those of more penetrating type. In experiments that have been described the secondary rays began to differ in penetrating power from the primary,—they were more easily absorbed ; they gave less evidence of polarization of the primary beam, the variation falling from about 20 per cent. to 6 or 7 per cent. in experiments made while the primary became more penetrating ; the ratio of intensity of secondary radiation in the direction opposite to that of propagation of the primary beam to that in one at right angles dropped considerably ; the ratio of ionization in the secondary electroscope to that in one testing the primary beam increased slightly.

These results might be explained qualitatively either by the introduction of a secondary radiation of different type superposed on the almost perfectly scattered, or by the scattering becoming more imperfect by the introduction of forces of considerable magnitude other than those produced directly by the primary pulses during the passage of those primary pulses over the electrons, or by the introduction of a greater proportion of tertiary rays due to

* 'Conduction of Electricity through Gases' (2nd edition) p. 321.

the emergence of the secondary rays from greater depths of the radiating substance, or by a combination of these.

Though the experiments, the results of which are stated above, were not performed concurrently, it was evident that the variation in intensity of secondary radiation exhibiting the polarity of the primary beam changed from about 20 per cent. to only about 6 or 7 per cent. as the X-ray tube became "harder," even before a difference between the penetrating powers of primary and secondary beams could be detected by direct comparison. It was not evident whether this change was actually one in the polarization of the primary beam itself or in the efficiency of the secondary rays in exhibiting a polarization of constant magnitude. The latter appeared the more probable when considered in conjunction with the changes that had been found in the ratio of intensities of secondary rays in a direction almost opposite to that of primary propagation and one at right angles. Further experiments were therefore made to determine if the changes were all attributable to the same cause. It was found, however, that although increasing the hardness of a given X-ray tube produced a decrease in the amount of polarization detected, the more penetrating portion of a primary which was transmitted through a sheet of aluminium did not exhibit less polarity but slightly more, indicating that the effect was due not merely to change in the penetrating power of the radiation but to some change in the polarity of the primary beam itself. This was supported by the fact that the secondary radiation did not become appreciably different in penetrating power from the primary producing it,—indicating a fairly perfect scattering. Finally, later experiments have shown that for a primary radiation proceeding from a tube in the state of hardness which has invariably been found to exhibit a minimum of polarity in the primary, the ratio of intensities of secondary rays in the two directions indicated has been such as would be given by an almost perfectly scattered radiation.

We cannot then attribute the decrease in the amount of polarization of a primary beam exhibited by the secondary rays to the scattered rays being only partially a scattered radiation or to imperfection of scattering, but it is almost

certainly a decrease in the polarity actually existing in a primary beam when the tube becomes harder.

These results are possibly due to the more swiftly moving cathode particles in the X-ray tube being productive of more secondary cathode rays in the anti-cathode. As the secondary cathode particles are not directed like the primary cathode rays, they produce radiation which is not polarized. The greater the number of secondary cathode particles produced, the less is the polarity of the complex radiation. As the X-radiation from the secondary cathode particles is probably less penetrating than that due to the primary cathode particles, the more penetrating portion of the complex X-radiation exhibits slightly more polarity than the more easily absorbed.

The small increase observed in the intensity of secondary radiation from air, paper, &c., as measured by the ionization produced in an electroscope, when the primary beam is made more penetrating, is possibly due to the superposition of a homogeneous unscattered radiation, such as is emitted by elements of higher atomic weight. This would account for the complex secondary radiation differing more and more in penetrating power from the primary as the latter became more penetrating; for it has a definite penetrating power characteristic merely of the element emitting it. Though experiments have not yet been made to analyse this secondary radiation set up by the more penetrating radiation, it appears, for reasons discussed later, exceedingly probable that such a radiation does appear when a penetrating primary beam is used.

A point still awaiting investigation is the change in the observed ratio of intensities of secondary radiation in directions approximately opposite and perpendicular to that of primary propagation. Experiments have been made to ascertain the amount by which this ratio is affected by a change in the polarization of the primary beam, by the superposition of tertiary rays in greater proportion, and by the superposition of homogeneous radiation characteristic of the radiating element. In these later experiments, however, the deviation from the theoretical ratio for perfect scattering has through all the changes made in the primary

beam been much less than in the first experiments ; and it has been found that even a fairly penetrating primary beam—much more penetrating than any used in the experiments referred to in a previous paper (*g*)—sets up secondary rays whose intensity distribution is within a few per cent. of that which would be given by perfect scattering. This matter is being further investigated *.

Cr–Zn Group.

The radiation proceeding from elements of atomic weight between those of chromium and zinc, when subject to X-rays of ordinary penetrating power, is of a very different type from that discussed, for from no two elements in this group is the penetrating power the same. The absorption by a thin sheet of aluminium ·0104 cm. thick is, between these limits of atomic weight, a decreasing function of the atomic weight, varying from 94 per cent. for the radiation from chromium to 64 per cent. for the radiation from zinc. [One of the primary beams used was absorbed to the extent of about 34 per cent.]

Homogeneity.—One of the most remarkable features about the radiation from any one of these elements is that though the primary rays incident upon the substance are very heterogeneous, that is consist of rays varying considerably in penetrating power, the secondary rays are homogeneous. This point has been briefly referred to in a previous paper (*f*).

To give a particular example :—The ionization produced in a given electroscope by a primary X-ray beam was diminished by 51 per cent. by placing a sheet of aluminium ·0208 cm. in thickness in its path ; after 77 per cent. had been absorbed by aluminium, a similar plate produced a

* It ought to be remarked, that though an elementary consideration of the production of secondary rays indicates that the intensity of radiation is the same in the forward and backward directions, and that each of these is double that in a direction at right angles, a more complete theory shows that these results can at best be only approximately true. It is only necessary to consider the action of the magnetic field in the primary pulses on the electrons as they begin to move under the action of purely electric forces, to see that dissymmetry must exist. A complete theory must also take into consideration the distribution of the tubes of electric force round each radiating electron.

further diminution of this ionization by 27 per cent.; after 91 per cent. had been absorbed the same plate cut off only 18 per cent., showing that the rays after each transmission became on the average more and more penetrating. This effect has been explained as due to the more absorbable constituents being sifted out.

Although such a primary beam produced in one of this class of substances the secondary radiation experimented upon, it was found that the secondary radiation was of an entirely different type, being equally absorbed after transmission through sheet after sheet of absorbing substance.

The radiation from thick copper was found not to differ appreciably from that from a very thin sheet which was only thick enough to absorb 14 per cent. of the primary rays. Thus the radiations from the deeper layers after transmission through the surface layers were of the same character as those from the surface layers. Consequently, in dealing with these secondary rays it is not necessary as in the H—S group to deal with very thin plates in order to determine the character of the radiation as emitted by the atoms themselves.

Using zinc as the radiating substance, the direct ionizing effect of the secondary rays was determined, and afterward the ionization produced by the same beam after transmission through thin sheets of zinc and aluminium placed at a distance of several centimetres from the detecting electroscope in order to avoid complications due to the more easily absorbed corpuscular secondary rays from the metal sheet.

The effect of the radiation from air was determined by separate experiments, and correction was then made for this in each observation, though when a large proportion of the secondary rays was absorbed, the air radiation was quite a considerable fraction of the whole and the possible error was as a consequence greatly increased.

Below are tabulated the percentage absorptions by a plate of zinc ·00131 cm. thick and one of aluminium ·0104 cm. thick of the secondary beam direct from zinc and of this beam after transmission through various thicknesses of aluminium. These results exhibit the striking homogeneity of the radiation from zinc.

TABLE I.

Radiation from Zinc (thick sheet).

I. Percentage Absorption by Al previous to absorption in column II. or III.	II. Percentage Absorption by Zn (·00131 cm.) after absorption in column I.	III. Percentage Absorption by Al (·0104 cm.) after absorption in column I.
0	*36·5*	*67·5*
22	36·2	67
67	35·4	—
0	*35·8*	—
89	35·4	67
97	33·9	66
0	*34·2*	—

Similar experiments were made on the absorption of the radiation from zinc by zinc when that radiation had been passed through various thicknesses of zinc to absorb different proportions of it.

The radiation from copper was examined in the same way.

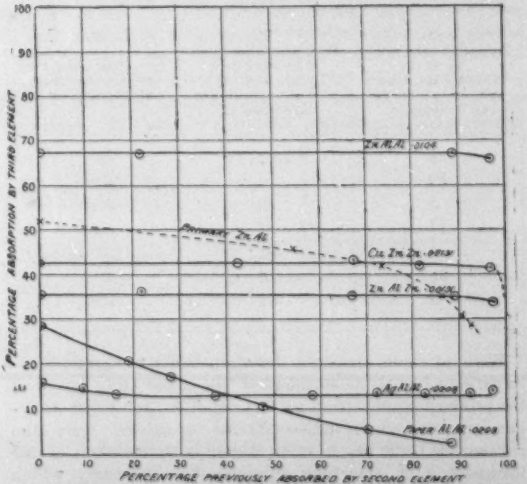
TABLE II.

Radiation from Copper (thick sheet).

I. Percentage Absorption by Zn previous to absorption in column II.	II. Percentage Absorption by Zn (·00131 cm.) after absorption in column I.
0	*43*
44	44
0	*41·4*
42·6	42·6
67·5	43·1
81·5	42
96·7	41·8
0	*42·5*

All these experiments show extremely little change in the percentage absorption even after almost complete absorption. The contrast between primary and secondary beams is strikingly shown in fig. 1, in which the absorptions are represented by ordinates and the amount previously absorbed by abscissæ. The corresponding curve for the secondary radiation from paper (in this case subject to penetrating primary rays) is given for comparison (fig. 1).

Fig. 1.


PERCENTAGE ABSORPTION BY THIRD ELEMENT
PERCENTAGE PREVIOUSLY ABSORBED BY SECOND ELEMENT
Zn Al Al .0104
Primary Zn Al
Cu Zn Zn .0013
Zn Al Zn .0013
Ag Al Al .0009
Paper Al Al .0009

Note.—Cu Zn Zn ·00131 indicates:—Copper radiation after transmission through Zn is absorbed by Zn ·00131 cm. thick to extent shown by ordinates.

Independence of Primary Rays.—To exhibit the independence of the penetrating power of the secondary radiation from one of these metals Cu, of that of the primary producing it, we have tabulated below the absorbability of various primary rays and that of the secondary rays produced

by these. Though the absorption of the primary by aluminium ·0208 cm. thick varied from 52 to 18 per cent., the absorptions of the corresponding secondary beams from copper by aluminium ·0104 cm. thick were as nearly as observable the same, the experimentally determined values being 58·3 and 58·1 per cent. respectively. This constancy in character makes accurate experiments on these radiations possible.

TABLE III.

Radiation from Copper (thick sheet).

I. Percentage Absorption by Al of Primary Radiation previous to incidence on Cu radiator.	II. Percentage Absorption of Primary Rays by Al (·0208 cm.) after absorption given in column I.	III. Percentage Absorption of Secondary Rays by Al (·0104 cm.).
0	..	58
52	..	58·6
0	52	58·3
77	27	58·8
0	51	58·4
91	18	58·1
0	51	58·8

Each of these elements therefore emits a characteristic radiation which is independent of the penetrating power of the primary beam setting up this radiation, and being homogeneous, the absorption-coefficient calculated from the relation $I = I_0 e^{-\lambda x}$ has a value which is independent of the thickness x of absorbing substance *—a property which appears to be unknown among X-ray beams hitherto experimented upon.

The absorption-coefficients for the radiations from Fe,

* After transmission through sheets of absorbing substance, secondary rays are superposed on the transmitted radiation, and the resultant radiation in some cases ceases to be even approximately homogeneous. This effect can, however, be readily distinguished from true heterogeneity by using as absorbers only those substances in which a radiation of different type is not stimulated.

Co, Ni, Cu, and Zn, when absorbed by Al, Fe, Cu, Zn, Ag, Sn, and Pt have been given in a previous paper (*f*).

Though it has been shown that the secondary radiation from some of the substances in this group (Cr–Zn) is remarkably homogeneous in comparison with the primary which produces it, the test applied is not one of extreme delicacy, and the presence of the scattered radiation similar to that of the first group (H–S) and of intensity given by the law found for that group would be exceedingly difficult to detect, as it would produce only about $\frac{1}{300}$ of the total ionization actually produced by the secondary rays from a very thin sheet of copper. Now we have seen that a radiation after transmission through metals may contain a considerable proportion of the radiation which is characteristic of the metal traversed and which was not in the incident radiation. This acquired radiation may even, if the substance

TABLE IV.

Radiation from Copper (thick sheet).

I. Percentage Absorption by Cu previous to absorption by Cu, Zn, or Al.	II. Percentage Absorption by Cu, Zn, and Al after absorption in column I.		
	Cu ·00295 cm.	Zn ·00262 cm.	Al ·0104 cm.
0	74	69·7	72
98·3	70	66	68

Radiation from Iron (thick sheet).

I. Percentage Absorption by Fe previous to absorption by Fe, Cu, or Al.	II. Percentage Absorption by Fe, Cu, and Al after absorption in column I.		
	Fe ·00313 cm.	Cu ·00296 cm.	Al ·0104 cm.
0	80·5	91·5	90
80	79·8	91	89·5
96	76·9	86	83·5

traversed has almost completely absorbed the incident radiation, constitute the bulk of the transmitted radiation. In such a case, experiments on the transmitted beam in order to analyse its constituents would be misleading, the constituents not being present in the original beam.

To eliminate the possibility of this error, we have tested the radiation from copper after transmission through thin sheets of copper by absorbing by further plates of copper. Thus radiation of a fresh type was not introduced.

It was found that after absorbing 98·3 per cent. of the copper radiation by copper, the absorption by copper ·00296 cm. thick had dropped from 74 to 70 per cent. The effect was more pronounced in the case of the radiation from iron, the numerical data for which are given below.

There was thus evidence of a slight heterogeneity even in these radiations.

Scattered Rays.—To test if this heterogeneity could be accounted for by the mixture of a scattered radiation, like that from light atoms, with the homogeneous radiation, a direct comparison was made between the ionization produced by the secondary beams from thin sheets of copper [·00067 cm.] and paper, subject to the same primary radiation. The paper which was used had ten times the mass of the thin copper, yet the ionization produced by the secondary rays from the copper was 19·5 times that produced by the secondary rays from paper even though a greater proportion of the radiation from copper was absorbed by the copper itself than that from the paper in the paper. As sheet after sheet of aluminium was placed in the path of the two secondary beams, the copper radiation was absorbed to a much greater extent than the radiation from paper; thus 34 per cent. of the radiation from paper was transmitted through aluminium ·0416 cm. in thickness, while only ·7 per cent. of the copper radiation got through. The ratio of the ionization due to the copper radiation to that due to the rays scattered from paper, after transmission through ·0416 cm. Al had dropped to ·401 : 1. After transmission through ·0782 cm. of Al the two radiations appeared approximately equal in penetrating power, the last ·26 per cent. of the

copper radiation being much more penetrating than the homogeneous radiation which had been practically all absorbed. When the two transmitted radiations were approximately of the same character, the ratio of their ionizing effects was about ·18 : 1 ; or from equal masses of copper and paper—disregarding all internal absorptions the intensities of the penetrating rays were in the ratio 1·8 : 1.

TABLE V.

I. Thickness of Al in path of Secondary beams.	II. Percentage of Cu radiation absorbed.	III. Percentage of Paper radiation absorbed.	IV. Ratio of ionizations due to Secondary beams from equal masses of Cu and Paper.
0	0	0	195·5 : 1
·0208	94·5	51·8	22·1 : 1
·0416	99·3	66	4·01 : 1
·0574	99·6	71	1·94 : 1
·0782	99·74	78	1·88 : 1
·0990	99·8	83	1·83 : 1

Allowing for the small extra absorption of the penetrating portion of the copper radiation in the copper plate itself above that of the corresponding rays in the paper—quantities which were determined by separate experiments—the radiation from copper must have contained approximately twice as much of the penetrating radiation as the radiation from paper.

By using a thicker copper plate as radiator the intensity of secondary radiation was increased, but the correction for absorption in the metal itself was also increased so that the result could not be regarded as more accurate, this correction not being obtainable with great accuracy. The conclusion was however practically identical with the above. We thus see that mixed with the homogeneous radiation from copper is a more penetrating radiation. The most penetrating portion of this is about twice as intense as the corresponding radiation from substances of the H–S group.

The radiations from other elements of the Cr–Zn group have not been examined so minutely as that from copper, but it has been seen that these also contain a small quantity of a more penetrating radiation which is probably scattered radiation. This more penetrating radiation was more evident in the radiation from thick iron, probably because the homogeneous iron radiation being very absorbable emerges from a thinner surface layer, while a scattered radiation emerges from greater depths than in the metals of higher atomic weight owing to the greater transparency of iron than substances of higher atomic weight. Consequently the scattered rays—if we may assume them to be such—are in reality from a much greater mass of iron than the homogeneous rays, and produce more than their appropriate portion of the total ionization.

It may be objected that in this case the radiation was transmitted through a second substance Al, and may have contained a considerable quantity of secondary radiation from Al of a kind not existent in the original radiation from copper. The production of a secondary radiation more penetrating than the primary producing it is, however, contrary to all experience.

Energy.—The ionization produced by the secondary rays from one of the elements of this group has been shown to be enormous in comparison with that produced by the secondary scattered rays from an equal mass of an element of low atomic weight. From a sheet of copper ·00067 cm. thick, absorbing 14 per cent. of the primary radiation, the secondary radiation produced an ionization in the detecting electroscope 200 times as great as that from an equal mass of paper. Correcting for absorption of primary and secondary rays, the ratio of ionization produced by the rays from equal masses of copper and paper was approximately 300 : 1. This is considerably greater than would have been found if all the radiation absorbed had been simply scattered as an untransformed radiation. This, however, by no means gives us a measure of the energy of the secondary rays, for these are of much more absorbable type than the primary. The percentage absorption by a thin sheet of aluminium is about five times as great for these secondary rays as for the primary.

If we assumed the same ratio for the ionizations produced in air by the two radiations if of equal intensity, we should be led to conclude that the energy of this homogeneous radiation is about 45 times that of the scattered radiation from an equal mass of paper, and about $\frac{1}{3}$ the total energy absorbed in the copper *.

Though it is impossible by such experiments to determine the energy with accuracy, we may safely conclude that the energy of the homogeneous radiation is many times greater than the energy of secondary radiation scattered from an equal mass of one of the light elements.

Comparisons of the ionizations produced by the rays from other elements of this group have been made. They are all of the same order of magnitude.

Distribution.—It has been shown by one of us that the secondary radiation from thick copper, when this is subject to a primary beam of ordinary penetrating power, is approximately equally intense in a direction almost opposite to that of propagation of the primary and in a direction at right angles. As probably 98 per cent. of the ionization produced by the secondary radiation from thick copper is due to the homogeneous rays, this may be said to be the result for the homogeneous rays alone.

From thick iron, however, the radiation varied in intensity by an amount represented by the ratio 1·1 : 1 in these two directions. But as we have shown, the heterogeneity of the radiation from thick sheets of iron is more marked, and this can be accounted for by the fact that from iron the homogeneous rays are very easily absorbed, consequently scattered rays emerge from a much thicker layer and appear in more than their normal proportion. The ratio 1·1 : 1 verifies this by showing the presence of a radiation which is controlled by the electric field in the primary pulses.

The result is of the order of magnitude that would be given by a mixture of scattered rays of about the same

* The homogeneous radiation has been assumed to be distributed uniformly in all directions and the scattered radiation to be proportional to $\sin^2\theta$, where θ is the angle between the direction of radiation considered and that of acceleration of the radiating electron.

intensity as found in the radiation from copper with the characteristic radiation uniformly distributed.

Polarization Experiments.—Though very careful experiments have been made with iron, copper, and zinc as secondary radiators placed in a partially polarized primary beam of Röntgen radiation, the secondary rays from these have not been found to give evidence of any polarity. Thus the intensity of secondary radiation in a given direction is independent of the position of the plane of polarization of the primary beam producing the radiation; in other words, the intensity of secondary radiation from members of this group is independent of the direction of electric force in the primary radiation. Again, this result may be taken as applicable to the homogeneous rays, as these constitute the bulk of the radiation from these metals.

Efficiency of Primary Rays as Secondary Ray Producers.—Although in addition to the relatively small amount of scattered radiation, rays of only one penetrating power were emitted by an elementary substance upon which a heterogeneous primary beam fell, it was still possible that only one constituent (rays of one penetrating power) in each heterogeneous primary beam was producing this radiation. It was therefore important to determine to what extent each constituent of the primary beam was effective in producing these secondary rays.

To do this, a portion of the primary beam direct from the X-ray tube was sent through one electroscope while another portion was incident on a secondary radiator, some of the rays from which passed through a second electroscope. The method was then simply to place absorbing plates in the primary beam before falling on the radiator, and to observe the extent by which the primary and the secondary radiations were reduced. It is obvious that the more penetrating constituents of the primary beam penetrate to greater depths than the absorbable constituents, and so are really transmitted through a greater mass than these. Hence, if two homogeneous constituents of primary radiation in passing through equal masses of radiating substance were equally efficient as secondary-ray producers, the radiation emitted by a thick plate would be produced principally by the more

penetrating constituents; consequently an absorbing plate placed in the position indicated, would, by cutting off the more easily absorbed constituents, produce less diminution of the ionization in electroscope E_2 than of that in E_1. It was therefore necessary to use as the radiator a sheet of metal which would absorb very little of the primary radiation, so that even the deepest layers would transmit different constituents in proportions approximately the same as those transmitted through the first surface-layer.

It was found that a sheet of Cu ·00067 cm. in thickness when placed in the primary beam produced an absorption of only 14·5 per cent., as measured by the ionization produced in an electroscope. This was considered sufficiently thin for use as a radiator. The deflexions of the electroscopes were first observed when no absorbing plate was used. Aluminium

TABLE VI.

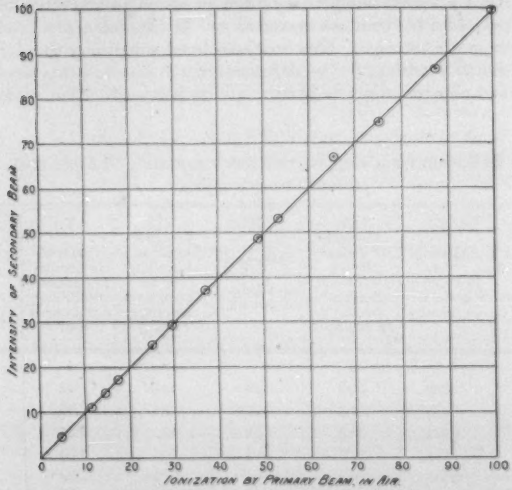
Radiation from Copper (thin sheet absorbing 14·5 per cent. of primary rays.)

I. Thickness of Al in Primary beam in centimetres.	II. Absorption of Primary by Cu (·00293 cm.) after passing through Al in column I.	III. Percentage of Primary absorbed by Al in column I.	IV. Percentage diminution of Secondary by Al in column I. absorbing Primary.	V. Ratio of Ionizations in Secondary and Primary electroscopes (relative).
0	84	0	0	1
·0032	82·8	12·2	13·6	·98
·0064	81·4	24·9	25·5	·99
·0104	80·3	34·2	33·1	1·01
·0168	77·5	46·7	46·6	1·00
·0208	75·9	51·1	51·4	·99
·0312	72·3	62·9	62·9	1·00
·0416	69·6	70·2	70·5	·99
·0520	65·2	74·6	75·0	·98
·0728	60·4	82·4	82·3	1·00
·0936	57 0	85 8	85 8	1·00
·124	51·2	88·6	88·9	·97
·248	41·7	95·4	95·3	1·02

plates of thickness shown in column I of Table VI. were placed in the primary beam, and the deflexions of the primary and secondary electroscopes were diminished by the amount given in columns III and IV.

In fig. 2 the ionization produced by the primary beam in air is indicated by abscissæ and that produced by the corresponding secondary beam from a thin sheet of copper as ordinates, when different portions of the primary beam have been absorbed.

Fig. 2.



As the secondary beam is practically homogeneous we have exhibited the relation between the ionization produced by a primary beam in the air, and its power of producing secondary rays in copper. We are thus led to the conclusion that the intensity of this homogeneous secondary radiation set up in a thin sheet of copper is proportional to the number of ions the primary beam would produce in a thin layer of air.

Thus if two beams of Röntgen rays, which in passing through a thin film of air would produce equal ionizations in that air, be sent through a thin sheet of copper, the intensity of secondary radiation produced on that sheet by one beam will equal that produced by the other, even though one is three or four times as penetrating to copper as the other one. As for such ranges of penetrating power as are possessed by the constituents of such a primary beam, the relative ionizations produced by those constituents in different substances are usually fairly constant, it is highly probable that the intensity of the homogeneous rays from copper is through wide ranges of penetrating power of the primary proportional to the ionization which takes place in the copper, and is independent of the character of the primary producing it.

It does not necessarily follow from this that the process of ionization produces the radiation, but it seems to indicate that the energy of the homogeneous secondary radiation is, for such ranges in penetrating power, proportional to the energy of the primary beam spent in the process of ionization.

When the primary radiation was transmitted through copper or iron before falling on the copper radiator, the ratio of intensity of secondary radiation to ionization produced by the primary beam dropped slightly, showing that the radiation transmitted through these substances was less efficient as a secondary-ray producer in comparison with its power of producing ionization in air. This may have been due to the fact that in transmission through these plates, a gradually increasing quantity of secondary radiation from the plates was superposed on the primary. This being in one case of the same penetrating power as the secondary emitted by the radiator, and in the other more easily absorbed than it, would not afterwards produce secondary rays in the copper (see later). When the radiation was absorbed by zinc, the ratio remained fairly constant. This result would, on the corresponding theory, be due to the fact that the radiation from zinc is slightly more penetrating than that from copper, consequently the zinc radiation acquired in the primary beam would be capable of stimulating a feeble secondary in copper. (See Table VII. p. 356.)

TABLE VII.

Radiation from Copper (thin sheet absorbing 14·5 per cent. of primary rays).

I. Thickness of plate in Primary beam.	II. Percentage of Primary absorbed by Zn in column 1.	III. Percentage diminution of Secondary by Zn in column 1 absorbing Primary.	IV. Ratio of Ionizations in Secondary and Primary electroscopes.
Zn ·00131 cm.	59·4	59·6	·99
·00262 ,,	77·5	77·6	·99
·00393 ,,	84·8	84·8	1·00
·00524 ,,	89·1	90·1	·91
·00786 ,,	93·5	93·6	·98
·01179 ,,	97·2	96·3	1·32
Cu ·00067 cm.	43·5	45·1	·97
·00134 ,,	65·0	66·5	·95
·00296 ,,	82	84·8	·84
·00592 ,,	90	91·6	·84
·01184 ,,	97	98	·66
Fe ·00315 cm.	77·2	80·8	·84
·00630 ,,	89·0	91·4	·78
·00945 ,,	93·0	95	·71

Whatever may be the true explanation, these results show that the proportionality exhibited after absorption by aluminium is not general even when copper is the radiating substance. This result may be contrasted with the corresponding phenomena exhibited by the homogeneous radiation from silver. In these experiments the homogeneous rays disappeared when the primary beam was made more absorbable and more efficient as an ionizer of air. The difference is almost certainly due to the fact that in the latter case the primary radiation passes from one more penetrating to one more easily absorbed than the secondary radiation characteristic of the element exposed to the primary rays. It seems highly probable therefore, that with sufficiently absorbable primary rays no such relation as that shown by fig. 2 would be obtained, but that the secondary would disappear while the primary still produced considerable ionization in air. It will be interesting to learn if a primary which ceases to stimulate the homogeneous radiation ceases also to produce ionization in the radiating substance. We, of course, know that elements of the H–S group are ionized, but do not

emit a homogeneous radiation which can be detected. This appears to indicate no necessary connexion between ionization and radiation, but is not conclusive for various reasons.

Special Penetrating Power.—In studying the absorption of these homogeneous radiations by a number of elements, it was found that the relation between the absorption by a given substance of the various homogeneous radiations from elementary substances and the atomic weight of those radiating substances was similar for all absorbing substances, except in the case of the radiating and absorbing elements being identical or possessing neighbouring atomic weights. Each substance appeared to be especially transparent to its own radiation, and to a less extent to that from elements of neighbouring atomic weight.

The transparency of various elements to an ordinary heterogeneous beam of X-rays has been investigated by Benoist, who studied the phosphorescence produced by the beam after passing through absorbing substances. He compared the transparencies by finding the mass of a prism of absorbing substance of definite cross-section, which when placed in the path of the beam absorbed it by a definite amount. The relation between the transparency so defined and measured, and the atomic weight of the absorbing substance was shown by a curve similar to that for paper given in fig. 3 (p. 358). It shows a rapid decline of transparency, with increase in atomic weight for low atomic weights, the rapidity of the fall of transparency diminishing with an increase of atomic weight. In the case of very soft rays this decline becomes a slight incline.

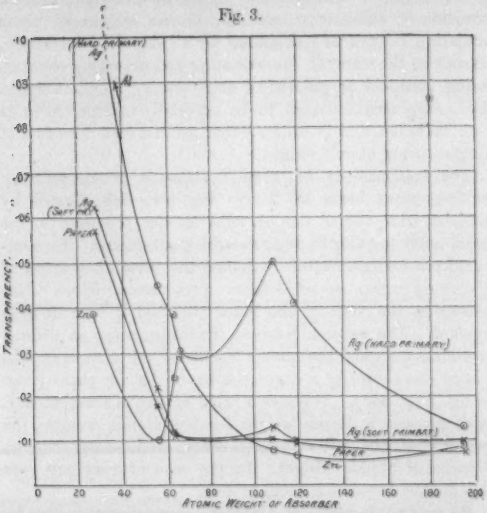
By using thin plates of absorbing elements and testing by the ionization method the percentage diminution of intensity of the secondary beams by transmission through these plates, it was easy to calculate the thickness and hence the mass per unit cross-section necessary to absorb a given proportion —in this case 75 per cent. Some of these results we have given in a previous paper (*f*)*. They show that in place of the usual relation between transparency and atomic weight there is a strongly marked deviation in each case in the

* In the paper referred to the numbers given in Table II. represent mass in grammes, not thickness in centimetres as stated.

neighbourhood of the atomic weight of the radiator, the rays from an element being especially penetrating to that element and to a less extent to elements of neighbouring atomic weight.

That this special penetrating power is not due to the constituent easily absorbed by the radiating substance having been sifted out before emergence through the surface layer

Fig. 3.



might be inferred from the fact of its homogeneous character, unless of all the constituents in the radiation as emitted from the atom itself this was the only one transmitted through even fairly thin layers of the substance. But we find other secondary rays transmitted with almost the same facility. We may conclude, therefore, that it is not merely from the surface of relatively thick sheets that the radiation emerges in a homogeneous state, but that it is so emitted by the atoms themselves. The special power of penetration is thus a specific property of the secondary rays and does not appear to be due to previous selective absorption.

Tertiary Rays.—Some of the most interesting phenomena in connexion with these homogeneous rays were those shown by experiments made in order to investigate the special penetrating power of these rays.

When the characteristic homogeneous radiation from iron was passed through a thin sheet of copper—a substance whose characteristic radiation is of more penetrating type—it was partially absorbed, and the transmitted radiation appeared unchanged in character.

When, however, the radiation from copper was passed through iron there was superposed on the copper radiation a considerable quantity of iron radiation; when the thickness of the absorbing plate of iron was sufficient to produce almost complete absorption, the bulk of the transmitted radiation was iron radiation.

Similar experiments were made on copper and zinc. The radiation from the former, which is more absorbable than that from the latter, when passed through zinc was transmitted without admixture of other radiations, but when the radiation from zinc was transmitted through copper it became more like the copper radiation. The effect was not so well marked as in the case of the transmission of copper radiation through iron.

Thus a characteristic homogeneous radiation was emitted by a metal when the primary beam to which the metal was exposed was of more penetrating type than the characteristic radiation. When the primary was of a more absorbable type, this characteristic secondary radiation was not emitted*.

Ag–I Group.

When subject to primary beams of moderate penetrating power such as those used in these experiments, the elements of atomic weights from that of silver to that of iodine are exceedingly susceptible to small changes in the penetrating power of that primary radiation, both as regards the character and the intensity of the secondary rays that they emit. This is indeed the most remarkable feature of this group of elements.

Silver and tin are the only elements of this group which have been examined in any detail, but they appear from

* Details of these experiments will be given later.

superficial observation to be typical of the whole group. In previous papers it has been shown that the radiation from these two when subject to a primary beam of only moderate penetrating power, is not a scattered radiation like that which proceeds from elements of the H–S group, for it differs much more in penetrating power from the primary producing it than the radiation from the elements of that group, though not so much as the radiation from Cr, Fe, Cu, &c. The radiation does not exhibit the polarity of a primary beam where such exists, and it is not distributed in the manner of the radiation from substances of low atomic weight, the intensity in a direction almost opposite to that of primary propagation being approximately equal to that in a direction at right angles. It was not possible, however, to perform this experiment with anything like the degree of accuracy with which it was done in the case of the radiation from Fe, Cu, &c., because the ionization produced by the secondary beams from Ag and Sn was much less intense.

Secondary Scattered Rays.—Experiments with the softest rays procurable from an X-ray tube of ordinary type, however, considerably simplified the secondary radiation, for it was found that the secondary rays from silver then differed very little in penetrating power from the soft primary. When the intensity of secondary radiation set up by a powerfully polarized primary was examined in the two principal directions at right angles, it was found that it varied by approximately the same amount as that from elements of the H–S group, thus exhibiting the same perfection of scattering. When tin was examined in the same way, the variation in intensity was found to be about half that exhibited by the rays from paper and from silver. Thus by using a very soft primary radiation, an almost purely scattered secondary radiation was emitted by silver, and a radiation consisting of a large proportion of scattered rays was emitted from Sn.

To compare the intensity of this scattered radiation from silver with that from elements in the H–S group, the ionization produced by the secondary radiation from a thin sheet of silver ·00064 cm. thick was compared with that produced by the radiation from paper. The masses of silver and paper emitting the secondary rays were ·2695 gr. and

2·72 gr. respectively. The relative ionizations produced by the secondary rays from these were 148 and 280. From other experiments on the absorption of the primary beam by sheets of silver and paper of different thicknesses, it was estimated that about 60 per cent. of the silver radiation and 75 per cent. of the paper radiation was transmitted through the surface layer. The intensities of radiation from these masses were, therefore, in the ratio 37 : 56 approximately. From equal masses this was 37 : 5·6, or 6·5 : 1 approximately.

As the primary radiation incident on silver became more penetrating, the ionization produced by the secondary rays increased enormously, the variation of intensity in the two principal directions at right angles to that of propagation of the primary rays gradually decreased from its original amount—about 14 per cent.—until it was inappreciable, though with the same primary the variation in the intensity of radiation from carbon had only dropped to about 6·5 per cent. The evidence of polarity of the primary given by the secondary rays from silver thus disappeared, while that given by the rays from carbon simply decreased from 14 per cent. to 6·5 per cent. It should, however, be noticed that the ionizing effect of the secondary rays from silver increased considerably, and when the evidence of polarity disappeared the total secondary ionization had increased about tenfold, so that such an effect would have been produced by the superposition of the homogeneous radiation, like that emitted by Cu, Fe, Zn, &c.; for such a radiation gives no evidence of polarity in a primary beam, being uniformly distributed around that beam *.

* An early experiment by one of us on the radiation from tin (*e*) indicated that the scattered radiation was not emitted in even the intensity that could be given by an element of the H–S group. The experiment was performed by comparing a very penetrating portion of the radiation from tin with that from paper. There are possibilities which make such an experiment inconclusive, and we feel that it requires verification. If the numerical values for the polarization given in Table VIII. (p. 362) could be taken as strictly accurate throughout, we should be led to conclude that the evidence of polarity disappeared more rapidly than could be accounted for by mere superposition of the homogeneous radiation, but the measurements in the final stages were too uncertain for such a conclusion to be based on them. We have, therefore, no conclusive evidence of a disappearance of the scattered radiation.

The evidence of polarity of the primary given by the radiation from tin disappears for even softer primary rays than in the case of silver. This must be connected with the fact that the characteristic homogeneous radiation from tin is less penetrating than that from silver, and is set up by a less penetrating primary radiation. It thus appears earlier in the process of hardening the primary and swamps the effect of the scattered radiation sooner.

TABLE VIII.

I. Absorption of Primary by ·01 Al.	II. Ratio of Ionizations in Secondary and Primary electroscopes.		III. Percentage Variation of intensity of Secondary radiation exhibiting polarity of Primary beam.	
	C radiator.	Ag radiator.	C radiator.	Ag radiator.
35·5	·303	·180	14	11·5
32·6	·340	·359	10·95	4·45
32·3	·318	·305	12·9	6·2
32·0	·315	·369	12·9	4·5
30	·349	·821	9·9	2·8
29·4	·353	·844	9·6	2·45
28·6	·364	1·12	9·15	·9
28·9	·380	1·35	7·5	·75
27·3	·405	1·62	7	·45
25·6	·425	1·82	6·5	0

Homogeneous Rays.—An analysis of the radiation from silver similar to that made of the radiation from elements of the Cr–Zn group shows that when the primary radiation is moderately penetrating, such a homogeneous secondary radiation constitutes the bulk of the rays emitted. Absorption by thin sheets of aluminium showed slight heterogeneity at first, such as would be evident if the scattered radiation were superposed on the more penetrating homogeneous radiation. After the absorption of this more easily absorbed scattered radiation the remainder appeared perfectly homogeneous.

The contrast between the constitution of the copper and silver radiations is shown by the curves given in fig. 1 (p. 345).

In the copper radiation the homogeneous rays are more easily absorbed than the scattered rays, so the curve is initially horizontal and finally slopes downwards. In the silver radiation the scattered rays are on the average more easily absorbed than the homogeneous rays, consequently the curve dips initially and finally becomes horizontal. The absorptions at successive states are shown in the following tables. With these results may also be contrasted those obtained for the scattered radiation from paper.

TABLE IX.—Radiation from Silver (thick).

Thickness of Al placed in Secondary beam from Ag.	Percentage diminution of Ionization produced by Secondary rays by transmission through Al in column 1.	Percentage diminution of Ionization by Secondary rays due to absorption by further sheet of Al (·0208 cm.).
0	0	15·9
·0104 cm.	9·2	14·8
·0208 „	16	13·8
·0574 „	37·7	13·1
·124 „	58	13·5
·182 „	72	14
·263 „	82·5	13·6
·387 „	92	13·8
·526 „	97	14·2
0	0	16

TABLE X.—Radiation from Paper.

Thickness of Al placed in Secondary beam from Paper.	Percentage diminution of Ionization produced by Secondary rays by transmission through Al in column 1.	Percentage diminution of Ionization by Secondary rays due to absorption by further sheet of Al (·0208 cm.).
0	0	28·1
·0104 cm.	18·7	20·9
·0208 „	28	17·4
·0574 „	47·5	10·9
·182 „	70	5·8
·584 „	88	2·5
0	0	27·4

W–Bi Group.

The radiations from the elements with atomic weights from that of tungsten to that of bismuth have not been examined minutely. They, however, appear to be very similar to the rays from Cu, Zn, &c. The ionization produced by these rays is of the order of magnitude of that produced by the rays from elements of the Cr–Zn group; but there appears slightly more variation in the character due to changes in the primary rays.

Though accurate observations have not been made, it appears probable that the radiation is a mixture of the scattered with the homogeneous rays, the proportional effect of the scattered being greater than in the radiation from Cu, Zn, &c.

Conclusions.

Secondary Röntgen rays of two distinct types are emitted by substances subject to a beam of X-rays. One, a scattered radiation produced by the motion of electrons controlled by the electric force in the primary Röntgen pulses, has been dealt with in previous papers by one of us, and has been further discussed in this paper with the H–S group of elements. The other, a homogeneous radiation characteristic of the element emitting it, and produced by the motion of electrons uncontrolled by the electric force in the primary pulses, has been but briefly mentioned.

All the phenomena of secondary X-rays so far observed by us may be explained by means of these two.

The experimental results of these investigations, both on the scattered radiation and the homogeneous radiation, are summarized below. In order to make the summary more complete, we have introduced several results which have been previously published. The references for these are given.

Experimental Results.—Scattered X-rays—those produced by the motion of electrons controlled by the primary pulses—constitute the bulk of the secondary radiation from elements of the H–S group when these are subject to a primary beam of low to moderate penetrating power.

Scattered X-rays are also emitted by many elements of

higher atomic weight—probably by all—when subject to such a primary beam; but unless the primary is very soft they are accompanied by homogeneous secondary X-rays (characteristic of the radiating element) which produce much greater ionizations.

The law of intensity of these scattered rays which holds for elements of the H–S group—that the intensity of radiation from an atom is proportional to its atomic weight—cannot be extended to include the elements of higher atomic weight. From some elements at least the intensity is greater than would be given by this law.

The scattered radiation from some elements whose characteristic homogeneous radiation is of comparatively penetrating type has been obtained free from admixture with this homogeneous radiation by the use of a primary beam consisting of less penetrating rays than the characteristic secondary.

Those scattered radiations not thus isolated have been accompanied by a homogeneous radiation more easily absorbed than the primary radiation producing them.

We have obtained no conclusive evidence that the relative intensity of secondary scattered and primary radiations changes with the penetrating power of the primary rays.

All elements of atomic weight greater than that of sulphur which have been examined emit a homogeneous secondary radiation when subject to a primary beam of X-rays of ordinary penetrating power. Cr, Fe, Co, Ni, Cu, Zn, Ag, have been examined. All other elements whose secondary radiations have been examined less minutely appear similar in this respect.

The penetrating power of this radiation from each element examined has been found independent of the intensity or the penetrating power of the primary radiation producing it; it is characteristic of the element emitting it.

The penetrating power of this radiation is a periodic function of the atomic weight of the radiating element (e).

The ionizing power of this radiation and almost certainly its energy is usually very much greater than that of the scattered radiation.

The homogeneous radiation has invariably been found more easily absorbed than the primary radiation producing it.

In all cases, when a primary was used which was softer than the characteristic homogeneous radiation, this radiation was not emitted. Also there is reason (from the curve connecting absorbability of a secondary radiation and the atomic weight of a radiator) for believing that those elements H–S which do not under ordinary circumstances emit such radiation, possess a characteristic radiation which is more penetrating than any primary beam used.

The intensity of this homogeneous radiation from copper is for a considerable range in the penetrating power of the primary merely proportional to the ionization produced by that primary in a thin film of air and is otherwise independent of the penetrating power of the primary. This is not general, as in many cases—probably all—the homogeneous radiation disappears when the primary radiation becomes more absorbable.

The intensity of the homogeneous rays in a given direction does not depend appreciably on the position of the plane of polarization of primary beam producing them.

The intensity of this radiation in a direction approximately opposite to that of propagation of the primary beam producing it, is within the small errors of experiment equal to that in a direction at right angles (*g*).

This radiation is specially penetrating to the element which emits it and to a less extent to elements of neighbouring atomic weight (*f*).

The fraction of the homogeneous rays from one element—copper—scattered by air, is the same as that for X-rays proceeding direct from an X-ray tube (within experimental errors) (*b*).

The absorbability of the secondary rays from copper which are scattered by air—tertiary rays—is the same as that for the direct secondary (*b*).

Theory.

The theory of the scattered X-rays has been dealt with in various papers, and the experimental evidence in support of that theory—briefly referred to in this paper—is so overwhelming that it need not be further discussed here.

It is important, however, to consider the evidence we have regarding the nature and origin of the homogeneous rays, which are characteristic of the elements emitting them.

The fact that the homogeneous secondary rays from copper are scattered by air in approximately the same proportion as the primary rays proceeding direct from an X-ray tube, and that the absorbability of these scattered rays is the same as that of the direct secondary rays, is strong evidence that they are of the nature of X-rays, for neither the observed intensity, nor the scattering without degradation, would have been expected on any corpuscular theory, whereas they are in perfect harmony with the æther pulse theory.

The relation between the absorption of a mixture of the homogeneous rays from a number of elements by various elements and the atomic weight of those absorbing elements is also very different from that found for any material radiation, while it is very similar to that obtained by experiments on a beam of Röntgen rays *.

The special powers of penetrating certain substances are such as have not been observed and are difficult to conceive of on any corpuscular theory.

Neither electrostatic nor magnetic deflexion of these rays has been observed.

Finally, the fact that the homogeneous rays are invariably produced by primary rays of more penetrating type, yet not necessarily more than just on the more penetrating side, appears some of the strongest evidence in favour of similarity in type between the primary and secondary rays. If the natures were different, the penetrating powers would represent totally different physical facts, and such connexion between them would be inconceivable. We can only conclude from consideration of this evidence that the nature of the homogeneous rays is similar to that of the primary X-rays.

As the homogeneous rays are of the nature of Röntgen rays, we must conclude that the radiation is set up by disturbance of electrons produced directly or indirectly by the passage of the primary pulses. That this motion of electrons

* Details of these experiments have not been given.

is not controlled by the electric forces in the primary pulses, is proved by the equality of the intensities of radiation in a direction approximately opposite to that of primary propagation and one at right angles, by the absence of evidence from the secondary rays of polarization of a primary beam in which such polarization exists, and by the absence of dependence of the penetrating power of the secondary beam on that of the primary.

The forces called into play which produce the accelerations resulting in radiation cannot then be directly due to the electric displacement in the primary pulses, but must be those called into play in the atom itself. The two possibilities that suggest themselves are that the radiations result from a disturbance of the atom, which quickly recovers its normal configuration, or that it is produced when the equilibrium of an atomic system is destroyed and forces of unusual magnitude are called into play.

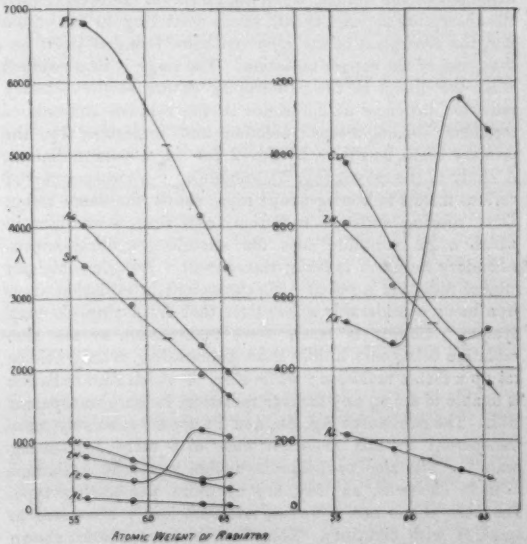
The homogeneity of the radiation and its independence of the primary rays suggest a regularity in the motion which is characteristic simply of the atom; and although the relation between intensity of secondary radiation from copper and ionization produced by the primary in air is striking, it does not follow that the radiation is due to ionization in the radiating substance. Indeed, this appears highly improbable, for the homogeneous radiation disappears when the primary radiation is made "soft" and appears in great intensity when the primary is "hard."

If, then, the radiation were emitted simply during the process of ionization and were proportional to it, a soft primary radiation would produce little or no ionization and a more penetrating radiation—(that is one more penetrating to most substances)—would produce an ionization at least hundreds of times as great. Though experiments have no been made on the ionizations produced in these substances investigated, such variations in ionization are of a higher order of magnitude than any observed. Again, ionization is undoubtedly produced in substances of the H–S group of elements when homogeneous rays are not emitted in appreciable intensity. We, however, do not know anything

of the homogeneous rays from these substances except that they are probably of penetrating type.

The relation between the absorption of a primary radiation and the intensity of secondary radiation emitted by the absorbing substance, also shows that the emission of homogeneous radiation necessitates a special absorption of the primary beam. Also, as far as we can estimate from experiments made, the energy of the homogeneous radiation is more than accounted for by the special absorption necessary to the production of that radiation.

Fig. 4.



Thus in fig. 4 we have plotted the coefficients of absorption of rays from Fe, Co, Ni, Cu, and Zn by Al, Zn, Ag, Sn, Pt In each case the radiations absorbed are not—if we generalize

from results given—capable of stimulating a homogeneous radiation characteristic of the absorbing substance, because these are of more penetrating type. The curves are similar. But when iron is used as the absorbing substance, cobalt radiation, which is just more penetrating, is unable to produce more than a feeble secondary in the iron, so the decline of the line for Fe is not quite so great as would have been expected, because of the little extra absorption necessary to set up this slight radiation from iron. The copper radiation, which is much more penetrating, is able to stimulate an intense radiation in iron and at the same time is enormously absorbed. This is exhibited by the sudden rise in the Fe curve. Beyond copper, the absorption appears to fall again according to the usual law, the absorption of the zinc radiation being a little less than that of the copper radiation. The stage is thus reached when the change in the penetrating power of the primary radiation produces no difference in the relative amounts of secondary homogeneous radiation and ionization by the primary beam, as shown by fig. 2 for the copper radiation. A study of the curves (fig. 3), exhibiting the transparency of various metals to homogeneous rays, shows the same thing. Thus, dealing with the radiation from zinc, aluminium, in which a Zn radiation does not stimulate, a homogeneous secondary radiation is fairly transparent. Iron, in which an intense radiation is set up—the characteristic radiation from iron being considerably softer than that from zinc—is very opaque. Copper is much more transparent, as the zinc radiation being only a little more penetrating, is only able to set up a feeble radiation ; while zinc, in which zinc radiation is unable to set up any further radiation, is more transparent still. The substances Ag, Sn, and Pt appear relatively more transparent to this radiation than to a more penetrating beam*. The zinc radiation is unable to set up radiations in these elements, as they are of more penetrating type. The absorptions are, however, not sufficiently abnormal to speak of with certainty. The effect is more clearly shown by the curve obtained from the homogeneous radiation from silver (fig. 3).

* Also Phil. Mag. Sept. 1907, p. 416, fig. 5.

These results show that a substance whose characteristic radiation is equally or more penetrating than the radiation incident upon it, does not absorb that radiation so much as when the incident is of more penetrating type and able to stimulate a secondary radiation in that substance. As the penetrating power of the incident radiation (as measured by most substances) increases, the absorption increases up to a certain point. A definite portion of the absorption thus appears to be connected with the secondary radiation, and may be proportional to it. But this does not conclusively show that ionization, or some kind of disruption in the atom is not the cause of the radiation, for the absorption of a certain amount of energy of the primary beam might be required to produce the instability which liberates more energy.

The facts that the homogeneous rays have invariably been found more easily absorbed than the primary rays producing them, and, in the cases investigated, that the homogeneous rays disappear when the primary becomes more easily absorbed, indicate a relation between primary and secondary which would be difficult to explain if this radiation were directly due to a disruption of any kind taking place in the atom. It appears rather that the radiation is due to what may be regarded as quite a normal behaviour of an atom after it has been passed over by Röntgen pulses, such as a free vibration of electrons.

Let us consider the passage of an electromagnetic pulse, in which the electric field is undirectional, over an electron with a free period of vibration much longer than double the time taken for the pulse to pass over it. (In this case the Röntgen pulse is thinner and more penetrating than the half-wave produced by the free vibration of the electron.) The electron receives an impulse and is left with kinetic energy after the pulse has passed. Consequently it is then acted upon by forces called into play in the atom itself. These produce a motion which is characteristic of the atom of which the electron forms a part, and this results in radiation—probably the homogeneous radiation discussed in this paper.

When the thickness of the primary pulse approaches the

half-wave length characteristic of the vibratory motion of the electron in the atom, the restoring force in the atom is brought into play before the primary pulse has passed and the absorption of energy and energy of subsequent radiation are diminished. This explains the diminished radiation when the penetrating power of the primary decreases and approximates to that of the radiation characteristic of the radiating substance. When the primary pulse is thicker than the half-wave characteristic of the motion of the electron, the electron is displaced a short distance and is gradually brought back by the restoring force against a gradually weakening electric force in the primary pulse, so that when this has passed, the electron is near its position of equilibrium again and the motion and radiation produced in the other cases are now absent. It is impossible to give an exact solution without some knowledge of the distribution of electric force in the primary pulses, of the forces binding the disturbed electron to the rest of the atomic system, and of the structure of that system; but we may consider this to be an approximation to the behaviour of each electron directly concerned with the phenomena discussed. The number of such electrons may not exceed one in each atom of radiating substance.

According to this theory, energy is taken from the primary beam and part, at least, appears as secondary homogeneous radiation, the rest being transformed into heat.

As the energy of this radiation is quite a considerable fraction of the total energy absorbed, we should expect that the difference between absorptions of primary rays more or less penetrating than the radiation characteristic of the absorbing element would be evident from a study of the absorption of the various homogeneous beams. But we have seen that there is a large absorption of a homogeneous radiation by an element which emits a much more easily absorbed radiation, because much of the energy is given to the electrons; that for other elements which emit a radiation only slightly softer, the absorption is much diminished because only a feeble disturbance and consequent secondary radiation is set up in the absorber; and when the radiation characteristic of the absorbing substance is more penetrating than

the absorbed radiation, the absorption is small and no homogeneous secondary radiation is produced. Thus what we have previously referred to as the special penetrating power of the homogeneous radiations may be explained by the small displacement produced in an atom by a radiation more easily absorbed, equally absorbed, or slightly more penetrating than the radiation characteristic of that atom, for reasons indicated.

Though according to such a theory, if the displaced electrons were merely held by a body of much greater mass, we should expect the emission of wave-trains instead of pulses, yet if the atomic system consisted of a number of interacting electrons, the energy of vibration would be rapidly communicated to other parts of the system and the motion of the displaced electron would be little more than half a complete vibration. The resultant radiation would in that case behave much as a number of isolated pulses.

An explanation on the disruption theory would be similar in many respects, but the displacement of electrons would on such a theory be sufficient to destroy the equilibrium of the atomic system and produce some change in its structure. The evidence against this is perhaps not conclusive, but there is no indication from the energy of secondary X-radiation of such a phenomenon, and the relation between the primary and secondary radiations points rather to the latter being due to the motion of the atomic system in regaining its normal configuration.

George Holt Physics Laboratory,
University of Liverpool,
30th May, 1908.

XXIV. *On the Coefficient of Diffusion.* By BASIL W. CLACK, *B.Sc., Lecturer in Physics at Birkbeck College**.

[Plate XXIII.]

§ 1. Introduction.
§ 2. Theory.
§ 3. Apparatus.
§ 4. Method of Experiment.
§ 5. Determination of Concentration.
§ 6. Determination of δ.
§ 7. Results for KCl and KNO_3.
§ 8. Earlier Method of Experiment [Method A].
§ 9. Length of the Tube.
§ 10. Conclusion.

1. *Introduction.*

THE following paper contains an account of a series of experiments on the diffusion of various salts through water. The objects with which they were undertaken were:—

1st. To test the practicability of a new method for the determination of the coefficient of diffusion;

2nd. To find how the value of the coefficient varies with the concentration of the solution.

Various methods have been employed from time to time to determine the value of the coefficient of diffusion. In some cases careful chemical analyses are necessary to obtain the amount of the substance diffused; in others it is the density of the solution which must be accurately determined at different points. Graham's, and kindred methods, involve serious disturbance of the solution in commencing an experiment, and also in withdrawing the various specimen layers for analysis.

The present investigation may be called a gravitational method, since the amount of the substance diffused is determined by means of the balance; but the method employed involves little disturbance at the commencement of an experiment, and during its progress the disturbance is quite negligible. Moreover, neither a series of density determi-

* Read May 22, 1908.

nations nor of chemical analyses is required to obtain the amount of diffusion which has taken place.

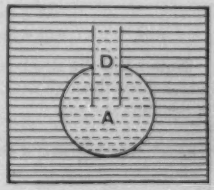
The method may be looked upon as an extension of that employed by Fick (Pogg. *Ann.* xciv. p. 59, 1855) and by Graham, but the method of obtaining the quantity of salt diffused is essentially different, as they determined this amount by chemical analysis.

Again, it does not appear that Fick took any account of the effect of the movement of the liquid along the tube, mentioned in the next section, on the quantity diffused. Although this omission would produce less effect in his method than in that about to be described, it would nevertheless cause a measurable difference in the calculated value of the coefficient of diffusion.

2. *Theory.*

In the diagram (fig. 1) let A represent a spherical vessel, to the upper side of which is fitted a vertical tube D of unit cross-section.

Fig. 1.



Suppose the bulb A and the tube D to be initially filled with a salt solution of uniform density, and to be suspended in a vessel containing pure water. Further, let it be assumed that the upper end of D is maintained in contact with pure water, and that the concentration of the salt solution at the lower end is also kept constant.

Under these conditions the salt will commence to diffuse up the tube D, and ultimately the diffusion current thus commenced will be found to become uniform.

Now when the steady state has been attained the quantities of water and of salt in tube D remain constant, and in dealing with variations in volume and weight we may confine our attention to the bulb A. When salt leaves A the volume of the liquid originally in it will decrease, since in general the addition of salt to water increases the resultant volume. Hence some of the water outside A will enter the apparatus

on account of this decrease in volume, with the result that as the salt diffuses up the tube D it meets a current of liquid which continually passes downwards, and which is due to the decrease in volume mentioned.

The weight of the suspended bulb A will decrease on account of the salt which leaves it, and will increase owing to the water which enters, and the resultant loss in weight is equal to the difference between the two.

Let d = density of solution at a point l cms. from top of tube D.

v = velocity downwards at the same point.

n = concentration in gms. per c.c. at the same point.

v_0 = velocity downwards at top of tube where $l = 0$.

N = concentration in gms. per c.c. at the end of tube where $l = \mathrm{L}$.

With the degree of accuracy obtained in the present work, it is found sufficiently correct to assume $d = 1 + an$ where a is some constant.

Since the same mass of water crosses each section of the tube

$$v(d-n) = v_0,$$

i. e. $$v(1-bn) = v_0, \text{ where } b = 1-a. \quad . \quad . \quad . \quad \text{(i.)}$$

Now when the steady state is reached, the weight of salt which flows through each cross-section of the tube per second is constant $= c$, say

i. e. $-vn + k\dfrac{dn}{dl} = c$, a constant, where k = coeff. of diffusion;

or $$-\frac{v_0 n}{1-bn} + k\frac{dn}{dl} = c.$$

The solution of this equation is

$$-\frac{b\mathrm{N}}{v_0 - bc} + \frac{v_0}{(v_0-bc)^2}\log_e\frac{(v_0-bc)\mathrm{N}+c}{c} = \frac{\mathrm{L}}{k}, \quad . \quad \text{(ii.)}$$

which, as $(v_0 - bc)\mathrm{N}$ is small compared with c, reduces to

$$\frac{\mathrm{N}}{c}\left(1 - \frac{v_0\mathrm{N}}{2c} + \frac{v_0(v_0-bc)\mathrm{N}^2}{3c^2}\ldots\right) = \frac{\mathrm{L}}{k}. \quad . \quad . \quad \text{(iii.)}$$

As already stated, there will be a contraction in volume produced in the vessel A when salt leaves it, and a corre-

sponding movement downwards will take place in the liquid in the tube D. The amount of water which enters the bulb owing to this movement may be calculated as follows:—

Mass of water in 1 c.c. of solution at bottom of tube D
= density of solution at bottom minus concentration of solution at bottom.
$= 1 + aN - N,$
$= 1 - (1-a)N,$
$= 1 - bN.$

Let δ = the ratio of the increment in volume produced to the increment in the mass of salt dissolved, for a solution of the strength employed in the bulb A; *i. e.* the increase in volume produced when 1 gram of salt is dissolved in a solution of the given concentration, the amount of the solution being so great that the addition of the salt makes no appreciable change in the concentration. The mass of water which enters the bulb due to a small diminution (dm) in the mass (m) of salt in the bulb

$$= \delta \times (1-bN)dm.$$

On integrating, the mass of water which enters the bulb due to a given diminution

$$= \int_{m_1}^{m_2} \delta \times (1-bN)dm.$$

Strictly both δ and N are functions of m, but the change in m is so small that they may both be taken as constants; thus the integral is equal to

$$\delta \times (1-bN)(m_2-m_1).$$

Now in one second $(m_2-m_1)=c$.

Hence the mass of water entering the bulb per second

$$= \delta \times (1-bN)c.$$

Now the decrease, i, in weight per second of the bulb = the weight of salt leaving per second minus the weight of water entering per second; or in symbols

$$i = c - c\delta(1-bN),$$

i. e. $$c = \frac{i}{1-\delta(1-bN)}.$$

Again, the volume of water entering per second, $c\delta$, is equal to the velocity of the liquid at the bottom of the tube

$$= \frac{v_0}{1-bN}, \quad \ldots \quad \text{(from i.)}$$

$$\therefore \frac{v_0 N}{2c} = \frac{N\delta}{2}(1-bN).$$

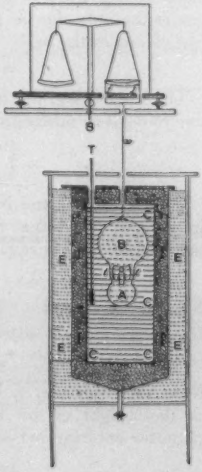
Again, the third term in the bracket in equation iii. is found to be negligibly small; hence this equation is written for practical use

$$k = \frac{Li}{N1 - \left\{\frac{N\delta}{2}(1-bN)\right\}\left\{1-\delta(1-bN)\right\}} \quad \ldots \quad \text{(iv.)}$$

3. *Apparatus.*

A and B are two hollow spherical glass bulbs respectively 5 cms. and 9 cms. in diameter, connected together as shown by two parallel glass quill-tubes t, t (fig. 2).

Fig. 2.



On the upper side of A the bulb is pierced by an aperture into which the diffusion-tube D is hermetically sealed.

The ends of this tube were ground off perfectly plane and perpendicular to its axis, and its dimensions were, in most of the experiments, 4 cms. long, and 1·694 sq. cms. in cross-section.

The complete apparatus AB was filled with the solution under investigation, and suspended in a cylindrical copper vessel C of about 7 litres capacity, containing distilled water, the temperature of which may be kept constant by means of a surrounding ice-jacket I, protected by a layer of cork-dust E, several inches thick. The mercury thermometer T had a small range on each side of 0° C. and was divided into $\frac{1}{100}$° C.

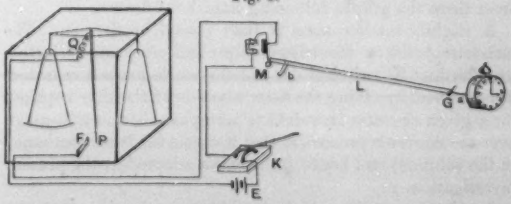
The Sartorius balance used was mounted upon a slate shelf firmly cemented into the stone wall of the laboratory.

Two holes were sand-blasted through the base of the balance-case, and a light frame, *f f*, passed through them as shown, and to this was attached a fine wire *w* which, passing through a third hole, in the slate shelf, sustained the apparatus already described. The movement of the balance beam was limited by means of stops shown at *s*, so that the pointer could only move over 1 scale-division, and the vertical movement which this permitted to the suspended system was only ·12 mm., so that any disturbance which might arise from this small and very slow movement will be quite negligible.

4. *Method of Experiment.*

As the salt diffuses up the tube, the stronger solution in the upper larger bulb, B, sinks by gravity, by means of the side tubes *t t*, and maintains the initial concentration of the solution at the lower end of the tube D; while the upper end is in contact with such a large volume of water that it may be considered to remain practically pure. The conditions studied in § 2 thus apply, and as diffusion proceeds the weight of the suspended system will decrease. If, then,

Fig. 3.



a slightly deficient weight be placed in the opposite scale-pan, the beam will turn until it reaches the supports. As time proceeds, however, the weight of the apparatus will ultimately become equal to that on the balance-pan, and the pointer will move to the zero of its scale. An automatic device was designed which would register the exact time at which this occurred. A very fine platinum wire P (fig. 3)

was soldered on to the lower end of the balance-pointer, and when the beam is in equilibrium, P makes contact with a fine strip of platinum-foil F, suitably mounted opposite the zero of the balance-scale. A delicate phosphor-bronze strip Q, such as is employed for galvanometer suspensions, served to connect the upper end of the pointer to a battery E, which was connected in series with a mercury lever-switch K, and an electromagnetic trigger M.

On contact being made between P and F the armature of the trigger is attracted, thus allowing a light wooden lath L, which is resting upon the trigger, to fall. This lath is pivoted at G, and is provided at this end with a flexible copper strip *a*, which, as the lath falls, is brought into contact with the balance-wheel of a common alarm-clock, from which the back has been removed, and which is suitably placed on the table.

The clock is thus stopped at the exact time at which the pointer reached the zero of the balance-scale.

In its fall another copper strip *b*, attached to the lath, knocks over the mercury switch K, and thus breaks the electric circuit.

The time recorded by the clock and the corresponding mass on the pan of the balance are recorded in tables, and from these the graphs following have been drawn.

A slightly smaller mass is then placed on the pan. The lath is replaced on the trigger, the clock reset and restarted, and the key K switched on, and the whole process repeated. This method, involving the determination of the time required for a given decrease in weight to take place, has an advantage over the converse process, in that it causes far less disturbance in the solution, and hence it has been adopted in the present investigation.

Another precaution which should be mentioned is that, the arrangement employed being extremely sensitive to changes in density, great care must be taken to prevent any but minute changes in temperature occurring in the solution.

The large amount of damping introduced, and the effect of the surface-tension of the liquid, prevented any accurate work being done until the effect of a film of oil on the surface of the solution was tested. This greatly reduces the surface-

tension, and increases the accuracy of weighing to a remarkable extent. Of a large number of oils tested, Fleuss pump-oil was found to be superior to any of the others, in that it always remained quite fluid.

As showing the effect of this oil-film a few figures might be quoted.

The apparatus was filled with water and suspended in water at a constant temperature. If set in oscillation, in no case would the balance-beam make more than two swings, and the resting place varied in different attempts between 10 and 15 on the scale for the same mass in the pan. Two drops of Fleuss pump-oil allowed to fall on the liquid near the suspending wire made a remarkable difference. Five turning-points could now be easily obtained, and assuming the ratio between any two consecutive deflexions to be constant, great consistency was found between the calculated positions of the resting place of the balance. For example,

Turning Points.		Resting Points.
20·8	3·2	10·0
14·4	7·8	10·2
11·5		10·1
		10·2
19·1	4·7	10·3
13·8	8·1	10·0
11·0		

Now in the diffusion experiments the beam does not oscillate, but slowly takes up its new position. The consistency obtained under these static conditions was tested, and the following is a sample of the figures obtained:—

Mass on pan in gms.....	37·105	37·104	37·103	37·105
Resting Point observed ...	10·2	11·9	13·5	10·0
Repeated.				
Resting Point observed ...	10·0	12·0	13·9	10·2

Moreover, it was found that the position of the pointer remained constant to within 1 scale-division for a week, the temperature being meanwhile kept at 0° C. by the surrounding ice-jacket.

The sensitiveness of the balance under the actual conditions holding in a diffusion experiment was found from a large number of readings to be ·00052 gm. per scale-division, *i. e.* a movement of 1 scale-division in the resting point was produced by ·00052 gm.

5. *Determination of the Concentration.*

Sodium chloride and potassium chloride prepared by Kahlbaum, of Berlin, were the first salts experimented upon. Solutions were prepared of approximately 0·2 gm., 0·1 gm., and 0·05 gm. of salt per c.c. of the solution, and in each case a check experiment was carried out to ensure experimental consistency. The concentration of these solutions was determined in most cases by three different methods: (1) From the density, making use of the results given in the *Physikalisch-Chemische Tabellen*, published by Landolt and Börnstein; (2) Precipitation by $AgNO_3$; (3) Evaporation to dryness in a platinum crucible.

The results obtained by all three methods usually agreed to within less than 1 per cent., as the following figures will show:—

Values of N in gms. of KCl per c.c. of Solution at 0° C.

Solution	2.	3.	4.	5.	6.	7.
By Density	·1990	·1009	·1004	·05122	·05147	·05106
By Precipitation ...	·1989	...	·1004	·05128	·05106	·05092
By Evaporation ...	·1988	·1008	·1004	·05100	·05097	·05113
Mean	·1989	·1008	·1004	·05117	·05117	·05104

The densities were obtained in the first place at the temperature of the laboratory; but the concentrations have been calculated at 0° C., the temperature of the experiments, by means of a determination of the absolute coefficient of expansion of the several solutions over this range of temperature, using for this purpose apparatus identical in principle with that employed by Dulong and Petit for a similar purpose.

6. *Determination of* δ.

It has been long known that when salts enter into solution, a change usually takes place in the volume occupied by the solvent.

F. Kohlrausch and Hallwachs (Wied. *Ann.* liii. 1894, p. 1) and Thomsen have studied these phenomena and have estimated the change in volume produced in several cases by the determination of the density of the solution formed, when a known mass of salt is dissolved in a known mass of water. W. F. Magie (Phys. Rev. xviii. 1904, pp. 449–452) and Macgregor have endeavoured to express in a formula the results obtained by previous experimenters.

We have seen that in the present investigation the phenomena referred to play an important part, and have deduced an expression for k involving the knowledge of the change in volume which accompanies a slight change in concentration.

Now δ is not a constant, but varies with the strength of the solution, and according to Magie depends on the relationship existing between the volumes occupied by the dissociated and the non-dissociated molecules of salt.

Its value can be easily deduced from the published tables giving the density of salt solutions of known concentration.

7. *Results for* KCl *and* KNO_3.

The accompanying graphs (Pl. XXIII.), which are numbered in accordance with the order of the experiments, show some of the curves obtained by plotting the weight of the suspended apparatus against the time as abscissæ.

The curves suggest that the method is at any rate capable of giving fairly consistent results, and they indicate a diminution in the coefficient of diffusion with a decrease in the concentration in the cases of both NaCl and KCl, a result which has been found by several other observers.

The actual figures obtained in this work are omitted for the reason mentioned below.

Before proceeding further, it was decided to test whether the outer vessel containing the salt solution was large enough. As the salt diffuses up the tube D the water outside becomes slightly denser, but the change in concentration is so small that it does not appreciably affect the quantity of salt diffused

up the tube; but the suspended apparatus has such a large volume, that a very slight variation in the density of the water outside has a considerable effect on the hydrostatic force upwards. When the earlier experiments were performed, the importance of this action was not fully appreciated.

It is impossible to calculate the change in the buoyancy of the water, but there is no doubt that the heavier liquid will tend to sink to the bottom, and that then diffusion will tend to make the density uniform. Any variation in the density of the liquid below the suspended apparatus will produce no effect on its weight, and so it is an advantage to have as deep a vessel as possible; to have the suspended apparatus as high as possible in the water; and to have the volume of the water as great as possible.

Experiments similar to the preceding were carried out with a deeper vessel, C C, shown in fig. 2, having also a volume about twice that of the original one, and they proved that the first vessel was undoubtedly too small. For this reason a number of results obtained with the earlier vessel must be discarded. The results given below have all been obtained with the larger vessel.

In the theory it has been assumed that the density of the solution at the bottom of the tube is constant over a horizontal plane, and that this density is the same as that of the mass of the solution in the neighbourhood. It has likewise been assumed that the density at the top of the tube in a horizontal plane is constant, and practically the same as that of pure water. These assumptions would only be strictly true if gravity were infinitely powerful, and in practice one would expect the density at the bottom of the tube to be rather less than that assumed, and at the top rather greater. These deviations from the assumptions will give a calculated value of k smaller than the true one, but even in the worse possible case studied, it can be shown by an approximate calculation that the error introduced is certainly much less than 1 per cent.

The narrower the tube the nearer will the assumptions be to the truth, but a narrow tube permits only a small amount of diffusion to take place.

In order to test whether it was necessary to have a tube of extremely narrow bore three diffusion-tubes were prepared, each 4 cms. long, but of different cross-section, as shown in the table below.

Using 10 per cent. KNO_3 solution, the following results were obtained using these tubes:—

Experiment.	$\frac{1}{t}A \times 10^6$ Mean Slope.	$k \times 10^5$.	Mean value $k \times 10^5$.	L.	A.
16	·2154	·843	·844	·00	1·694
17	·2159	·845			
18	·1294	·836	·843	3·990	1·024
19	·1316	·850			
20	·0956	·852	·853	3·993	0·743
21	·0959	·855			

The results obtained, using different tubes, do not vary more than the individual experiments on the same tube, so that for the present at any rate we may neglect any end correction to our tubes of 4 cms. length. The results also suggest that the vessel containing the water is large enough: for, although the quantity of salt diffused into the water varies considerably, the quantity in the last case being less than half that in the first, yet we get approximately the same calculated value for the coefficient of diffusion.

The next table gives the results for 5 per cent. solution of KNO_3.

L=4·00 ms.; A=1·694 sq. cms. N=gms. KNO_3 per c.c. solution at 0° C.

Experiment.	N.	$\frac{1}{t}A \times 10^6$.	$k \times 10^5$.	Mean $k \times 10^5$.
22	·0521	·1174	·870	·871
23	·0521	·1177	·872	

Comparing this result with the value obtained in Experiments 16 and 17 shown in the previous table, we see that the coefficient of diffusion of KNO_3 increases as the concentration decreases, as was noticed by Scheffer in 1888, a phenomenon contrary to that which holds for KCl and NaCl.

The results obtained with solutions of KCl are tabulated below.

L=4·00 cms. A=1·694 sq. cms. N=gms. KCl per c.c. solution at 0° C.

Experiment.	N.	$iA\times10^6$.	$k\times10^6$.	Mean $k\times10^6$.	Approx. concentration.
24	·1975	·4748	·966	·972	20 p. cent.
25	·1975	·4810	·978		
26	·1034	·2518	·953	·954	10 p. cent.
27	·1021	·2491	·956		

8. *Earlier Method of Experiment.*

It might be mentioned that a number of experiments were carried out by a somewhat different method, denoted in the following section as Method A.

The diffusion apparatus, filled with distilled water, was inverted, so that the smaller bulb A (fig. 2) was uppermost, in a solution of the salt under investigation, thus allowing the salt to diffuse into instead of out of the apparatus, and the resultant rate of increase in weight was determined.

Some of the results obtained with solutions of KCl by this method are given in the next table in order to indicate the agreement.

L=4·00 cms. A=1·694 sq. cms. N=gms. KCl per c.c. solution at 0° C.

Experiment.	N.	$iA\times10^6$.	$k\times10^6$.	Mean $k\times10^6$.	Approx. concentration.
8	·1997	·5729	1·011	1·009	20 p. cent.
9	·1997	·5703	1·007		
10	·1014	·2786	·952	·953	10 p. cent.
11	·1016	·2799	·955		

If the results of Experiments 26 and 27 be compared with those of 10 and 11, it will be seen that the agreement in the case of the 10 per cent. solution is almost perfect, but it is

not quite so satisfactory in the case of the 20 per cent. solution, for there is a difference of 3 per cent. The actual measurements are believed to be more accurate than this.

Some of the difference may be due to uncertainty in the value of δ, but this would not produce so great an error as 3 per cent. It may be that part of the difference is due to condensation of water from the air on the salt solution in Method A, so that part of the increase in weigh of the suspended apparatus may be due to the diminution of buoyancy of the solution, on account with its dilution with this condenced water.

The error produced by this cause would probably not be great, but no quantitative experiments were actually carried out on the subject, as the experience gained in the research convinces us that the first method described in this paper is distinctly superior to Method A.

For this reason no further experiments are being carried out by the earlier Method A.

Some of the points in which the method mentioned in § 4 shows superiority over Method A may be summarized. First, in this better method, the condensation of water mentioned above, instead of being harmful, is even somewhat of an advantage.

Again, this method necessitates the preparation of much smaller amounts of solution; then it requires much less manipulation in commencing an experiment, as the apparatus, filled with the solution and having the open tube temporarily closed by a thin glass plate, may be simply lowered into the cooled distilled water.

Instead of this simple process, in the Method A in commencing an experiment, the solution under investigation was slowly run in through a glass tube passing to the bottom of the vessel C (fig. 2), while the distilled water, which is thus displaced, was removed from the top by means of a syphon. Great care was taken to ensure the complete removal of the water, so that the vessel C was filled with a solution of uniform density.

Thus the whole of the suspended system remains filled with distilled water, but is immersed in a salt solution of uniform density, so that the diffusion-tube, initially filled

with pure water, is brought into contact at its lower end with a salt solution whose concentration, on account of the large volume present, may be considered to remain constant.

Again, the better method enables an experiment to be repeated with precisely the same solution, so that the working details are much more under control. Moreover, it has another advantage in that greater accuracy is introduced in the determination of the quantity δ.

It is only in very dilute solutions that the value of δ is doubtful, and as only a very slight decrease occurs in the concentration of the solution inside the diffusion apparatus as the salt diffuses out, these very dilute solutions are not produced in this method.

Since no experiments were made by Kohlrausch with solutions as dilute as those produced by the diffusion of the salt into the suspended apparatus in Method A, it was necessary to make a special series of experiments to determine the value of δ. Moreover, as the solutions were so dilute, it was thought advisable to use a larger flask than than that employed by Kohlrausch. In place then of his flask of about 130 c.c. capacity, one was taken having a volume of about 300 c.c. and it was loaded with shot, so as to weigh about 4 gms. when immersed in water. The neck was drawn off and sealed up in the blowpipe, and the flask was then suspended by means of a pair of fibres of unspun silk attached to the pan of the balance, in distilled water cooled to 0° C.

It is very necessary that the fibres should be thoroughly wetted before use. They must also be smooth, and dust should be removed as far as possible from the water, otherwise the balance-beam refuses to swing, or only swings in a very erratic manner.

When the flask has been weighed in distilled water at 0° C. it is clamped by means of a suitable device, and a standardized solution of the salt is added to the distilled water (over 6 litres) 10 c.c. at a time. A series of very dilute solutions of known strength are thus obtained, and the corresponding density may be found, from which the value of δ easily follows.

NaCl.		KCl.		KNO_3.	
Concentration.	δ.	Concent.	δ.	Concent.	δ.
·0008453	·2152	·0004153	·3054	·0004146	·3097
·001265	·2208	·0008290	·3072	·0006211	·3072
1684	·2252	·001241	·3079	·0008267	·3061
2102	·2281	1652	·3074	·001032	·3042
2516	·2308	2061	·3089	1440	·3038
2931	·2329	2469	·3091	2047	·3037
3343	·2354	2875	·3094	2449	·3052
3754	·2363	3280	·3095	3046	·3061

It might be mentioned that a comparison of the weighings from which the above figures were obtained with the earlier ones mentioned in § 4, using a wire suspension, show that they are apparently equally consistent and trustworthy. It seems probable that the disadvantage of using such a delicate suspension as a fibre of unspun silk may be overcome by using a fine wire in its place, provided that the surface of the liquid be covered with a film of oil. When it is remembered that the fibre supports the flask inside a metal vessel surrounded by ice, and that clamping and stirring must be carried on without the experimenter being able to see what is happening inside the vessel, the advantage of using a somewhat stronger suspension will be apparent.

In the present experiments, however, the cocoon suspension was adhered to, and with this no oil-film should be used.

In order to find what value must be taken for δ in the various diffusion experiments, the method of successive approximations was used. The final rate of increase in weight is first assumed to be due entirely to the salt which enters the apparatus. In this way a rough value of the concentration inside the apparatus at the end of, say, a fortnight, is deduced. This gives us a rough value for δ, by means of which our first assumption may be corrected, and so yield a more exact value of δ.

9. *The Length of the Tube.*

Some experiments were also carried out with tubes of various lengths. It has already been shown that it is likely that to our present degree of accuracy no end correction is requisite to the tube of 4 cms. length. It may be pointed out that the end correction must depend on both the length and the diameter of the tube. The longer the tube the less is the rate of diffusion through it, and therefore the more perfectly can gravity regulate uniformity of density in a given horizontal plane.

Three tubes were employed, the lengths being 4 cms., 2 cms., and 1 cm.; the areas of cross-section being respectively 1·697, 1·025, and 1·059 sq. cms., and the diameters being 1·465, 1·14, and 1·16 cms. respectively; and the solution employed was a 5 per cent. solution of KCl. The results are plotted on the accompanying graph (Pl. XXIII. fig. 6).

Assuming no appreciable end correction to the tube of 4 cms. length, it is found that to give the same value for the Coefficient of Diffusion in each case, 0·036 cm. should be added to each end of the 2 cm. tube (*i. e.* ·032 of the diameter), and 0·045 cm. to each end of the 1 cm. tube (*i. e.* ·039 of the diameter). The results prove conclusively that it will not be possible to accelerate the work by using tubes much shorter than 4 cms., unless perhaps a battery of short and narrow tubes can be employed.

10. *Conclusion.*

My chief object in bringing this paper before the Physical Society of London in its present state has been the hope of obtaining from the Fellows some useful suggestions for future work, or improvements in the method.

In all 43 experiments have been performed; many of them, described in this paper, have had the sole object of endeavouring to obtain information of the best experimental conditions.

The numerical results of some of the others have not been stated, as improvements in the method, which they have suggested, have rendered the numerical result unsatisfactory.

The values of the coefficient of diffusion at 0° C., which the investigation indicates to be most trustworthy, are those obtained with the suspended apparatus filled with the salt solution, in the larger copper vessel containing distilled water. These results are

Experiment.	Salt KNO_3.		Experiment.	Salt KCl.	
	Concen.	$K \times 10^5$.		Concen.	$K \times 10^5$.
16	10 p. cent.	·843	24	20 p. cent.	·966
17	10 „	·845	25	20 „	·978
22	5 p. cent.	·870	26	10 p. cent.	·953
23	5 „	·871	27	10 „	·956

The results show that the method is capable of yielding consistent values, and if compared with those obtained by other observers in the same subject, although the concentration of the solutions and the temperatures at which they were employed vary considerably, thus making comparison difficult, yet the agreement, as far as can be judged, appears to be satisfactory.

For instance, results expressed in C.G.S. units, obtained by Schuhmeister and by Scheffer for KCl and for KNO_3 respectively are shown below

Salt.	Temp.	N.	$k \times 10^5$.	Observer.
KCl.....	17°·5	·022	1·47	Schuhmeister.
	10°	·007	1·27	
KNO_3.....	7°	·031	·98	Scheffer.
	7°	·0093	1·06	
	10°	·015	·92	Schuhmeister.

The absence of any trustworthy values for the temperature

coefficients, and more especially the fact that the concentrations shown above are so much lower than those employed in the present paper, render further comparison difficult.

In conclusion, I should like to offer my thanks to the Principal and Council of Birkbeck College for assistance in carrying the experiments into effect, and especially to Dr. A. Griffiths, who not only suggested the research, but has been always ready to assist me when in trouble or difficulty.

Discussion.

Dr A. Griffiths congratulated the Author upon his paper, and referred to the importance of keeping the temperature constant during the experiments. The effects of temperature changes were very marked. The importance of a deep outer vessel was also surprising, and Dr Griffiths pointed out how it was possible to place a superior limit to the error introduced by the passage of the salt into the outer vessel. He also suggested the use of a differential method, using two glass vessels, to overcome some of the difficulties encountered in the experiments.

Prof. C. H. Lees said that all who had experience with diffusion experiments would congratulate the Author. The idea of using a balance was an old one, but the Author had made an old method a thoroughly reliable one. The question was still to be solved whether an increase of concentration increased or decreased the diffusion constant, and he hoped the Author would be able to carry on his experiments and settle this point.

XXV. *The Spectrum Top.* *By* F. Peake Sexton, *A.R.C.S., Lecturer in Physics, Central Technical Schools, Truro* *.

[Abstract.]

The colours formed by the Benham spectrum top are here explained. The explanation rests on the fact that the red, green, and blue colour sensations have different rates of growth and decay.

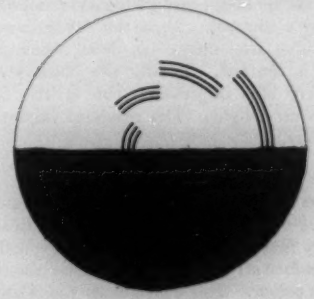
When the top or disk shown in the figure is slowly rotated anticlockwise, it will be noticed that there is a slight red extension to the innermost sector lines. This indicates that the growth of the red sensation is the most rapid. If the rotation is reversed these lines present a blue colour, thus

* Read May 22, 1908.

showing that this is the last to decay. Thus there is good ground for assuming that the rates of growth are red, green, and blue, and of decay—blue, green, and red; where the first is the most rapid in both cases.

Experiments showed that the radial distance of the sector-lines was without effect on the colour of the lines, and the disk, as illustrated, shows that the colour depends on their angular positions.

These colour effects are dependent on the angular widths of the sector-lines, being best for lines of 1 mm. observed from 1 metre. This phenomenon is thus entirely dependent



on a contrast. With broad lines the colour produced is too faint to be observed, but when the line is narrow, the immediate surrounding white part will appear to have the complementary colour, and thus apparently increase the effect.

The length of the lines naturally affects the colour.

The speed of rotation has an important influence on the colour, but the best effects are obtained at about six revolutions per second.

When the illustrated disk is given an anticlockwise rotation, the colours are—purple—dull orange—yellow-green—violet-blue; counting from the centre, and this order of colours reverses with the rotation.

In order to obtain an explanation of the top, it is convenient to assume the eye as directed to one part and consider the succession of the different parts before it.

I. *The Innermost Sector-lines.*

The black sector will pass giving an impression of black lines on a red ground, but as there is not a contrast the effect is small. Then the sector-lines will pass giving a red impression behind them, because of the rapid growth of the red sensation, and this is reinforced by a contrast. Finally the disappearance of the white sector will give a bluish impression, which is not reinforced by contrast. Thus the resultant impression on the eye is *red lines*.

II. *The Outermost Sector-lines.*

The black sector disappears giving a red impression, which is unaided by a contrast. Then the sector-lines come into view with a blue after-image on them, which latter is reinforced by a contrast. Finally the black sector appears giving another blue impression, but in the absence of a contrast it is very slight. The resultant impression on the eye is *blue lines*.

III. *The Middle Sector-lines.*

The colours in these cases are simply due to a combination of the causes considered under I. and II. In all cases, the causes II. act first to produce a blue image, and those of I. second to produce a red image; and these combine to give the resultant colour. This depends on the relative intensity of the impressions, which vary with the size of the white sectors producing the effects. Thus the inner set of lines is orange because the red impression is about double that of the blue: and the outer set is green because the blue in this case is double the red.

The above experiments were all made in daylight, but Abney's experiments made under various illuminations fully confirm the theory which the author deduced.

The following references are to the past work on this subject:—

Prof. Liveing	Nature, vol. li. 1894–5.
Mr. C. E. Benham . . .	" " "
Sir W. de W. Abney . .	" " "
Mr. Shelford Bidwell . .	Proc. Royal Society, 1896.

Discussion.

Mr F. E. Smith asked if the Author had viewed the top when illuminated with lights of different colours, and if so with what result.

The Author replied that experiments had not been tried in different coloured lights, as there was difficulty at the time of obtaining a coloured illumination of sufficient strength. The bands were visible in sodium light; but this did not affect the theory, because sodium light, although practically monochromatic, affects the three colour sensations.

XXVI. *A Modified Theory of Gravitation.*

By C. V. Burton, *D.Sc.**

1. So many questions arise in connexion with any theory which aims at assigning a dynamical basis for gravitation, that it may conduce to clearness if some elements of this paper are first presented without mathematical treatment.

The theory now put forward is pulsatory, and may in fact be regarded as a development of that which has been suggested by Prof. Hicks †. The æther, even where modified by the presence of atomic matter, is assumed to be so nearly incompressible that the bulk-modulus of elasticity enormously transcends every other elastic modulus; so that, in dealing with any compressional-rarefactional disturbances, we may treat the medium as a fluid.

In two particulars, however, there is an essential difference between the views now tentatively put forward and the more familiar form of pulsatory hypothesis. These differences are indicated in the following five paragraphs.

2. It is a well-known hydrodynamical result ‡ that if two spheres, placed at some distance apart in a frictionless incompressible fluid, are by any means caused to execute periodic pulsations (increase and decrease of volume), the pulsation-period being the same for the two spheres, the

* Read May 8, 1908.

† W. M. Hicks, "On the Problem of two Pulsating Spheres in a Fluid," Proc. Camb. Phil. Soc. iii. p. 277 (Oct. 1879).

‡ Hicks, *loc. cit.* *Cf.* also Basset's "Hydrodynamics," vol. i. chap. xi., where further references are given.

average force which either sphere causes to be exerted upon the other will be an attraction, a repulsion, or zero, according as the phase-difference of the pulsations is less than, greater than, or equal to a quarter-period. Hence, if we are to construct on these lines a model illustrative of gravitation, we must assume agreement of phase amongst all the pulsating centres, or at least we must suppose that the phases of the various centres are more nearly in agreement than would be the case if the phase-distribution were purely random. This seems to suggest that the source of pulsatory motion is not to be sought for in the free vibrations of individual atoms or electrons, but rather in something external to these, and acting on them all in common.

Let us suppose, for example, that free æther is very nearly incompressible, and that every region where atomic matter exists is a region of somewhat enhanced compressibility. On this supposition, an increase of pressure applied by any ideal means to all the æther within a given volume would cause only a very minute contraction except where atomic matter was present; the expansion resulting from a diminution of the pressure applied to the æther being similarly localised.

3. It appears that we might thus account for attractions of gravitational type by supposing the ætherial pressure to be undergoing a *secular* change. A gradually increasing (or decreasing) pressure would cause every region containing atomic matter to behave as a sink (or as a source) in the medium. For our purpose, however, *accelerated* inflow (or outflow) of æther at each sink (or source) would be necessary; that is to say, the strength of each sink (or source) must be steadily increasing or steadily decreasing with the time—no matter which. The suggested possibility of accounting for gravitation by means of a slow secular change taking place throughout the universe appeared at first sight very alluring; but calculation shows at once that, feeble as gravitational forces are, they are incomparably greater than could be accounted for in this way.

4. What, then, would be the result of periodic or quasi-periodic fluctuations of the ætherial pressure? Every

portion of atomic matter would constitute a centre of pulsatory motion in the æther; and throughout any region not too immense the pulsations would agree sensibly in phase. It is by assuming the existence of a fluctuating ætherial pressure that I have here sought to obtain a consistent illustration of gravitational attraction; and, as furnishing most readily an assignable cause for such fluctuations of pressure, the propagation of compressional-rarefactional waves through the nearly incompressible æther is suggested. These waves are supposed to travel with so great a velocity that, even though all effective periods involved be very small, the effective wave-lengths, measured even by astronomical standards, are very great. Though the origin of the assumed wave-trains were unexplained, these might still be regarded as contributing to the explanation of gravity; for when once the constitution of a dynamical system has been defined, the mere supposition that the system is in motion is hardly to be viewed as a piling-up of hypotheses. And if we regard the question broadly, extending our consideration to that wider range of space and motion wherein the whole of our explorable universe must needs be treated as an infinitesimal volume-element, sensibly homogeneous as regards its ætherial content, we may admit our complete ignorance of this greater universe, and of the forms of disturbance which might emanate from it. One might perhaps conceive of compressional waves as proceeding from a more primitive and chaotic condition of the "primal æther," from which an æther such as we know, with electromagnetic qualities, may be in process of formation. As regards the sensible uniformity in time which must characterize the primary disturbance, if so unchanging a phenomenon as gravitation is thus to be accounted for, we may think of it as a quality naturally to be looked for in any activities whose scale is sufficiently vast; just as the energy of the radiation passing into space from the sun varies but little in the course of an hour, or as the turbulence of an ocean presents much the same aspect from minute to minute.

In any case it is instructive to trace out some of the consequences which might be expected to follow from the propagation of compressional waves through a very slightly

compressible æther; the more so as we are led to certain conclusions which, although essentially based on the ordinary postulates of dynamics, are at variance with some traditional or instinctive views.

5. The second innovation referred to concerns the nature and mode of progression of the centres of enhanced compressibility, which are assumed to be associated with atomic matter. If the nucleus of an electron is taken to be a vacuous cavity, the question of free mobility of this nucleus through the æther presents some difficulty, even when we suppose that the electron itself is essentially constituted by a self-equilibrating distribution of strain in the æther surrounding the nucleus. But now, discarding for the moment a too minute scrutiny of ætherial constitution, so that we may regard the æther as a continuous medium, let us suppose that the nucleus of an electron *, instead of being vacuous, is merely a region of somewhat diminished density. (Though increased compressibility may not be inevitably associated with diminished density, it is convenient to assume such a relation for preliminary descriptive purposes. In the sequel this question is treated more generally.) This implies that, in the complete strain-distribution which constitutes the electron, there is included some degree of expansion in the nuclear region. An electron thus constituted may more readily be conceived as freely mobile through the æther, no vacuous cavity being present to complicate the problem: so that any infinitesimal displacement of the electron is simply equivalent to the impressing of a differential strain upon the medium.

6. This assumption as to the nature of an electronic nucleus is admittedly gratuitous, but apart from the difficulty regarding mobility which it was designed to remove, it has the advantage of greatly simplifying the dynamics of the problem proposed. So far as we are concerned with the distribution of resultant ætherial motion from point to point, we may treat each moving electron as a doublet comprising a source and a neighbouring sink, while it is shown that the

* The necessity for distinguishing, in this connexion, between positive and negative electrons is considered in Appendices A and B.

resultant of the hydrodynamical (gravitational) forces * tending to accelerate the motion of the electron simply follows the gradient of diminishing pressure and is proportional thereto. It further appears that the assumption referred to might enable the elementary requirements of the gravitation problem to be satisfied with the contribution of only a very minute non-electromagnetic term to the total effective inertia of an electron (Appendix D).

7. Let F be a physical magnitude characteristic of all electrically neutral (atomic) matter, and defined as follows: if m is the mass of atomic matter included in any given region, then $Fm\rho$ is the defect of mass of ætherial substance contained within that region, as compared with the mass included in an equal region filled with free æther; the density of the æther being denoted by ρ. (As we shall suppose the average density of ætherial substance to be but little different in a vacuum and in the densest atomic matter, no more precise definition of ρ is needed in this connexion.) Otherwise thus: if, within a region bounded by a fixed ideal surface and originally free from atomic matter, a mass m of matter could be created, a volume Fm of æther would flow outward across the bounding surface. F may be called the extrusion of æther per unit mass of matter.

8. Consider further the application of a small additional normal pressure δp to a surface bounding some definite portion of the ætherial medium, which is modified by the presence of a mass m of atomic matter. A reduction of volume will result, and we shall define another constant H by the statement that the reduction of volume is *greater* by $Hm\delta p$ than if the original volume had been wholly occupied by free æther. Thus evidently

$$H = -d\mathrm{F}/dp. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

9. Let a particle of atomic matter be present where from any cause there is a pressure-gradient in the æther. When

* That is to say of those forces which are concerned in palpable ætherial motions; our scrutiny being minute enough to take account of variations of ætherial qualities through the region occupied by an electron, but not so minute that the æther ceases to appear to us as a continuous medium.

the particle is displaced from one point to a second neighbouring point, there is a flow of æther inward at the first point, and a flow outward at the second point. If the ætherial normal pressure is greater at the first point than at the second, work on the whole will be done by the pressure during the displacement.

Consider a particle m of matter at a point (x, y, z) where the pressure is p, and let the particle be displaced to $(x+\delta x, y, z)$ where the pressure is $p+\partial p/\partial x \,.\, \delta x$. Also let the particle be acted upon by such ideal forcive as may be necessary to ensure that the displacement takes place very slowly. There is work done by the pressure p at (x, y, z) corresponding to a reduction of volume $\mathrm{F}m$; this work being $p\mathrm{F}m$. There is also work done against the pressure $p+\delta p$ at $(x+\delta x, y, z)$ corresponding to an increment of volume $(\mathrm{F}+\delta\mathrm{F})m$; this work is, to a first order,

$$(p\mathrm{F}+p\delta\mathrm{F}+\mathrm{F}\delta p)m,$$

where $$\delta p = \partial p/\partial x \,.\, \delta x,$$

and $$\delta\mathrm{F} = -\mathrm{H}\delta p = -\mathrm{H}\partial p/\partial x \,.\, \delta x.$$

On the whole, then, the work done by the normal pressure of the medium in the virtual displacement δx is

$$\left(\mathrm{H}\frac{\partial p}{\partial x} \,.\, p - \mathrm{F}\frac{\partial p}{\partial x}\right)m\delta x.$$

Of this the first term represents the work done in compression, the potential energy of the system being increased to that extent, while the remaining term is the work done against the ideal forcive which was applied to the particle to prevent its moving with appreciable acceleration. Thus if a particle of mass m is at rest in the æther (or, as below, in motion) at a point where an ætherial pressure-gradient exists, there will be a force

$$-m\mathrm{F}\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

exerted on the particle. If the particle were allowed to move with a finite acceleration, the associated movements of expansion and contraction would be correspondingly accelerated, and work would thus be done in imparting this

form of kinetic energy to the æther. This only implies, however, the existence of a corresponding term * in the total effective inertia of the particle, the remaining terms being probably of electromagnetic origin. Since here, as in the usual acceptation of the term, the mass m signifies the *total* inertia of the particle in question, we may remove the restriction as to the particle being at rest, (2) being in any case the force-components acting on m. It should be particularly noticed that the virtual displacement from which (2) was deduced was a displacement of m with respect to the æther, and that consequently

$$-m\mathrm{F}\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right) \text{ is the force tending to accelerate the particle } m \text{ with respect to the æther} \quad . \quad (3)$$

It will be necessary later to examine the forces which material bodies might be expected to exert on one another owing to their motion through the æther.

10. Properly speaking, there are two distinct ways in which matter may be conceived of as moving with respect to the æther. For simplicity, consider only a single electron, and let us agree to give the name "nucleus" to a certain definite central region of the electron. At any instant the nucleus comprises a certain identical portion of ætherial substance, and if this portion were bodily displaced, with respect to the surrounding æther (which thereby became strained), it might be said that the electron had suffered a displacement of like amount. But such displacement, even through distances far less than the diameter of the electronic nucleus, would presumably be met by an opposing stress of enormous magnitude, arising from the deformation of the circumnuclear æther; while the alternative type of electronic motion, involving only transference of strain from one part of the æther to another, can take place freely without evoking any opposing stress. (The forces mutually exerted by electrons and assemblages of electrons not coming here into consideration.) We shall suppose that when, from any cause, matter experiences a tendency to move, or to change

* The possible magnitude of this term is considered in Appendix D.

its motion with respect to the æther at large, the resulting motion may, with sufficient accuracy, be treated as exclusively of the strain-transference type assumed in the last paragraph. It should be remarked that, whereas vortex motion would be involved in a bodily displacement of that identical portion of ætherial substance which is instantaneously co-extensive with the nucleus of an electron, the type of motion dealt with in § 9 above is irrotational. The latter motion may be supposed to be entirely distinct from that which, arising from the charge of the moving electron, constitutes a surrounding magnetic field. Such a supposition is justifiable if, as I imagine, a magnetic field involves no bodily motion of the æther.

11. Let us now conceive the æther to be traversed by trains of compressional-rarefactional waves, and consider first how the motion of matter will be *directly* affected thereby. As already stated, the æther is regarded as so slightly compressible that, where compressional waves are concerned, any stresses which may be involved, over and above the variations of normal pressure, are relatively insignificant: the æther, in regard to such waves, behaving like a compressible fluid, whose motion is sensibly irrotational. To fix ideas, let the wave-motion be limited to a single harmonic train of plane-waves, propagated with velocity $\mathfrak{V}$ in the direction of x-increasing. It will be convenient to define the waves in terms of the variations of pressure taking place in the æther; thus

$$p = \bar{p} + \mathrm{B} \sin \frac{2\pi}{\lambda} (x - \mathfrak{V}t + \epsilon); \quad . \quad . \quad . \quad (4)$$

so that B is the semi-amplitude of pressure-variation, $\mathfrak{V}$ the velocity of the disturbance, λ the wave-length, and $\lambda/\mathfrak{V}$ the period; while ϵ is a linear magnitude serving to fix the otherwise undefined question of phase.

12. As a particular case, suppose that F, the extrusion of æther per unit-mass of atomic matter (§ 7), is zero, any given volume of space thus containing precisely the same amount of ætherial substance, whether atomic matter is present in that space or not. It is then evident from general considerations that the æther affected by the wave-motion, in

surging to and fro, will merely carry with it whatever atomic matter may be present; no tendency arising for those strain-distributions which constitute matter to lag behind or to outrun the bodily excursions of the ætherial plenum. The same conclusion is reached when we put $F = 0$ in (3); there are then no forces belonging to the class there concerned, or, in other words, the wave-motion has no tendency to accelerate the motion of atomic matter with respect to the æther; so that every portion of atomic matter will continue in its state of rest, or of uniform or accelerated motion *with respect to the æther*, precisely as if no wave-motion were taking place. Under this particular assumption that $F = 0$, the wave-motion (at least in the case of sufficiently great wave-lengths) would appear to be without influence on any of the phenomena which come within the range of our observation, and to belong to that class of activities for which the name *aphenomenal* has been suggested *. Apart from any question as to the correspondence between our special assumptions and the conditions obtaining in the physical universe, it is instructive to realize that the picture here presented—of vibratory motions affecting all matter without influencing the phenomena of ordinary dynamics—is in no way incompatible with the fundamental concepts of dynamical science.

13. If, instead of $F=0$, as in § 12, we suppose F to be finite, so that the mean density of ætherial substance in a region containing atomic matter is different from the density in free æther, it is evident that a general bodily acceleration of the æther will tend to produce an acceleration of atomic matter with respect to the æther. From (3) and (4) we find for the acceleration of any free mass m of neutral matter with respect to the æther

$$a = \left\{ -FB\frac{2\pi}{\lambda}\cos\frac{2\pi}{\lambda}(x - Vt + \epsilon),\ 0,\ 0 \right\}; \quad . \quad (5)$$

while the absolute acceleration A of the æther itself is given by

$$A = -\frac{1}{\rho}\left(\frac{\partial p}{\partial x},\ \frac{\partial p}{\partial y},\ \frac{\partial p}{\partial z}\right) = \frac{1}{\rho}\left\{ -B\frac{2\pi}{\lambda}\cos\frac{2\pi}{\lambda}(x - Vt + \epsilon),\ 0,\ 0 \right\}. \quad (6)$$

* But see also Appendix B.

In this case then, and (as we easily see) generally for small compressional disturbances of the æther, the ratio

$$\textit{acceleration of free matter with respect to the æther : acceleration of the æther} = F\rho \quad . \quad . \quad . \quad . \quad . \quad (7)$$

14. There is another aspect of the hypothetical wave-motion which deserves some attention. At any point fixed with respect to the æther, changes of pressure are supposed to occur, accompanied by changes of density, and as we consider the compressibility of the medium in bulk to be very slight, the proportional changes of pressure and of constitutive (potential) energy will be great compared with the proportional changes of density. From this it may be surmised that corresponding changes in the velocity of radiation would be involved, and if the amplitude of these changes were not a quite insignificant fraction of the whole velocity, effects might result which would be palpable to observation. On the other hand, all effective wave-lengths are assumed to be so immense that, throughout even an astronomically considerable region, the ætherial pressure, and therewith the velocity of radiation, may be treated as uniform at each instant. It is perhaps conceivable that the effect of increased ætherial pressure would be to accelerate not only radiation but *all* phenomena in the same proportion; so that, for example, two universes identically constituted otherwise, but differing from one another as to the pressure prevailing throughout the æther and the rapidity of phenomena in general, might present identical aspects to their respective inhabitants. But even so, in the question whether the pressure of the æther throughout our universe could be gradually or suddenly changed without giving rise to observable effects, some nice points of dynamics would seem to be involved *.

15. The primary effects of the hypothetical compressional waves having been for the moment disposed of, the secondary

* Among the values which, in § 33 below, are tentatively assigned to the various physical magnitudes involved, the variations of potential energy of the æther are set down as so small in comparison with the whole constitutive energy that the special difficulty above referred to seems to be evaded.

or gravitational effects may be considered. Every portion of atomic matter, in so far as its presence implies altered compressibility of the ætherial plenum ($H \neq 0$), behaves as a centre of pulsatory motion, or as an aggregation of such centres. Thus the motion of the æther, to a second approximation, is to be found by superposing on the wave-motion proper a pulsatory motion due to a distribution of alternating source-sink centres, to which should be added (except, as will appear, when $F=0$) a distribution of source-sink doublets arising from the *motion* of atomic matter through the æther. It will be assumed that the whole region of space under consideration is so circumscribed that its greatest dimension is very small compared with the shortest effective wave-length of the compressional disturbance, so that, at any given instant, the primary pressure-variation is sensibly in the same phase throughout, while, to the same order of approximation, the source-sink centres just referred to may be treated as if they existed in an incompressible medium. In accordance with the assumptions explained in §§ 1, 11 above, the variations of normal (hydrostatic) pressure are taken to outweigh in importance all other stresses evoked by varying compression of the medium, and the whole motion to be investigated is accordingly irrotational.

16. Let σ be the density of atomic matter at any point (x, y, z); then the total compressibility of matter-encumbered æther contained in the volume-element $dx\,dy\,dz$ exceeds the total compressibility of a like volume of free æther by

$$H\sigma dx\,dy\,dz, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

in accordance with our definition of H (§ 8). Further, so far as the primary disturbance is concerned, let the pressure p be expressed by

$$p = \bar{p} + \mathfrak{p}\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

the presence of atomic matter in the volume $dx\,dy\,dz$ is thus equivalent to a *source* of strength

$$-H\frac{\partial \mathfrak{p}}{\partial t}\sigma dx\,dy\,dz\,; \quad . \quad . \quad . \quad . \quad . \quad (10)$$

which expression must therefore be equated to $\nabla^2\phi\,dx\,dy\,dz$, ϕ being the velocity-potential of the secondary motion now considered, while $\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$.

With due regard to boundary conditions, this leads to

$$\phi = \frac{H}{4\pi}\frac{\partial \mathfrak{p}}{\partial t}\iiint \frac{\sigma'}{r}dx'dy'dz' \; ; \quad . \quad . \quad . \quad (11)$$

σ' being the density of atomic matter at any point (x', y', z'), while $r^2 \equiv (x-x')^2+(y-y')^2+(z-z')^2$. The integration must be conducted through a sufficiently extended region to include all pulsatory centres which contribute appreciably to the value of ϕ at (x, y, z). In accordance with the assumption mentioned in § 15 above, the factor $\partial \mathfrak{p}/\partial t$ is placed outside the sign of integration; while sources and sinks other than those corresponding to the expression (10) are assumed to be absent from the secondary motion.

17. Treating the primary wave-motion more generally than hitherto, (4) may be taken as expressing a single typical constituent, no restriction being imposed on the direction of propagation of the waves, or on their periods or phases, provided that all effective wave-lengths are supposed sufficiently great, and all limited sources of the primary disturbance sufficiently remote. The form of our equations of course involves the implicit assumption that, in the primary wave-motion, all relations are linear. It is now similarly assumed that, in dealing with the secondary disturbance expressed by (11), non-linear terms may be omitted from the equations, the principle of superposition being accordingly applicable to the primary and secondary motions. Thus, to our order of approximation, we may write

$$p = \bar{p} + \mathfrak{p} + \varpi \; . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and

$$\frac{\partial \psi}{\partial t} + \frac{\partial \phi}{\partial t} = -\int \frac{dp}{\rho} + \text{func. } (t)$$

$$= -\int \frac{d(\mathfrak{p}+\varpi)}{\rho} + \text{func. } (t) \; ; \quad . \quad . \quad (13)$$

where ϕ, ϖ are respectively the velocity-potential and pressure-increment corresponding to the secondary motion, ψ, $\mathfrak{p}$ being the velocity-potential and pressure-increment corresponding to the primary disturbance. Hence

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial t} + \frac{\partial \phi}{\partial t}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{1}{\rho}\left(\frac{\partial \mathfrak{p}}{\partial x} + \frac{\partial \varpi}{\partial x}\right); \quad . \quad (14)$$

and subtracting from this the equation proper to the primary motion, there remains

$$\frac{\partial^2\phi}{\partial t\,.\,\partial x} = -\frac{1}{\rho}\frac{\partial \varpi}{\partial x}; \quad . \quad . \quad . \quad . \quad . \quad (15)$$

which with (11) gives

$$\frac{\partial \varpi}{\partial x} = -\frac{H\rho}{4\pi}\cdot\frac{\partial^2 \mathfrak{p}}{\partial t^2}\iiint \sigma' \frac{\partial}{\partial x}\left(\frac{1}{r}\right) dx'\,dy'\,dz'$$

$$= \frac{H\rho}{4\pi}\frac{\partial^2 \mathfrak{p}}{\partial t^2} I_x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where

$$I_x = \iiint \frac{\sigma'}{r^3}\frac{x-x'}{r}\,dx'\,dy'\,dz'. \quad . \quad . \quad . \quad (17)$$

Hence the secondary motion gives rise to a force upon a mass m of atomic matter at (x, y, z), the x-component of which is, by (2),

$$-mF\frac{\partial \varpi}{\partial x} = -m\frac{HF\rho}{4\pi}\frac{\partial^2 \mathfrak{p}}{\partial t^2} I_x = mX \text{ (say)}, \quad . \quad (18)$$

with corresponding expressions for the y- and z-components.

In (18) F may be written $\overline{F} + \delta F$, where $\overline{F}$ is the mean value of F, and δF, to a sufficient approximation, represents a variable term, arising from and proportional to the variable pressure-term $\mathfrak{p}$, which corresponds to the primary disturbance. Thus

$$F = \overline{F} + \delta F = \overline{F} - H\mathfrak{p}, \quad . \quad . \quad . \quad . \quad (19)$$

by (1); so that (18) becomes

$$X = \frac{\rho}{4\pi}\left(-HF\frac{\partial^2 \mathfrak{p}}{\partial t^2} + H^2\mathfrak{p}\frac{\partial^2 \mathfrak{p}}{\partial t^2}\right) I_x. \quad . \quad . \quad (20)$$

The mean value of X is therefore given by

$$\overline{X} = \mathfrak{av.}\left[\frac{\rho}{4\pi}H^2\mathfrak{p}\frac{\partial^2 \mathfrak{p}}{\partial t^2} I_x\right]. \quad . \quad . \quad . \quad (21)$$

18. As the primary wave-motion is now supposed to involve only very great wave-lengths, so that the (primary) pressure-variation throughout the region of space considered is sensibly a function of the time only, and not of x, y, z, we may write

$$\mathfrak{p} = \Sigma B_s \sin(st + \epsilon_s); \quad . \quad . \quad . \quad . \quad . \quad (22)$$

each B being of the nature of a pressure (in general infinitesimal), while the ϵ's are phase-terms. Hence

$$\frac{\partial^2 p}{\partial t^2} = -\Sigma B_s s^2 \sin(st+\epsilon_s); \quad . \quad . \quad . \quad . \quad (23)$$

and (20) becomes

$$\overline{X} = \frac{\rho}{4\pi} H\overline{F}\Sigma B_s s^2 \sin(st+\epsilon_s) \,.\, I_x$$

$$-\frac{\rho}{4\pi} H^2 \Sigma B_s \sin(st+\epsilon_s) . \Sigma B_s s^2 \sin(st+\epsilon_s) \,.\, I_x. \quad (24)$$

There is thus a gravitative field at (x, y, z) defined by the average values

$$(\overline{X}, \overline{Y}, \overline{Z}) = -\frac{H^2\rho}{8\pi}\Sigma B_s^2 s^2 (I_x, I_y, I_z)$$

$$= -\frac{H^2\rho}{8\pi}\Sigma B_s^2 s^2 \iiint \frac{\sigma'}{r^2}\frac{x-x', y-y', z-z'}{r}\,dx'dy'dz'$$

$$. \quad . \quad . \quad (25)$$

Within the limits of our assumptions, there is seen to be universal mutual attraction of electrically neutral matter, the Newtonian constant being

$$G = \frac{H^2\rho}{8\pi}\Sigma B_s^2 s^2. \quad . \quad . \quad . \quad . \quad . \quad (26)$$

19. It has been pointed out above, as an assumption essential to the theory proposed, that every effective wave-length of the primary ætherial disturbance must be very great compared with the distance between any mutually attractive bodies for which the Newtonian law of inverse squares has been closely verified. It may be worth while to indicate very briefly what would result if this condition were not realized. By way of illustration, let the primary disturbance take the form of plane progressive waves of harmonic type and of definite wave-length λ. Two bodies whose line of centres was perpendicular to the direction of propagation of the waves, and which were at a distance r apart, would attract one another with a force proportional to

$$r^{-2}\cos 2\pi r/\lambda,$$

which, as r is increased, changes sign periodically at intervals of $\frac{1}{2}\lambda$. (If, in place of a single wave-length λ, there were a continuous distribution of wave-energy over a wide range of wave-lengths, there would simply be a falling off of attraction

between the two bodies at a rate more rapid than that of the inverse square of the distance.)

With the primary disturbance in the form of progressive waves travelling in one direction (as above) the case of two bodies in a line not perpendicular to that direction would be more involved; but if the primary waves were travelling indifferently in all directions, there would merely be a gradual extinction of gravitational attraction as the bodies concerned were removed to distances apart not wholly inconsiderable in comparison with the smallest effective wave-length of the primary disturbance.

It will be remarked that, when the distance between two bodies is small compared with the mean effective wave-length of the primary disturbance, the attraction between them exhibits only a *second order* deviation from the Newtonian law of gravitation.

Motional Forces.

20. The next question to be considered concerns the forces arising from the motion of neutral matter with respect to the æther. The expressions for these forces, as will appear, contain a factor F^2, F being (as defined in § 7) the extrusion of æther per unit mass of matter: so that if the mean value F were zero, these forces—which correspond to nothing so far observed—would disappear.

Let (x, y, z) be the co-ordinates of a particle of mass m at time t, and let (x', y', z') be the simultaneous co-ordinates of another particle of mass m', all referred to axes fixed with respect to the general mass of æther. As in § 10, the type of ætherial motion which here concerns us, and which arises from the motion of m and m', with respect to the æther, is sensibly irrotational.

Let us consider the forces exerted on m owing to the motion of m', and let ϕ be the velocity-potential at (x, y, z) due to the motion of m' only, while $r^2 \equiv (x-x')^2 + (y-y')^2 + (z-z')^2$. Then, the effect of the motion of m' being representable by a source-sink doublet at (x', y', z') we readily find

$$\phi = -\frac{\mathrm{F}m'}{4\pi} \cdot \frac{d'}{dt}\left(\frac{1}{r}\right), \quad . \quad . \quad . \quad . \quad . \quad (27)$$

where d'/dt operating on any quantity represents the time-flux of that quantity due to variations of x', y', z' only. Hence also,

$$\dot{\phi}=-\frac{\mathrm{F}m'}{4\pi}\frac{d'^2}{dt^2}\left(\frac{1}{r}\right)$$
$$=\frac{\mathrm{F}m'}{4\pi}\left\{\left(\frac{1}{r^3}-\frac{3(x-x')^2}{r^5}\right)\left(\frac{dx'}{dt}\right)^2+\ldots+\ldots\right.$$
$$-\frac{6(y-y')(z-z')}{r^5}\frac{dx'}{dt}\cdot\frac{dy'}{dt}-\ldots-\ldots$$
$$\left.-\frac{x-x'}{r^3}\frac{d^2x'}{dt^2}-\frac{y-y'}{r^3}\frac{d^2y'}{dt^2}-\frac{z-z'}{r^3}\frac{d^2z'}{dt^2}\right\}. \quad . \quad . \quad (28)$$

21. In accordance with (3), the force components on the particle m at (x, y, z) are

$$-m\mathrm{F}\left(\frac{\partial\rho}{\partial x},\frac{\partial p}{\partial y},\frac{\partial p}{\partial z}\right)=m(\mathrm{X},\mathrm{Y},\mathrm{Z}),\text{ say;}$$

or, to a first approximation, neglecting squares and products of ætherial velocities,

$$m(\mathrm{X},\mathrm{Y},\mathrm{Z})=-m\mathrm{F}\rho\left(\frac{\partial\dot{\phi}}{\partial x},\frac{\partial\dot{\phi}}{\partial y},\frac{\partial\dot{\phi}}{\partial z}\right)$$
$$=-\frac{mm'\mathrm{F}^2\rho}{4\pi}\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\frac{d'^2}{dt^2}\left(\frac{1}{r}\right);$$

that is

$$m\mathrm{X}=-\frac{mm'\mathrm{F}^2\rho}{4\pi}\left\{\left(\frac{9(x-x')}{r^5}-\frac{15(x-x')^3}{r^7}\right)\left(\frac{dx'}{dt}\right)^2\right.$$
$$+\left(\frac{3(x-x')}{r^5}-\frac{15(y-y')^2(x-x')}{r^7}\right)\left(\frac{dy'}{dt}\right)^2+\left(\frac{3(x-x')}{r^5}-\frac{15(z-z')^2(x-x')}{r^7}\right)\left(\frac{dz'}{dt}\right)^2$$
$$+\left(\frac{6(y-y')}{r^5}-\frac{30(x-x')^2(y-y')}{r^7}\right)\frac{dx'}{dt}\frac{dy'}{dt}+\left(\frac{6(z-z')}{r^5}-\frac{30(x-x')^2(z-z')}{r^7}\right)\frac{dx'}{dt}\frac{dz'}{dt}$$
$$-\frac{30(x-x')(y-y')(z-z')}{r^7}\frac{dy'}{dt}\frac{dz'}{dt}+\left(\frac{1}{r^3}-\frac{3(x-x')^2}{r^5}\right)\frac{d^2x'}{dt^2}-\frac{3(y-y')(x-x')}{r^5}\frac{d^2y'}{dt^2}$$
$$\left.-\frac{3(z-z')(x-x')}{r^5}\frac{d^2z'}{dt^2}\right\},$$

with similar expressions for $m\mathrm{Y}$, $m\mathrm{Z}$. . . . . (29)

22. From these equations it follows that, when two bodies are moving in the same direction through the æther, and

with the same unaccelerated velocity, the motional forces experienced by the bodies will be equal and opposite. They will not, however, have the same line of action, except when the motion is either in the line of centres or perpendicular thereto. When the common velocity $\mathbf{v}$ of the two bodies is *in the line of centres*, there is *repulsion* between the bodies amounting to

$$\frac{3}{2\pi}\frac{mm'\mathrm{F}^2\rho}{r^4}\mathbf{v}^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

When the common velocity $\mathbf{v}$ is *perpendicular to the line of centres* there is *attraction* amounting to

$$\frac{3}{4\pi}\cdot\frac{mm'\mathrm{F}^2\rho}{r^4}\mathbf{v}^2 = \mathrm{K}\frac{mm'}{r^4}\mathbf{v}^2, \quad . \quad . \quad . \quad . \quad (31)$$

where

$$\mathrm{K} \equiv \frac{3}{4\pi}\mathrm{F}^2\rho. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

There is neither attraction nor repulsion when the common velocity of the two bodies is inclined at an angle $\tan^{-1}\sqrt{2}$, or approximately 54° 44′ to the line of centres. But in addition to the forces in the line of centres, there are in general forces perpendicular to this line, and if we imagine the two bodies connected by a rigid immaterial (or material) bar, the tendency will be to set the line of centres *perpendicular* to the direction of motion; the couple acting on the system only vanishing when the direction of motion is parallel or perpendicular to the line of centres. When the direction of motion is inclined 45° to the line of centres, the couple has its maximum value, namely,

$$\frac{3}{4\pi}\cdot\frac{mm'\mathrm{F}^2\rho}{r^3}\mathbf{v}^2 \quad \text{or} \quad \mathrm{K}\frac{mm'}{r^3}\mathbf{v}^2. \quad . \quad . \quad . \quad (33)$$

23. If we consider a bar made of given material and of definite proportions, and so suspended as to have a given period of oscillation about an axis through its centre of figure, then the angular deviation of the bar arising from a given velocity of translation through the æther will, at its maximum, be inversely proportional to the square of the linear dimensions of the bar. Though no experiments may have been specially made with the object of detecting motional forces of this kind, it seems certain that if more than a very slight

residual effect existed, it must have become apparent in other observations. None the less it would be interesting to make some definite experiments to put the question to the test, the apparatus required being simple. Incidentally, if the couples proved to be of measurable magnitude, we could readily obtain the data necessary for a determination of our motion relatively to the æther. This would involve no contradiction of the principle of relativity in electromagnetism; the phenomena with which we are here concerned not being electromagnetic in character.

24. Though we do not know what may be the value of the constant K defined by (32), or even whether it be different from zero, we can make some attempt to assign an upper limit to its value. Imagine two compact masses of 1 gram each, connected by a bar of negligible mass, the distance between their centres being 2 centimetres. Let this system be suspended by a quartz fibre, so that its axis is horizontal, and the period of its oscillations 30 seconds. Let us further suppose that we are able to detect any deviation as great as one second of arc on either side of a mean position, when the suspended system turns about a vertical axis. Now in virtue of the earth's orbital motion, the highest value attained by the velocity $\mathbf{v}$ must be at least as great as 32 kilometres per second ($=3{\cdot}2\times10^{6}$ cm. per second), so that by (33) our suspended system, when most favourably placed and oriented, could experience a couple as great as

$$\tfrac{1}{8}(3{\cdot}2\times10^{6})^{2}\mathrm{K}\doteqdot1{\cdot}28\times10^{12}\mathrm{K}. \quad . \quad . \quad . \quad (34)$$

Also the restoring moment when the system is displaced through one second of arc about a vertical axis is $4{\cdot}4\times10^{-7}$ dyne-cm.; and for the experiment to give a null result, we should accordingly have

$$\mathrm{K} \not> 3{\cdot}4\times10^{-19} \text{ in c.g.s. measure.} \quad . \quad . \quad (35)$$

Assuming, in advance, that a null result would be obtained in this case, the value which, in the sequel, will be provisionally assumed is

$$\mathrm{K}=2\times10^{-20}. \quad . \quad . \quad . \quad . \quad . \quad (36)$$

25. This tentative value of K may be compared with one suggested by another way of regarding the question. If the

negative electron could be regarded as having a vacuous spherical nucleus of radius 10^{-13} cm., and if we assume the atom of hydrogen to contain only one negative electron, while the number of molecules in 1 c.c. of gas at 0° C. and 1 atmosphere pressure is taken to be $3{\cdot}9\times10^{19}$; then it appears that, on the score of negative electrons alone, F, the extrusion of æther per unit mass of matter (§ 7), would be 4×10^{-15} c.c. of æther per gram of matter; equivalent to ·004 gram of æther per gram of matter, when an ætherial density of 10^{12} is assumed*. Though the existence of a vacuous nucleus in an electron does not appear reconcilable with some of the views adopted in this paper, it may be worth while to remark that the value of F just mentioned leads to $K=4\times10^{-18}$, which is 200 times as great as the estimate (36), and about 12 times as great as might just suffice to give positive results in the hypothetical experiment of § 24.

Conjectural Estimates.

26. The values of a further number of physical magnitudes will have to be conjectured before a quantitative view of the theory can be attempted. Some of these magnitudes are introduced in the following seven paragraphs, which are devoted to establishing certain simple relations amongst the quantities concerned; the immediate object being to discover whether there is any consistent set of values which does not bring our theory into conflict with experience.

27. From (32)

$$F=2\sqrt{\left(\frac{\pi K}{3\rho}\right)}. \quad . \quad . \quad . \quad . \quad . \quad (37)$$

As regards the primary disturbance, the problem (as in § 11) may be simplified by limiting our consideration to a single train of plane waves of definite wave-length λ, the greatest linear dimension of the region with which we are concerned being but a small fraction of λ, so that, to a sufficient degree of approximation, the pressure variations at any point within that region are given by

$$p=B\sin(2\pi nt-\epsilon), \quad . \quad . \quad . \quad . \quad . \quad (38)$$

* *Cf.* § 35 below.

n denoting the frequency of the disturbance. Equation (26) now becomes

$$G = \frac{H^2\rho}{8\pi} B^2 s^2 = \frac{\pi}{2} . H^2 \rho B^2 n^2, \quad . \quad . \quad . \quad (39)$$

since we are now limited to a single s, which is identical with $2\pi n$.

F, the extrusion of æther per unit mass of matter (§ 7) suffers variations given by

$$\delta F = -Hp = -HB \sin (2\pi nt - \epsilon). \quad . \quad . \quad (40)$$

Let the greatest numerical value of δF be ζF; then by (40), (39),

$$\zeta^2 F^2 = H^2 B^2 = 2G/\pi\rho n^2; \quad . \quad . \quad . \quad . \quad (41)$$

or, remembering (37),

$$n = \frac{1}{\pi\zeta} \sqrt{\left(\frac{3G}{2K}\right)}. \quad . \quad . \quad . \quad . \quad . \quad (42)$$

28. Let κ denote the bulk-modulus of elasticity of the æther, or $1/\kappa$ its compressibility. By § 8 above, $H\sigma$ is the *additional* ætherial compressibility where there are σ grams of atomic matter per cubic centimetre; let this be equivalent to $\mu\sigma/\kappa$, so that $\mu\sigma$ is a pure number. Thus

$$H = \mu/\kappa. \quad . \quad . \quad . \quad . \quad . \quad (43)$$

In virtue of our assumption that κ is overwhelmingly greater than any other elastic modulus of the æther, we have for the square of the velocity of compressional waves

$$n^2\lambda^2 = \kappa/\rho,$$

whence

$$\kappa = n^2\lambda^2\rho. \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

From (39), (43), (44),

$$B = \frac{n\lambda^2}{\mu} \sqrt{\left(\frac{2\rho G}{\pi}\right)}, \quad . \quad . \quad . \quad . \quad . \quad (45)$$

$$\frac{B}{\kappa} = \frac{1}{\mu n} \sqrt{\left(\frac{2G}{\pi\rho}\right)}, \quad . \quad . \quad . \quad . \quad . \quad (46)$$

and

$$\frac{B^2}{2\kappa} = \frac{\lambda^2}{\mu^2} \frac{G}{\pi E} . E, \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

where E is the constitutive energy per cubic centimetre of the æther. Thus the factor of E in the right-hand member

of (47) expresses the maximum potential energy per unit volume due to the wave-motion as a proportion of the constitutive energy of the medium. Moreover, this maximum potential energy per unit volume corresponds to an increment δF, equal to ζF, above the mean value F, and is thus, to our order of approximation, proportional to ζ^2. We may write, then,

$$\tfrac{1}{2}\zeta^2 = \vartheta \frac{\lambda^2}{\mu^2} \cdot \frac{G}{\pi E},$$

or

$$\zeta = \frac{\lambda}{\mu}\sqrt{\left(\frac{2G\vartheta}{\pi E}\right)}; \quad . \quad . \quad . \quad . \quad (48)$$

which serves to define a numerical constant ϑ now first introduced. This constant might, not improbably, be unity, or a number of that order of magnitude, and for illustrative purposes it is later assumed that

$$\vartheta = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

29. From (42), (48),

$$n = \frac{\mu}{2\lambda}\sqrt{\left(\frac{3E}{\pi K\vartheta}\right)}, \quad . \quad . \quad . \quad . \quad (50)$$

the velocity of compressional waves in free æther being

$$V = n\lambda = \frac{\mu}{2}\sqrt{\left(\frac{3E}{\pi K\vartheta}\right)}. \quad . \quad . \quad . \quad (51)$$

Using (50) in conjunction with (44) and (45) respectively,

$$\kappa = \frac{3\mu^2\rho E}{4\pi K\vartheta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

$$B = \frac{\lambda}{\pi}\sqrt{\left(\frac{3EG\rho}{2K\vartheta}\right)}; \quad . \quad . \quad (53)$$

while from (43)

$$H = \frac{4\pi K\vartheta}{3\mu\rho E}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

30. An expression is readily found for the maximum translational velocity attained by any element of æther in the primary wave-motion, half the square of this velocity multiplied by the density of the æther being equal to the maximum kinetic or maximum potential energy per unit volume; that is, equal to $B^2/2\kappa$. From (47) the desired expression is at once obtained, and only requires to be divided

by $2\pi n$ to give the semi-amplitude of vibration (maximum displacement), and multiplied by $2\pi n$ to give the maximum acceleration of any æther-element. Remembering (50) the three expressions are:—

$$\text{Maximum displacement from mean position} = \frac{\lambda^2}{\pi\mu^2}\sqrt{\left(\frac{2KG\mathfrak{d}}{3\rho E}\right)}, \quad (55)$$

$$\text{Maximum velocity} = \frac{\lambda}{\mu}\sqrt{\left(\frac{2G}{\pi\rho}\right)}, \quad (56)$$

$$\text{Maximum acceleration} = \sqrt{\left(\frac{6GE}{\mathfrak{d}\rho K}\right)}. \quad (57)$$

31. Bearing in mind § 9, and in particular equation (3), we have also, *for any particle of neutral matter considered with respect to the æther*,

$$\text{Maximum displacement from mean position} = \frac{2\lambda^2 K}{3\mu^2}\sqrt{\left(\frac{2\mathfrak{d}G}{\pi E}\right)}, \quad (58)$$

$$\text{Maximum velocity} = \frac{2\lambda}{\mu}\sqrt{\left(\frac{2GK}{3}\right)}, \quad (59)$$

$$\text{Maximum acceleration} = 2\sqrt{\left(\frac{2\pi GE}{\mathfrak{d}}\right)}; \quad (60)$$

these three equations of course concern only that motion of matter with respect to the æther which is directly due to the primary wave-motion.

32. From (46), (50) we have for the range of proportional changes of volume of the æther, above and below mean value,

$$\frac{B}{\kappa} = \frac{2\lambda}{\mu^2}\sqrt{\left(\frac{2GK\mathfrak{d}}{3\rho E}\right)}; \quad (61)$$

while for the rate (in ergs per second) at which wave-energy is being propagated through any square centimetre of surface whose plane is parallel to the wave-fronts, we must take the product of the wave-energy per c.c. multiplied by the velocity of propagation. Now throughout our purely progressive wave-train the *total* energy per unit volume is $B^2/2\kappa$, the maximum value attained by the potential energy per unit volume. Thus multiplying (47) by (51) the rate at which wave-energy is being propagated, in ergs per second per

square centimetre, is found to be

$$\mathcal{V}B^2/2\kappa = \frac{\lambda^2 G}{2\mu}\sqrt{\left(\frac{3E}{\pi^3 K\vartheta}\right)} \quad . \quad . \quad . \quad (62)$$

33. All the quantities with which we are now concerned are expressed in terms of the following seven quantities:—

(i.) G the Newtonian gravitation constant $= 6{\cdot}66 \times 10^{-8}$.
(ii.) K the motional constant, defined by (31) [2×10^{-20}].
(iii.) ρ the density of the æther [10^{12}].
(iv.) E the constitutive energy per c.c. of the æther [10^{33}].
(v.) λ the wave-length of the primary disturbance [10^4 astronomical units $= 1{\cdot}5 \times 10^{17}$ cm.].
(vi.) μ the proportional increase of ætherial compressibility arising from the presence of 1 gram of atomic matter per c.c., as defined at the beginning of § 28 [20].
(vii.) ϑ a numeric defined by (48) [1].

Of these seven, only the first, the gravitation constant, is known, the values indicated (in square brackets) for the remaining six being merely conjectural. If no set of values could be found which did not lead to demonstrably false results, we should have to conclude that the theory in the form here suggested was untenable. The considerations leading to the choice of the values in question will be best understood after the corresponding values of some related quantities have been computed. The table (p. 92) gives in a collected form the relations obtained in §§ 27–32.

34. We may now review briefly the tentative values attributed in the last paragraph to K, ρ, E, λ, μ, and ϑ. The value 2×10^{-20} is assumed for the motional constant K, in accordance with (35); the supposition being that K is small enough for a null result to be obtained in the test suggested in § 24, but not with any great margin to spare. It might be that the test referred to would give a positive result, and in any case a much more sensitive disposition could easily be devised. But since no such positive result has yet been recorded, we cannot on this score assign any lower limit to the value of K, nor is our choice of something near the highest

TABLE.

Values of certain quantities connected with the hypothetical (primary) wave-motion, expressed in terms of G, K, ρ, E, λ, μ, and $\mathfrak{I}$, together with the numerical values for the same quantities derived from those already assigned to G, K, &c.

Nature of quantity.	Symbol.	Equivalent expression.	Value in C.G.S. units.	Value otherwise expressed.
For any element of æther: Maximum displacement.	...	$\frac{\lambda^2}{\mu^2}\sqrt{\left(\frac{2\mathfrak{I}KG}{3\rho E}\right)}$	$6{\cdot}9\times10-5$	·69 micron.
For any element of æther: Maximum velocity.	...	$\frac{\lambda}{\mu}\sqrt{\left(\frac{2G}{\pi\rho}\right)}$	$5{\cdot}6\times10^{8}$	56 kilom. per sec.
For any element of æther: Maximum acceleration.	...	$\sqrt{\left(\frac{6GE}{3\rho K}\right)}$	$4{\cdot}5\times10^{17}$	
For any element of atomic matter, with respect to the æther: Maximum displacement.	...	$\frac{2\lambda^2K}{3\mu^2}\sqrt{\left(\frac{2\mathfrak{I}G}{\pi E}\right)}$	$2\times10-8$	1/5000 micron.
For any element of atomic matter, with respect to the æther: Maximum velocity.	...	$\frac{2\lambda}{\mu}\sqrt{\left(\frac{2GK}{3}\right)}$	$1{\cdot}6\times10^{3}$	16 metres per sec.
For any element of atomic matter, with respect to the æther: Maximum acceleration.	...	$2\sqrt{\left(\frac{2\pi GE}{\mathfrak{I}}\right)}$	$1{\cdot}3\times10^{14}$	
Frequency of primary disturbance.	n	$\frac{\mu}{2\lambda}\sqrt{\left(\frac{3E}{\pi\mathfrak{I}K}\right)}$	$1{\cdot}29\times10^{16}$	
Velocity of compressional waves in æther.	$\mathfrak{V}$	$\frac{\mu}{2}\sqrt{\left(\frac{3E}{\pi\mathfrak{I}K}\right)}$	$2{\cdot}19\times10^{27}$	$2{\cdot}3\times10^{9}$ light-years per second.
Extrusion of æther per unit mass of matter (§ 7).	F	$2\sqrt{\left(\frac{\pi K}{3\rho}\right)}$	$2{\cdot}9\times10-16$	
Mass of æther extruded per unit mass of matter.	F_ρ	$2\sqrt{\left(\frac{\pi\rho K}{3}\right)}$	$2{\cdot}9\times10-4$	
Maximum proportional deviation of F from its mean value.	ζ	$\frac{\lambda}{\mu}\sqrt{\left(\frac{2\mathfrak{I}G}{\pi E}\right)}$	$5{\cdot}6\times10-5$	
Bulk-modulus of elasticity of the æther.	κ	$\frac{3\mu^2\rho E}{4\pi\mathfrak{I}K}$	$4{\cdot}78\times10^{56}$	$4{\cdot}72\times10^{50}$ atmospheres.
Additional ætherial compressibility per unit density of matter present.	H	$\frac{4\pi\mathfrak{I}K}{3\mu\rho E}$	$4{\cdot}2\times10-56$	
Maximum deviation of ætherial pressure from its mean value.	B	$\frac{\lambda}{\pi}\sqrt{\left(\frac{3EG\rho}{2\mathfrak{I}K}\right)}$	$3{\cdot}9\times10^{45}$	$3{\cdot}8\times10^{39}$ atmospheres.
Maximum proportional deviation of ætherial density from its mean value.	B/κ	$\frac{2\lambda}{\mu^2}\sqrt{\left(\frac{2\mathfrak{I}GK}{3\rho E}\right)}$	$8{\cdot}1\times10-22$	
Total energy of primary disturbance per cubic centimetre.	$B^2/2\kappa$	$\frac{\lambda^2G}{\pi\mu^2}$	$1{\cdot}6\times10^{24}$	$1/(6\times10^{8})$ of the constitutive energy of the medium.
Energy propagated through 1 sq. cm. of normal surface per second.	$\mathfrak{V}B^2/2\kappa$	$\frac{\lambda^2G}{2\mu}\sqrt{\left(\frac{3E}{\pi^3\mathfrak{I}K}\right)}$	$3{\cdot}4\times10^{33}$ ergs.	$3{\cdot}4\times10^{41}$ kilowatts per sq. cm.

of seemingly admissible values made necessary by the quantities given in the table. The experimental determination of the motional constant K, or of an upper limit to its value, would be interesting quite apart from its bearing on this theory; the question involved being, essentially, whether or not the total of ætherial substance comprised in any given volume is modified by the presence of atomic matter within that volume.

35. The values assumed for ρ and E (the density and the constitutive energy per unit volume of the æther) are those lately suggested by Sir O. Lodge *, who has given very convincingly his reasons for supposing that, in order of magnitude, the true values may not be very different from these estimates. Incidentally it may be remarked that Lodge's estimate of the constitutive energy per unit volume of the æther is about 6 times the electrostatic energy per unit volume, close to the surface of a negative electron, calculated on the usual assumptions that practically the whole inertia of the electron is electromagnetic, and that the ordinary linear relations of the electromagnetic field hold good right up to the surface of the electron.

36. For the wave-length of the primary disturbance, we must assume a value so great that, even at distances of at least several astronomical units, the inverse square law of gravitational attraction is sensibly accurate. By putting $\lambda = 10^4$ astronomical units, this condition seemed to be amply fulfilled, since for distances small compared with λ, the deviations from the inverse-square law would be only of the second order (§ 19). The chief consideration against assigning a very much greater value to λ is that ζ must be kept small to preserve linearity of relations, and also, I imagine, for general plausibility.

37. The same consideration (keeping ζ small) leads us to assign a fairly high value to μ (defined in § 28). It is difficult to see what sort of value μ should be expected to have: whether the proportional increase of ætherial compressibility due to a space being "filled" with water instead of being vacuous should be a large number or a small fraction. The value -20—chosen for μ would make the phase-difference

* Phil. Mag. vol. xiii., 1907; Nature, lxxv. p. 519, 1907.

of pressure-fluctuation of about the same order between the surface and the centre of the sun as between the surface of the sun and that of the earth, under maximum conditions.

The somewhat vague grounds on which the value unity is assigned to $\mathfrak{d}$ were indicated at the end of § 28.

38. Some of the derived quantities in the table above may now be considered. In the first place, take ζ, which represents the maximum deviation from its mean value of F the "specific displacement" of atomic matter (§ 7). If the primary disturbance is slight enough to be adequately represented by linear equations, we should expect on general grounds that ζ would be small, and the value for ζ derived from our assumptions is $5{\cdot}6 \times 10^{-5}$, while the energy of the primary disturbance per unit volume, under the same assumptions, appears as $1/(6 \times 10^8)$ part of the constitutive energy of the æther.

39. For n, the frequency of the primary disturbance, $1{\cdot}29 \times 10^{10}$ per second is found, leading to $2{\cdot}19 \times 10^{27}$ cm. per second, or over two thousand million light-years per second, for $\mathfrak{V}$, the velocity of propagation of a compressional-rarefactional disturbance through the æther. This would amply suffice for the sensibly instantaneous character of gravitational attraction.

40. The bulk-modulus of elasticity of the æther, denoted by κ is, with our assumptions, represented by $4{\cdot}78 \times 10^{66}$ dynes per sq. cm., or $4{\cdot}72 \times 10^{60}$ atmospheres. Comparing this with the assumed constitutive energy of the æther (10^{33} ergs per c.c.), which is taken to be the basis of dielectric elasticity (the reciprocal of dielectric capacity) in free æther, the comparison appears to be consistent with our assumption that the bulk-modulus is enormously greater than any other elastic modulus of the æther; an assumption in accordance with which the æther, in relation to compressional-rarefactional motions, has been treated as a fluid.

Direct Effects of the Primary Disturbance.

41. Considering next the motion of the æther constituting the primary disturbance, and the directly resulting motion of matter with respect to the æther, the question arises: what phenomena, if any, would present themselves to our

observation if such motions were actually taking place? In the first place, bearing in mind § 12 above, it will be evident that a bodily vibratory motion of the æther, of sufficiently great wave-length, *with atomic matter equally partaking in such motion*, could give rise to no observable phenomena; if any effects are to be made manifest, as a direct consequence of the primary disturbance, these must arise from the motion of matter with respect to the æther. So far as I have been able to see, after long and careful consideration, there would be nothing capable of affecting interference phenomena, no heating effect, and no production of electromagnetic waves, provided only that one condition were realized. That condition is that the positive and negative electrons should have identical accelerations impressed on them by the direct action of the primary disturbance. The discrimination between positive and negative electrons in relation to gravitational agency is a question presenting many aspects for consideration. Some of these are touched upon in Appendices A and B.

42. Perhaps the most surprising of the tentative numerical values tabulated above, is that suggested as the rate at which energy is being propagated through the æther, per square centimetre of surface normal to the direction of propagation; the energy so propagated in one second exceeding by many million times the Sun's entire store of available heat. The problem has of course been simplified by limiting the primary disturbance to a single progressive wave-train of definite wave-length; but the result would not have been greatly different if a series of wave-lengths had been included as in § 18, and it is of course quite immaterial whether we assume the primary waves to be travelling indiscriminately in all directions, or predominantly or exclusively in a single direction. It must indeed be admitted that the basis of the estimates put forward in § 33 is nothing better than guesswork; but, after making all allowances for the very great uncertainty attaching to such conjectures, it appears to be an inseparable feature of the theory proposed that energy should be travelling through the æther on a prodigious scale.

43. The theory of gravitation which forms the subject of this paper has a good deal in common with its predecessors. In the first place, it is made to appear that gravitational attraction is not an essential and inseparable attribute of matter, the Newtonian constant being theoretically susceptible of increase or diminution, or even of entire suppression through a change of external circumstances. In this respect the present theory is comparable with Le Sage's hypothesis of ultramundane corpuscles, as well as with a scheme put forward by Challis *, which likewise assumed the existence of compressional waves in the æther, although the function attributed to these waves was very different. But the theory now advanced is much more akin to that of Prof. Hicks, which indeed suggested it, and of which it may be regarded as a development. The introduction of ætherial compressional waves of great wave-length to actuate the pulsatory movements of atomic matter, and thus secure the necessary agreement of phase amongst the pulsating centres, is one of the main modifications suggested; another being the manner in which the capacity for pulsatory motion, assumed to be associated with the electrons, is represented as mobile through the æther. In this way we avoid the necessity for supposing that any element of ætherial substance ever deviates by more than a minute amount from its mean position, the motion throughout being of "stationary" type, while at the same time the problem with which we have to deal is essentially one of hydrodynamics, capable of being worked out to a first order by means of a simple analysis. We are enabled to treat the æther as if it were a fluid, not because we assume it to yield freely to certain types of stress, but because the distortions involved in the motions considered are so excessively minute that no appreciable opposing stresses are called up, save only the changes of hydrostatic pressure, which owe their importance to the enormous value attributed to the bulk-modulus of elasticity of the æther. Apart from the simplicity thus attained, it appears to me most desirable that we should discard, if possible, the conception of matter, or of

* See a review by Maxwell, 'Nature,' vol. viii.; Scientific Papers, vol. ii. p. 338.

any parts or properties associated with matter, as grossly ploughing a course through a reluctant æther. All that we know of æther and of matter seems to indicate that the mobility of matter is absolute, and that the elastic properties of the æther remain perfect, notwithstanding the motions of material bodies, or even of detached negative electrons travelling with velocities approaching that of radiation. And the only way in which I have been able to conceive of matter as travelling through the æther, without doing unwarrantable violence to the structure of the medium, is by supposing the entire phenomenon of motion to be reducible to a transference of strain, so that no event in our universe involves a progressive yielding of the æther.

44. By Maxwell * it was felt to be an objection to such hypotheses as those of Le Sage and of Challis, that they involved a continual expenditure of energy for the maintenance of gravitational attraction, the conservation of energy in such cases being "apparent only." The present theory is equally open to this criticism; but the objection is a metaphysical rather than a physical one, and in view of recent developments it has hardly the force which it might have appeared to have some forty years ago. We have become accustomed to the idea that our rapid motion through a medium of enormous density not only fails to provide us with a useful source of energy, but defies the most refined attempts to detect it by means of terrestrial observations.

And there is nothing more inherently improbable in the notion that our universe may be traversed by waves of enormous energy, perceptible to us only by means of a minute secondary effect—gravitation. That we should be unable, even in our dealings with this feeble residual phenomenon, to extract with continual profit the minutest portion of the energy so abundantly propagated, is a view which may appear to us somewhat ironical, but which is not out of harmony with the trend of modern physical conceptions.

45. Whatever may be the difficulties of the theory, this attempt to contribute something towards the explanation of gravitation has appeared to me sufficiently suggestive to be

* *Loc. cit.*

worth publication; the more so as incidentally certain questions are raised, some of which, it may be hoped, are capable of being decided by experiment.

Appendix A.

Electromagnetic Phenomena which might conceivably arise from Gravitation.

46. Since in this paper gravitation is regarded as a secondary or residual effect, due to the influence of atomic matter on the propagation of compressional ætherial waves, it might have seemed more logical to discuss the electromagnetic phenomena possibly produced by such waves, before considering effects of a like character which might be supposed to arise from gravitational attraction. In the first place, however, gravitation is known to exist, and in the second place, the effects to be considered, being statical, are comparatively simple. Accordingly gravitation may be dealt with first in this connexion.

47. Independently of any theory as to the cause of gravitation, if we suppose that ordinary matter is made up of electrons, then it seems reasonable to conclude that any influence exerted on matter can be analysed into influences exerted on electrons. Again, experiment indicates unmistakably that, whatever the nature of positive and negative electrons may be, they are far from being symmetrical opposites. Thus it is natural to inquire what effects are to be expected when we discriminate between positive and negative electrons under gravitational influence.

48. To take the simplest case, consider a body moving without constraint in a uniform field of gravity, the acceleration of the body f thus agreeing in direction and magnitude with the strength of the field. Suppose that, electrically, the condition of the body is not changing; then on the whole both positive and negative electrons are moving with the acceleration f. Now if the forces exerted in a field of gravity are in the same sense on positive and negative electrons, and of magnitudes proportional respectively to the masses of such electrons, the identical acceleration of the two denominations will follow, without additional forces of electrostatic type

being called into play. But if the gravitational forces on positive and negative electrons do not conform to the condition just referred to, they can always be resolved into two sets: one set of like forces proportional respectively to the masses of the positive and negative electrons on which they act, and another set of forces acting equally and oppositely on the positive and the negative. This latter set produces on the body under consideration the same effect as would result from a uniform electrostatic field, the body becoming electrically polarized if of dielectric material, or, if a conductor, acquiring a surface charge of electrification without internal polarization.

49. The state of things here suggested is somewhat different from anything ordinarily contemplated in electrostatics; the gravitational quasi-electromagnetic intensity which acts throughout the substance of a conducting body being balanced by the true electromotive intensity arising from the surface distribution of electricity, so that we have an electrostatic field of force exerted in a conducting medium in equilibrium. In these circumstances, it may not be superfluous to point out that, in estimating the electric field-intensity from the surface charges, no dielectric constant other than that of free æther comes into play. For in an electrostatic field, the intensity at any point is determined jointly by the signs and positions of all the electrons concerned, and by nothing else, each electron contributing to the field-intensity the same component as if the other electrons were non-existent. As is well understood, the electronic theory affords on these lines an account of the dielectric qualities of different media; such qualities, however, being without any bearing on the problem, so long as *on the whole* there are no forces tending to displace relatively to one another the positive and the negative electrons which together make up the medium under consideration.

50. Let μ_1, μ_2 denote the *masses* of the positive and negative electrons respectively, contained in unit mass of matter; then

$$\mu_1+\mu_2=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63).$$

Also let the total *charge* of the positive electrons in unit

mass of matter be $e\chi$, the total charge of the negative electrons being accordingly $-e\chi$: where e is the quantity of electricity required to liberate one gram of hydrogen electrolytically, and χ is a numeric.

Further, in a field of gravity whose intensity is measured by the acceleration f, let

gravitational attraction on mass μ_1 of positive electrons be $\mu_1\gamma_1 f$,
" " " μ_2 of negative electrons be $\mu_2\gamma_2 f$.

We then readily find, for a mass moving freely in the field f, that the electrical condition is the same as if an electrostatic field were exerted, whose intensity, measured in the same direction as f, is

$$(-\mu_1+\mu_1\gamma_1)f/e\chi=(\mu_2-\mu_2\gamma_2)f/e\chi\ ; \quad . \quad . \quad (64)$$

this equivalent field-intensity being expressed in electrostatic or electromagnetic measure, according as e is in electrostatic or electromagnetic measure.

51. The (single) condition that this quasi-electromotive-intensity should vanish is $\gamma_1=1=\gamma_2$, in which case the gravitational forces acting on masses μ_1, μ_2 of positive and negative electrons in a field of intensity f are respectively $\mu_1 f$, $\mu_2 f$; that is to say, in the proportion of the masses. As will be seen in Appendix B, the gravitation theory of this paper appears almost certainly to demand that the condition just referred to should be satisfied; but putting that theory for the time being on one side, it is interesting to trace out some consequences of a less restricted view as to the relative behaviour of positive and negative electrons under the action of gravity.

52. When the body with which we have to deal is not moving freely in a sensibly uniform gravitational field, the corresponding results cannot be written down without making some assumption as to the relative mobility of positive and negative electrons in a conducting or dielectric mass. For example, let us trace the consequences of supposing that the negative electrons alone possess sufficient mobility to enable them to migrate through unrestricted distances within the mass of a solid conducting substance; the negative electrons thus serving to convey the whole of any current which may

be flowing in the substance. In a cylindrical conductor whose axis is vertical, let a current be flowing upward; consider what happens when negative electrons whose aggregate charge is $-\epsilon\chi$ pass downward through the cylinder, and in particular through a certain stratum of the cylinder, bounded above and below by horizontal planes, whose distance apart is h. According to our notation, $-\epsilon\chi$ is the aggregate charge of the negative electrons which are comprised in one gram of neutral matter, and which have collectively the mass μ_2, experiencing in a gravitational field of intensity g the force $\mu_2\gamma_2 g$, measured downward, so that the work done by gravity while the quantity $-\epsilon\chi$ moves downward through the stratum h is $\mu_2\gamma_2 gh$. There is thus virtually, corresponding to a height h of the conductor, an electromotive force

$$-\mu_2\gamma_2 gh/\epsilon\chi \text{ (measured downward)}; \quad . \quad . \quad (65)$$

the quasi-electromotive-intensity due to the action of gravity being therefore

$$-\mu_2\gamma_2 g/\epsilon\chi \text{ (measured downward)}. \quad . \quad . \quad . \quad (66)$$

53. If, in place of gravity acting, an acceleration f is impressed on an isolated conductor, then the negative electrons of total charge $-\epsilon\chi$, having an aggregate mass μ_2, must experience, when there is electrical equilibrium, a resultant force $\mu_2 f$, and in virtue of the assumed mobility of the negative electrons, this resultant force $\mu_2 f$ must be due to an electrical distribution on the conductor in question. But an electrical distribution which would exert the force $\mu_2 f$ on certain electrons is precisely that distribution which would arise from such an external field as would by itself exert the force $-\mu_2 f$ on the same electrons. In other words, the conducting body caused to move with acceleration f, becomes electrified precisely as if it were placed in an electrostatic field of intensity

$$\mu_2 f/\epsilon\chi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

If we take f to be vertically downward, and identical in value with g, we find by combining (66) and (67) an expression for the virtual electrostatic field due to the action of gravity on a freely falling body; and this expression agrees

with (64) which was obtained independently of any assumption regarding the respective mobilities of positive and negative electrons.

54. On the other hand, if we were to assume that, while currents of conduction were carried exclusively by the negative electrons, it was the positive electrons alone which were acted upon by gravitational attraction, we should be led to the conclusion that bodies at rest relatively to (say) the earth would acquire no electrical charges through the action of gravity.

55. Reverting now to the illustrative assumption of § 52, we may attempt to form some idea of the magnitude of the electromotive effects to be expected when the negative electrons are supposed to be not only the exclusive carriers of currents of conduction, but also the only objects influenced by gravity. A further assumption is needed as to the total of positive or of negative charges carried by the electrons in unit mass of matter. Let it be assumed, for example, that $\chi=1$ (see § 50), so that the negative electrons comprised in one gram of hydrogen (or of other substance) will have an aggregate charge equal to ϵ, the quantity of electricity required to liberate one gram of hydrogen by electrolysis. The expression (66) for the downwardly-directed quasi-electromotive intensity resulting from the direct action of gravity becomes $-\mu_2\gamma_2 g/\epsilon$, where $\mu_2\gamma_2 g$ is the downward force exerted in the earth's gravitational field upon the mass μ_2 of negative electrons comprised in one gram of matter. But under our present assumptions this latter force is simply the weight of one gram of matter, and is equivalent to g; so that finally the downwardly-directed quasi-electromotive intensity is measured by $-g/\epsilon$. Now ϵ is roughly 10^4 e.m. units of quantity per gram, and g may be taken as about 980 cm./sec.^{2}; consequently, for the quasi-electromotive intensity affecting stationary bodies at the earth's surface the estimate obtained is 980×10^{-4} e. m. units of potential per cm., or $9{\cdot}8\times10^{-10}$ volt per cm.

56. On applying the axiom that gravity cannot be made to furnish an unlimited supply of energy, it is evident that the total gravitational electromotive force round any closed circuit of conducting bodies must vanish, however small may

be the conductivity of those bodies; so that any attempt to detect such electromotive effects galvanometrically must fail. The only methods conceivably available would be those depending upon the convection of electric charges by insulated conductors moving between one level and another, and making contact intermittently with the extremities of an elongated vertical conductor. It would be easy to devise a machine for performing continually a cycle of operations, whereby any electromotive influence due to gravity would be rendered effective in gradually imparting contrary charges to a pair of insulated hollow conductors. But apart from the excessive minuteness of the effect to be looked for, and the thermoelectric and other disturbing influences almost necessarily encountered, another consideration steps in to render any such attempt nugatory. To guard against stray electric influences, our whole apparatus would require to be enclosed in a conducting envelope, the charge induced on which by gravitational influence would exactly neutralize, throughout the interior of the envelope, the quasi-electromotive intensity due to the direct action of gravity. Thus, even if we suppose positive and negative electrons to be oppositely acted upon by a gravitational field (in which case, for example, the aggrgate attractive force on the negative electrons in a body might greatly exceed the weight of the body), it would still be impossible, by means of laboratory experiments, to detect any electromotive effects due to the earth's gravity.

57. The case of the earth moving in its orbit under the sun's attraction may be regarded as that of a body moving freely under a sensible uniform field of gravity. Making any assumptions that suggest themselves as to the number of electrons in a gram of matter, the relative masses of positive and negative electrons, and the forces experienced by these under gravitational influence (64), may be used to estimate the resulting electrical distribution on the earth's surface (or in the upper strata of the atmosphere) arising directly from the gravitational attraction of the sun. But with such assumptions as I have tried, it appears that the effect to be expected is very minute. For example, making the same assumptions as in § 52, and in addition assuming that nearly all the inertia of neutral matter is the

inertia of positive electrons, the extreme difference of potential between opposite poles of the earth comes out as about 80 microvolts. Moreover, the shifting of this feeble electrical distribution, owing to the earth's (solar) diurnal rotation, could not have any sensible influence on the phenomena of terrestrial magnetism.

Appendix B.

Electromagnetic and Thermal Effects produced by Compressional Ætherial Waves.

58. Returning more particularly to the view adopted in the body of this paper, the results obtained in §§ 16–18 above may be readily revised so as to discriminate between the possibly different gravitative behaviour of positive and negative electrons.

As in Appendix A, let unit mass of neutral matter contain mass μ_1 of positive and μ_2 of negative electrons (so that $\mu_1+\mu_2=1$), the aggregate charge of mass μ_1 (or μ_2) of positive (or negative) electrons being as before denoted by $\epsilon\chi$ (or $-\epsilon\chi$).

Let

volume of æther "extruded" by mass μ_1 of positive electrons $=F_1\mu_1$,
" " " " μ_2 " negative " $=F_2\mu_2$;

then

mass " " " μ_1 " positive " $=F_1\mu_1\rho$,

and

mass " " " μ_2 " negative " $=F_2\mu_2\rho$.

Also let $\quad -dF_1/dp=H_1, \quad -dF_2/dp=H_2; \quad . \quad . \quad (68)$

then evidently

$$F_1\mu_1+F_2\mu_2=F(\mu_1+\mu_2)=F \quad \text{and} \quad H_1\mu_1+H_2\mu_2=H. \qquad (69)$$

Precisely as in (7), we can write down, in terms of the acceleration of the æther, the corresponding forces on positive and negative electrons. Writing A for the acceleration of the æther at any instant, arising from the propagation of the primary waves, the corresponding force tending to accelerate the mass μ_1 of positive electrons relatively to the æther is $F_1\mu_1\rho A$, while the mass μ_2 of negative electrons

similarly experiences a force $F_2\mu_2\rho A$. Now

$$\left.\begin{aligned} F_1\mu_1\rho A &= A\{F\mu_1\rho + (F_1 - F)\mu_1\rho\} \\ F_2\mu_2\rho A &= A\{F\mu_2\rho + (F_2 - F)\mu_2\rho\} \end{aligned}\right\} \quad . \quad . \quad (70),$$

the aggregate forces acting respectively on the positive and on the negative electrons in unit mass of matter being thus resolvable into forces $F\mu_1\rho A$, $F\mu_2\rho A$ proportional to the masses on which they act, together with the pair of forces $(F_1 - F)\mu_1\rho A$ and $(F_2 - F)\mu_2\rho A$, which by (69) are equal and opposite. The first two constituents named tend to accelerate each element of matter with respect to the æther as a whole, the total force exerted per unit mass of matter being $F\mu A$; while the remaining equal and opposite forces, acting respectively on positive and negative electrons whose aggregate charges are $\epsilon\chi$, $-\epsilon\chi$, have the same effect on the motion of those electrons as would be produced by an electromotive intensity

$$T = \frac{(F_1 - F)\mu_1\rho A}{\epsilon\chi} = \frac{(F - F_2)\mu_2\rho A}{\epsilon\chi}, \quad . \quad . \quad (71)$$

measured in the direction of the acceleration A.

59. As in Appendix A, let it be assumed (merely by way of example) that $\chi = 1$, ϵ being as before the quantity of electricity required to liberate one gram of hydrogen electrolytically. Assume also that the extrusion of æther F per unit mass of neutral matter is due entirely to negative electrons; the æther extruded by one gram of matter being therefore that extruded by a mass μ_2 of negative electrons, whose total charge is $-\epsilon\chi\ (= -\epsilon)$. Assume, that is to say,

$$F_1 = 0 \quad \text{and} \quad F_2\mu_2 = F\mu = F(\mu_1 + \mu_2). \quad . \quad . \quad . \quad (72)$$

If we further assume (as in Appendix A, § 57) that nearly all the inertia of neutral matter is the inertia of positive electrons, so that $\mu_1 = 1$ approximately, then (71) becomes

$$T = -\frac{F\rho A}{\epsilon}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

which, with our present special assumptions, is the quasi-electromotive intensity due to an acceleration A of the æther.

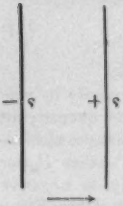
60. As regards the thermal effects arising from compressional waves in the æther, let us simplify the problem to

be considered by reducing the disturbance (as in § 27) to a single plane wave-train of very great wave-length λ, and of frequency n. Putting s in place of $2\pi n$, we may write for any region of dimensions very small compared with λ,

$$T = T_0 e^{ist}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

where it is to be understood that T and T_0 are expressed in absolute *electromagnetic* measure.

61. Consider in particular the case of a flat parallel-faced slab of conducting material, whose length and breadth are very great in comparison with its thickness, the main faces of the slab being perpendicular to the direction in which the compressional ætherial waves are being propagated, and perpendicular therefore to the direction of the quasi-electromotive intensity T, which we shall suppose to be measured positively from left to right in the figure, as indicated by the arrow.



At any instant let ς be (in electromagnetic measure) the surface-density of electrification on the right-hand face, $-\varsigma$ being that on the left-hand face; ς will evidently be a periodic function of the time of the form

$$\varsigma = \varsigma_0 e^{ist}, \quad . \quad . \quad . \quad . \quad (75)$$

where ς_0 is a constant, in general complex.

Between the faces of the slab there is, at any time t, a conduction current whose density, referred to unit of area, is $d\varsigma/di$, and a polarization current whose density is $-d\varsigma/dt$*, the total effective current being thus zero. The same result is otherwise evident; for at any point exterior to the slab the total electromotive intensity arising from the surface-densities ς and $-\varsigma$ vanishes at each instant. Hence outside the slab there is no polarization and no polarization-current, and of course no conduction-current; so that the total effective current is zero outside the slab. The solenoidal or "stream" character of the total current accordingly necessitates a zero value for this vector within the substance of the slab also.

* *Cf.* Appendix A, § 49.

62. Thus everywhere, except near the edges of the slab (which we ignore) magnetic force is absent, and electromagnetically speaking our system is without kinetic energy *. There must of course be some kinetic energy involved in the reciprocating relative motions of positive and negative electrons, but this kinetic energy, even though a close enough analysis would reveal its electromagnetic character, has no place in the ordinary equations of electromagnetism, and a moment's consideration suffices to indicate its excessive minuteness.

63. Within the substance of the slab there is at each instant a quasi-electromotive intensity T and a true electromotive intensity $-4\pi V^2\varsigma$, V being the velocity of radiation. For simplicity it will now be supposed that the only opposing forces are due to electrical resistance; thus in the absence of appreciable inertia effects, the relation

$$(T-4\pi V^2\varsigma)c=d\varsigma/dt \quad . \quad . \quad . \quad . \quad . \quad (76)$$

is always satisfied; c being the conductivity of the substance of the slab. Using (74) and (75),

$$\varsigma_0=T_0c/(4\pi V^2c+is). \quad . \quad . \quad . \quad . \quad . \quad (77)$$

This gives for the current-density

$$I=is\varsigma_0e^{ist}=\frac{Ac(s^2+i\,.\,4\pi V^2sc)}{16\pi^2V^4c^2+s^2}e^{ist}; \quad . \quad . \quad (78)$$

or, separating the real from the imaginary problem, we have, corresponding to

$$T=T_0\cos st, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

$$I=\frac{T_0cs}{16\pi^2V^4c^2+s^2}(s\cos st-4\pi V^2c\sin st). \quad . \quad . \quad (80)$$

* There will accordingly be no production of electromagnetic waves, except near the edges of the slab where the same simple conditions are not realized. But *in general* electromagnetic waves will be produced wherever there are bodies of dielectric or conducting material in the path of the primary compressional waves. If the primary disturbance, as in the foregoing table, is assumed to be periodic and of frequency $1{\cdot}29\times10^{10}$, the electromagnetic waves, being of the same frequency, will in free æther have a wave-length of 2·33 cm. Without having considered these waves in any detail, I think it may be said that they will be quite insignificant, provided the electromotive effects of the primary waves are too slight to give rise to appreciable thermal phenomena.

64. The rate, in ergs per second, at which energy is being dissipated as heat in each cubic centimetre of the conductor is accordingly

$$\frac{I^2}{c} = \frac{T_0^2 c s^2}{16\pi^2 V^4 c^2 + s^2} (s \cos st - 4\pi V^2 c \sin st)^2; \quad . \quad (81)$$

so that

$$\text{average of } \frac{I^2}{c} = \frac{1}{2} \frac{T_0^2 c s^2}{16\pi^2 V^4 c^2 + s^2}. \quad . \quad . \quad . \quad (82)$$

The last-written expression is a maximum with respect to c when $c = s/4\pi V^2$, in which case the average of I^2/c becomes

$$T_0^2 s/16\pi V^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

64 *a*. Remembering (73), we find that, with the particular set of numerical values adopted in the foregoing table, $T_0 = 130$ volts per cm., or in absolute e.m. units $1{\cdot}3 \times 10^{10}$; $s = 2\pi n = 2\pi \times 1{\cdot}29 \times 10^{10}$; the average rate at which energy is being dissipated in the conductor being in that case

2×10^8 *ergs per sec. per c.c.* or about 7·5 *calories per sec. per c.c.*

This is with the conductivity most favourable to dissipation, the value of the conductivity being then given by $c = s/4\pi V^2$; so that the specific resistance would be $4\pi V^2/s = 1{\cdot}4 \times 10^{11}$ in absolute measure, which is equivalent to 140 ohms as the resistance of a centimetre cube. If the slab were of much higher or much lower conductivity, the dissipation of energy (corresponding to the same set of assumptions) would of course be much less.

65. Thus with the special assumptions of § 59 the rate of development of heat, through the action of the primary waves on matter, though excessively minute compared with the rate at which we have supposed energy to be propagated in those waves, is nevertheless, at its maximum, much greater than could be reconciled with experience. But in obtaining the estimates of the previous paragraph, it was assumed that the "extrusion" of the æther per unit mass of matter * was entirely due to the negative electrons, and that the inertia of matter was almost entirely due to the positive electrons. Unless we were to suppose the extrusions due to

* *Cf.* § 7.

positive and negative electrons to be of opposite signs, this would be the assumption corresponding with the greatest electromotive effects from the action of the primary waves on atomic matter, and with the greatest consequent dissipation of energy; the inference being that, if we are to retain our tabulated estimates, or any others not wholly different from these in order of magnitude, we must adopt some more favourable assumption as to the relative "extrusions" and masses of positive and negative electrons. Now it is evident from (71) that if the "extrusions" of positive and negative electrons were proportional to their masses (that is $F_1 = F = F_2$), the electromotive effects of the primary waves would vanish. In that case there would be no heating of matter through dissipation of the energy of the primary disturbance, nor would there be any corresponding production of electromagnetic waves. Experience suggests, then, that the relation

$$F_1 = F = F_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

is exactly or very nearly fulfilled.

66. If indeed the æther is traversed by waves of other than electromagnetic type, to the penetration of which no sort of matter forms a barrier, it would be interesting to test for any residual heating effect due to the interaction of such waves with atomic matter. To devise the most sensitive arrangement within given limits of size wound need some thought, but evidently a body of large bulk would be best. Such a body (perhaps hollow) of suitable conductivity, well jacketed with a material of the highest possible thermal and electrical resistance, and further enveloped in a double-walled vessel containing ice, might be examined for a gradual rise of temperature. Even if all laboratory experiments on these lines should give negative results, it might still be held possible that some part of the earth's heat was due to the electromotive effect of compressional ætherial waves. This is only mentioned as a possibility suggested by the theory of this paper, and not as tending to remove any existing difficulty in the domain of geophysics.

67. Some attempt having already been made in Appendix A to trace the possible electromotive effects of a gravitational

field, the modifications which must be made in the equations of § 17 above, when we wish to discriminate between positive and negative electrons, will now be very briefly indicated. With the notation explained in § 58, the source whose strength is expressed by (10) must now be represented by the more complete expression

$$-(H_1\mu_1+H_2\mu_2)\frac{\partial \mathfrak{p}}{\partial t}\sigma\,dx\,dy\,dz, \quad . \quad . \quad (85)$$

which is to be equated to $\nabla^2\phi\,dx\,dy\,dz$; hence (11) becomes

$$\phi=\frac{H_1\mu_1+H_2\mu_2}{4\pi}\frac{\partial \mathfrak{p}}{\partial t}\iiint\frac{\sigma'}{r}dx'\,dy'\,dz'; \quad . \quad . \quad (86)$$

while (16) is replaced by

$$\frac{\partial \pi}{\partial x}=\frac{(H_1\mu_1+H_2\mu_2)\rho}{4\pi}\frac{\partial^2 \mathfrak{p}}{\partial t^2}I_x, \quad . \quad . \quad . \quad . \quad (87)$$

where I_x, as before, has the meaning indicated by (17).

In place of (18) we have now two equations, expressing the forces acting on positive and negative electrons respectively. The force on the positive electrons in unit mass of matter

$$=-\mu_1F_1\frac{\partial \varpi}{\partial x}=-\mu_1F_1\frac{(H_1\mu_1+H_2\mu_2)\rho}{4\pi}\frac{\partial^2 \mathfrak{p}}{\partial t^2}I_x, \quad . \quad (88)$$

the force on the negative electrons being similarly

$$=-\mu_2F_2\frac{\partial \varpi}{\partial x}=-\mu_2F_2\frac{(H_1\mu_1+H_2\mu_2)\rho}{4\pi}\frac{\partial^2 \mathfrak{p}}{\partial t^2}I_x; \quad . \quad (89)$$

(19) being replaced by

$$\left.\begin{aligned}F_1&=\overline{F}_1+\delta F_1=\overline{F}_1-H_1\mathfrak{p}\\F_2&=\overline{F}_2+\delta F_2=\overline{F}_2-H_2\mathfrak{p}\end{aligned}\right\}. \quad . \quad . \quad . \quad . \quad (90)$$

Hence the gravitational force acting on the positive electrons in unit mass of matter is

$$-\frac{\rho}{4\pi}\mu_1(\overline{F}_1-H_1\mathfrak{p})(H_1\mu_1+H_2\mu_2)\frac{\partial^2 \mathfrak{p}}{\partial t^2}I_x; \quad . \quad (91)$$

and since the average value of $\partial^2\mathfrak{p}/\partial t^2$ is zero, the only part of this expression which does not disappear on averaging is

$$\text{average of }\left\{\frac{\rho}{4\pi}H_1\mu_1(H_1\mu_1+H_2\mu_2)\mathfrak{p}\frac{\partial^2 \mathfrak{p}}{\partial t^2}I_x\right\}. \quad . \quad (92)$$

Similarly, the average force on the negative electrons in unit mass of matter is

$$\text{average of } \left\{ \frac{\rho}{4\pi} H_2\mu_2(H_1\mu_1 + H_2\mu_2)p \frac{\partial^2 p}{\partial t^2} I_x \right\}. \quad . \quad (93)$$

On adding (92), (93) and comparing with (21), it is seen that $(H_1\mu_1 + H_2\mu_2)^2 = H^2$, which is otherwise immediately evident.

68. Now we have seen that the only escape from inadmissibly great heating effects of the primary waves lies in supposing the relation (84) to be at least very approximately true; and unless we make the very special assumption that the relation in question is only fulfilled for one particular value of the general ætherial pressure, we must likewise conclude that

$$dF_1/dp = dF/dp = dF_2/dp; \quad . \quad . \quad . \quad . \quad . \quad (94)$$

or, remembering (68), that

$$H_1 = H = H_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

In this case the expressions (92), (93) for the forces exerted in a given gravitational field on the positive and the negative electrons in unit mass of matter, become proportional respectively to the aggregate masses of the electrons in question. As was pointed out in § 51 (Appendix A), this is the condition that no electromotive effects shall be experienced by a body moving freely under a sensibly uniform field of gravity.

The relation between the notation of this Appendix and that of Appendix A is evident: μ_1 and μ_2 having the same meaning in each case, while H_1, H_2 are respectively equivalent to $\gamma_1 H$, $\gamma_2 H$.

Appendix C.

A Non-Electromagnetic Term in the Inertia of an Electron.

69. It will readily be realised that, just as an electron possesses mass or inertia of electromagnetic origin, so on our theory the total effective mass of the electron comprises what may be called a gravitational term. The motions which are, on this view, to be classed as of gravitational type are wholly

of the nature of irrotational bodily movements of the volume-elements of æther. What has here been called the primary disturbance is assumed to consist of compressional waves, while the secondary disturbance—the supposed cause of gravitation—takes the form of a pulsatory movement whereof every particle of matter may be regarded as a centre. Finally, the motion of matter with respect to the æther appears in general to involve (over and above electromagnetic phenomena) an irrotational distribution of ætherial motion. This necessarily implies the addition of a corresponding term to the inertia of the matter in question; but as will now appear this term may, on our assumptions, be relatively insignificant. By way of illustration, expressions for the gravitational inertia of an electron, obtained on two further alternative assumptions, will now be given.

70. In the first place, let us suppose that the free æther is a strictly continuous medium, and that the constitution of an electron involves a modification of ætherial density expressible as a continuous function of coordinates. Let the modification of density be such as corresponds with a radial displacement δR (measured outwards) of any element distant R from the centre of the electron. It is understood, of course, that the unmodified state of the æther, expressed by $\delta R = 0$, is one of uniform density. In particular, let *

$$\left.\begin{aligned} &\delta R = CR \quad \text{from} \quad R=0 \quad \text{to} \quad R=R \\ &\delta R = CR_1^3/R^2 \quad \text{from} \quad R=R_1 \quad \text{to} \quad R=\infty \end{aligned}\right\}; \quad . \quad (96)$$

then it can be shown that the corresponding term in the inertia of the electron is

$$\frac{20}{3}\pi\rho C^2 R_1^5, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (97)$$

where ρ as before is the density of the æther.

* This distribution of δR suffers from the disadvantage that motion of the electron through the æther involves impulsive changes of velocity of those æther elements for which, instantaneously, $R = R_1$. A slight modification of (96) would remove the objection, but at a great sacrifice of that analytical simplicity which must always be a leading consideration in the choice of purely illustrative examples.

71. Now the mass of æther "extruded" by the electron is $4\pi\rho CR_1^3$ and the expression (97) bears to this the ratio $\frac{1}{3}$ C, which may be very small provided C is small enough. Moreover, in the table given above, the mass of æther "extruded" by one gram of neutral matter is estimated at $2{\cdot}9\times10^{-4}$ gram; and it therefore seems possible that the theory proposed may involve only a very minute non-electromagnetic term in the total inertia of a body.

72. Alternatively let the æther be regarded (for present purposes of illustration) as made up of thin coreless vortex-filaments pervading a frictionless liquid* of density ρ; the sum total of the vacuous cores comprised in any considerable volume of æther being so small a proportion of that volume that ρ may sensibly be identified with the average density of the æther. Let us further suppose, as in § 70, that the constitution of an electron involves ætherial displacements expressible by (96), and that such differences of density as exist from one volume-element of æther to another are due solely to altered diameter of the vacuous cores, no alteration in the total length of cores per unit volume being involved. It then appears that what we have called the gravitational part of the inertia of an electron, in addition to the quantity (97) contains a term

$$\tfrac{2}{3}\rho\times\text{volume of electron}\times\text{average value of}\left\{\text{volume of vacua per unit volume of æther}\times\left(\frac{d\delta r_0}{dR}\right)^2\log\frac{r_1}{r_0}\right\};\quad .\quad (98)$$

where $r_0+\delta r_0$ is the radius of a vacuous core at a distance R from the centre of the electron, and r_1 is a length corresponding in order of magnitude with a scale of structure of the æther. In (98) "volume of electron" must be understood as extending to all that region wherein $d\delta r_0/dR$ differs sensibly from zero. The addition of the term (98) does not disturb the conclusion that the gravitational part of the inertia of an electron may be very small compared with the mass of æther which the electron "extrudes."

* *Cf.* Appendix D.

Appendix D.

A Kinetic Model of a Slightly Compressible Medium.

73. The representation of the electromagnetic properties even of free æther by means of a turbulently moving liquid is beset by difficulties which may well prove to be insuperable; but it is nevertheless interesting to remark that a perfectly incompressible liquid, through which vortices are distributed, constitutes a kinetic model of a slightly compressible medium such as we have assumed the æther to be, provided some or all of the vortices are coreless. One way in which a very minute degree of compressibility may be represented, is by supposing only a small proportion of the vortices present to be coreless. When such a medium is subjected to increased pressure, so as to diminish the volume occupied by the vacuous cores, the circulation around each vortex remaining unaltered, the energy of the turbulence will be increased by an amount equal to the work done by the pressure, and this additional energy is to be regarded as potential when the turbulently moving liquid is treated as a continuous medium.

74. If we suppose each core, whether vacuous or consisting of rotationally moving liquid, to be of very small diameter in relation to the radius of curvature of its "curved axis," and very small also in comparison with the distance between neighbouring vortices, an expression may readily be obtained for the compressional elasticity of the medium. If p is the mean pressure, we shall have, for any point in the liquid sufficiently remote from vortices,

$$p = \frac{\Omega_1^2 \rho}{8\pi^2 r_1^2} = \frac{\Omega_2^2 \rho}{8\pi^2 r_2^2} = \ldots ; \quad . \quad . \quad . \quad (99)$$

where $r_1, r_2, \ldots$ are the radii of the various vacuous cores, $\Omega_1, \Omega_2, \ldots$ the circulations around them, and ρ the density of the liquid. Hence, extending the summation to the unit of volume, and calling $l_1, l_2, \ldots$ the lengths of the respective coreless vortices comprised in that volume,

$$\frac{\rho \Sigma l \Omega^2}{8\pi p} = \Sigma l \pi r^2 = \mathrm{U}, \text{ say}; \quad . \quad . \quad . \quad . \quad (100)$$

so that U is the total volume of vacuous cores comprised in unit volume of the turbulent liquid. Also, assuming the l's to be invariable,

$$-\frac{dU}{dp}=\frac{\rho}{8\pi p^2}\Sigma l\Omega^2=\frac{8\pi^2}{\rho}\Sigma\frac{v}{\Omega^2}; \quad . \quad . \quad . \quad (101)$$

where, corresponding with any one vortex of strength Ω, there is a volume v of vacuous core comprised within the particular unit volume now considered. The left-hand member of (101) represents evidently the reciprocal of the bulk-modulus of elasticity appertaining to the medium as a whole, while the velocity V of compressional waves is given by

$$V^2=\frac{1}{8\pi^2\Sigma(v/\Omega^2)}. \quad . \quad . \quad . \quad . \quad (102)$$

These expressions are given merely by way of illustration; they are of course far from being the most general, in the first place because of our assumption that all the vortices may be treated as linear, and in the second place because it has been supposed that the core of any one vortex either consists wholly of rotationally moving liquid or is wholly vacuous.

Discussion.

Mr C. S. Whitehead asked if the amount of energy stored in the æther was greater or less than it was in the past.

Prof. C. H. Lees asked if the æther considered was the old one or a new one. He referred to an interesting point in the Author's representation of gravitation, namely, that matter was produced by the absence of something from the æther. In the theory advocated by Osborne Reynolds matter was produced by a deficiency in the normal piling, and recent theories seek to explain matter and gravitation by the absence of something which is present in free æther.

Dr C. V. Burton, in reply to Mr Whitehead, said he was not aware whether the amount of energy in the æther varied with time or not.

XXVII. *On a Simple Physical Method of illustrating the Principles of Geometrical Optics.* By R. M. Archer, *A.R.C.Sc., B.Sc. (Lond.)**.

[Plate XXIV.]

The principles of geometrical optics are frequently illustrated by experiments in which the images of narrow obstacles are obscured by similar obstacles arranged in the line of vision. This method is somewhat tedious and is open to the objection that the path of a shadow is traced rather than that of a beam. The following method is free from these objections, and will be found interesting and convincing †.

The procedure is to allow light from a narrow rectilinear source to pass through a slit and fall upon a flat white surface at almost grazing incidence. In most cases the source may be the straight filament of a glow-lamp, and should be arranged parallel to the slit and not less than about 18 inches from it. Alternatively, a batswing burner with the flame "edge on" to the slit may be employed, but must be screened from draughts by a suitable enclosure.

It will be found easy to obtain upon the white surface a long narrow streak of light with sharp edges; and if a mirror be placed with its plane approximately normal to the surface another streak corresponding to the reflected ray will be seen. Similarly, the path of the beam after its emergence from a glass block or prism may be traced. When it is desired to trace the refracted beam before emergence, a glass tank of the required shape should be filled with water and the internal beam allowed to graze a white surface (*e. g.* that of a glazed tile) supported almost horizontally in the water. Provided, however, the glass block has a transparent base, all that is necessary is to coat the latter with white enamel paint and allow the internal beam

* Read May 8, 1908.

† My thanks are due to Mr. R. S. Bowman, B.Sc., for the valuable assistance he has rendered during the preparation of the photographs.

to strike this at a low angle. If in the above cases of refraction the incident light be bright, emergent beams which have been internally reflected several times will be observed.

Very beautiful effects can be obtained by using many slits and casting the beam from a distant optical lantern upon them. This mode of illumination was adopted in the case of the photographs reproduced (Pl. XXIV.). Perhaps the most striking case to select is that of a narrow concave strip cut from a cylindrical mirror. If this be illuminated by a very wide beam, a general convergent effect will be produced. The broad beam may then be analysed into many narrow ones by interposing a cardboard comb, and the formation of a caustic demonstrated. For this latter purpose it is better to skew the mirror until its main axis is oblique to the incident light. The broad beam may then be used and the skew caustic traced upon the surface by a black crayon. When the comb is interposed all the reflected rays will touch the curve, and the whole reflected set will sweep round it when the comb is given a transverse displacement. The best effects are obtained when the radiant is small, distant, and brilliant. In lecture demonstrations the prism or mirror may be fixed on a sheet of white cardboard attached to a vertical board.

When a divergent pencil is required a broad beam should be cast upon several narrow plane mirrors, and the latter turned until they converge the light upon the back of the slit. For laboratory purposes, however, it is generally sufficient to put the glow-lamp into different positions and trace the successive paths of the single beam.

Cylindrical lenses do not give such good effects as mirrors; but if they are allowed to project through a narrow aperture in the cardboard the results are fairly satisfactory.

Darkening the whole room renders the demonstrations more effective, but for individual work local screening is sufficient.

In conclusion it may be remarked that quantitative results can be obtained of an accuracy comparable with that usually reached in experiments with the ordinary optical bench.

Discussion.

Prof. C. H. Lees expressed his interest in the neatness of the experiments, and said the Author's method of dividing up a beam was a very useful one.

Mr A. Campbell congratulated the Author and remarked that the methods described could be applied to other optical experiments. In using a vibration galvanometer or an oscillograph, an easy way to obtain a curve of the wave-form was to put obliquely in the path of the beam a sheet of white paper and vibrate it.

Dr W. H. Eccles also congratulated the Author upon his paper.

XXVIII. *Note on the Amount of Water in a Cloud formed by Expansion of Moist Air.* *By* W. B. Morton, *Professor of Natural Philosophy, Queen's College, Belfast* *.

The calculation of the total mass of water condensed, when a volume of air saturated with vapour receives a given sudden expansion, is an important step in Prof. J. J. Thomson's determination of the charge on a gaseous ion †. In conjunction with the size of a single drop, as found from the rate of fall of the cloud, it gives the number of drops in unit volume and so the number of the ions which form the nuclei of the drops. In making the calculation Prof. Thomson, following C. T. R. Wilson, makes the assumption that the air is cooled down to the full extent by the adiabatic expansion before the drops begin to form. Then condensation proceeds and the latent heat liberated warms the air until the density of the remaining vapour is such as to saturate the air at the increased temperature. This process is, of course, irreversible. There is no reason to doubt that it represents closely the actual experimental conditions; but with a view to estimate the amount of error which would be introduced by a small departure from the assumed process, it seems of some interest to calculate, for comparison, the result of the other extreme

* Read June 12, 1908.

† 'Conduction of Electricity through Gases,' p. 122.

assumption, viz., that the expansion and condensation exactly keep pace with each other, the density of vapour at each stage being the saturation-density corresponding to the instantaneous value of the temperature. In this case the process would be reversible, and we can equate the initial and final values of the entropy of the combined mass of air and water-substance. Since an irreversible change is accompanied by an increase of entropy, it is easy to see that on the new assumption we shall arrive at a lower value for the density of the remaining vapour and higher value for the mass of water in the cloud.

In both cases the solution is most easily obtained by a graphical method. We plot a curve connecting the vapour density ρ and temperature t at any stage of the condensation, and find where it intersects the curve representing the maximum density of saturated vapour at different temperatures. In the irreversible case the former curve is the straight line.

$$\rho = \frac{\rho'}{x} - \frac{\mathrm{CM}}{\mathrm{L}}(t - t_2)^*;$$

where (using the notation of the authors quoted) ρ' is the initial saturation density, x the expansion ratio, t_2 the calculated temperature after adiabatic expansion, M the mass of unit volume of air after expansion, C specific heat of air at constant volume, L latent heat of the vapour. The heat-capacity of the drops is neglected in comparison with that of the air. For the reversible case the equation of constant entropy comes out

$$(x\mathrm{MC} + s\rho') \log \frac{273 + t}{273 + \mathrm{T}} + \mathrm{L}\left\{\frac{x\rho}{273 + t} - \frac{\rho'}{273 + \mathrm{T}}\right\} + x\mathrm{MR} \log x = 0,$$

where, in addition to the symbols already define, s is the specific heat of water, T the temperature before expansion, and R the gas constant for one gram of air.

I have plotted this curve for the particular case discussed in detail by Prof. Thomson, viz., starting from an initial tem-

* Equation (2), *loc. cit.*

perature of 16° C., and giving an expansion of 1·36. The graph comes out practically a straight line running parallel to the irreversible line, both lines being nearly perpendicular to the saturation-density curve with which their intersections are sought. The numerical values come out on the reversible hypothesis,

$\rho = 4{\cdot}87 \times 10^{-6}$, $t = 0°{\cdot}5$, mass of cloud per cm.^{3} $5{\cdot}06 \times 10^{-6}$grm.

as compared with the values obtained on the accepted assumption by Prof. Thomson,

$\rho = 5{\cdot}15 \times 10^{-6}$, $t = 1°{\cdot}2$, mass of cloud per cm.^{3} $4{\cdot}77 \times 10^{-6}$ grm.

So that even on the extreme assumption here made we arrive at a value differing by only 6 per cent. from the other. We can infer that no error, comparable with the other inaccuracies attached to the methods of measurement, will be involved if the assumption of complete adiabatic cooling of the air does not exactly correspond with the facts.

27th May, 1908.

FIG. 4.

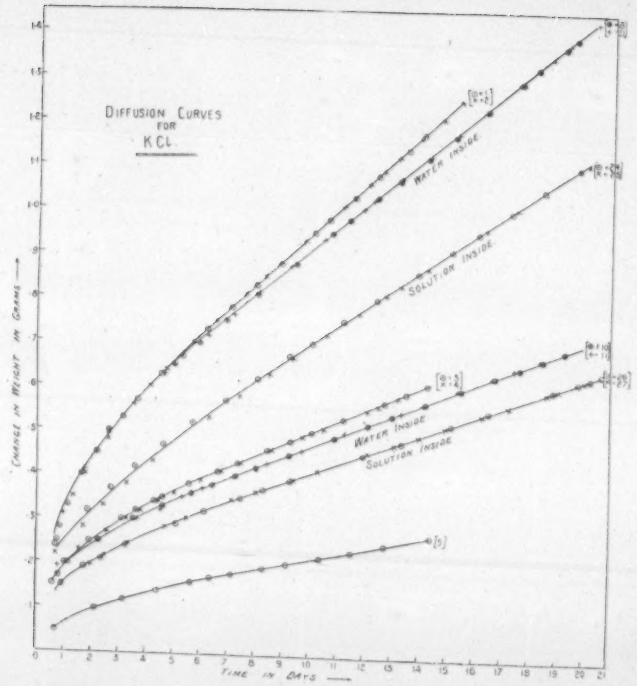


FIG. 5.

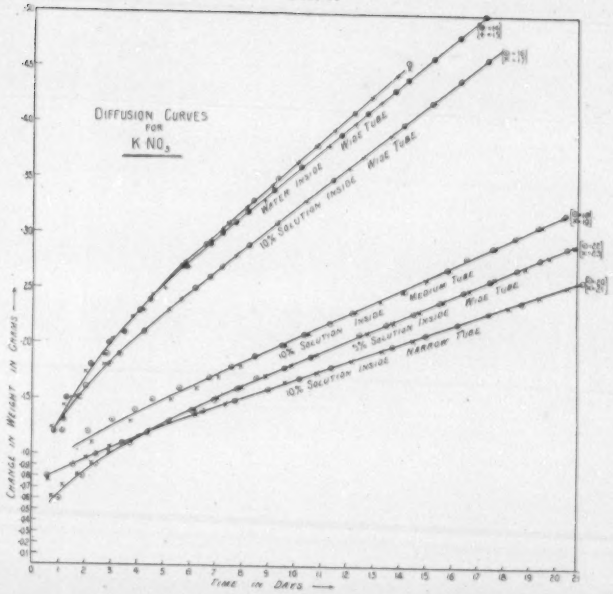


FIG. 6.

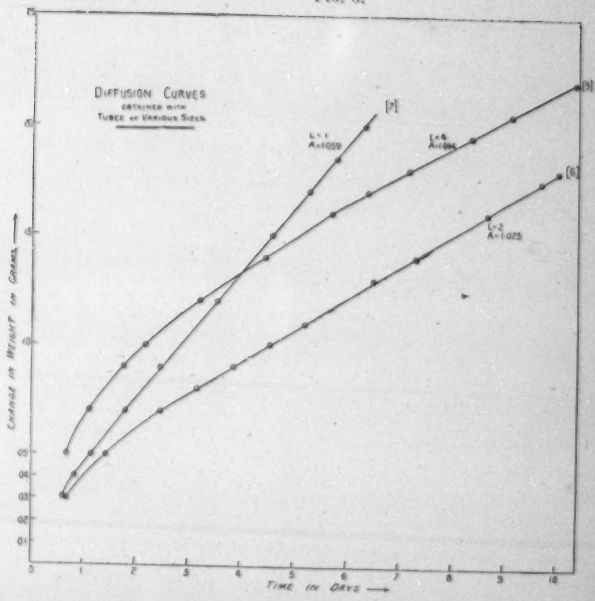
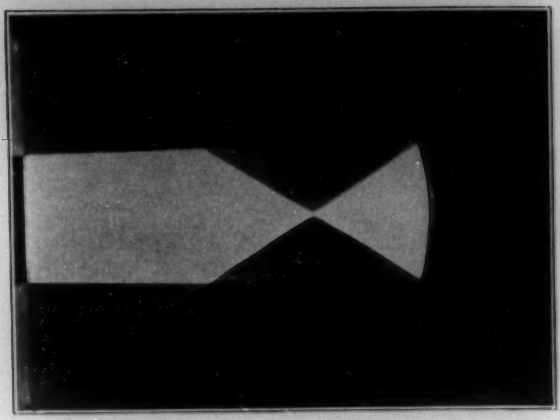
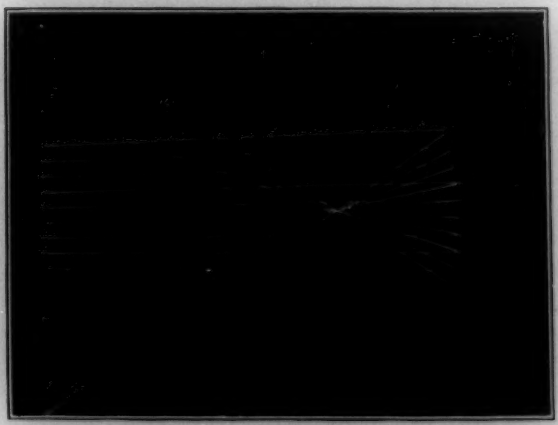


FIG. 1.



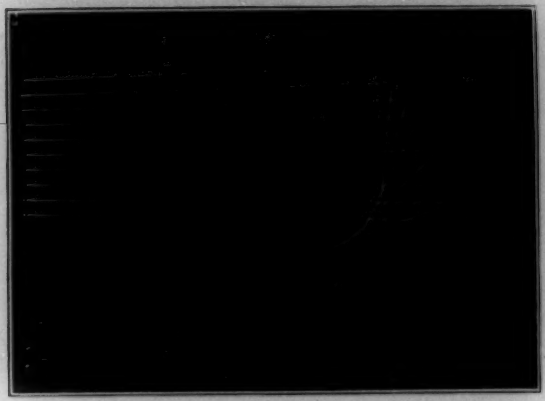
General convergent action of concave (cylindrical) mirror.

FIG. 2.



Analysis of fig. 1 into narrow beams.

FIG. 3.



Concave cylindrical mirror. Rays touching caustic curve.

FIG. 4.

Dispersion by train of prisms.

XXIX. *The Inductance of Two Parallel Wires.* By J. W. Nicholson, *D.Sc., B.A., Isaac Newton Student in the University of Cambridge* *.

When direct and return currents flow in two wires of great length, and the alternation is not rapid, the effective self-induction L per unit length of the system may be calculated readily by the method of geometric mean distances† or by simple integration‡. If the wires have radii (a, b) and permeabilities (μ_1, μ_2), and if c be the distance between their axes,

$$L=2\log\frac{c^2}{ab}+\frac{1}{2}(\mu_1+\mu_2). \quad . \quad . \quad . \quad . \quad (1)$$

But this formula ceases to be of any practical utility in many cases when the frequency of alternation is several thousands per second. Such frequencies are of constant use in practical work. For example, in the measurement of small inductances by Mr. Albert Campbell's method§, it is necessary to employ long leads in order to keep them at some considerable distance from bridge and other circuits. The self-induction of these leads must be small, and a calculation of its value is very desirable. It was therefore suggested to me that I should attack this problem. The general case presents apparently insuperable mathematical difficulty, but the solutions given below appear to include all cases of practical importance. A short statement of these results was given by the author in 'Nature,' Jan. 30th, 1908, but the limitations were not emphasized.

Let the axis of z be chosen parallel to those of the two wires. Any point in a section defined by a constant value of z may be conveniently specified by means of polar coordinates in two ways. Let these coordinates be (r, θ) and (ρ, ϕ), where (r, ρ) are the distances of the point from the two axes respectively, and (θ, ϕ) its orientations measured from a line perpendicular to both axes.

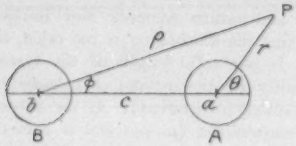
* Read June 12, 1908.

† Maxwell, 'Electricity and Magnetism.'

‡ Russell, Alt. Currents, i. p. 50.

§ Phil. Mag. Jan. 1908.

In the figure A and B are the projections of the axes of the two wires, and $AB=c$. A and B will be referred to as the first and second wires respectively. If the first wire were not disturbed by the presence of the second, the electric and



magnetic vectors would be derivable from a vector potential H parallel to the axis, independent of θ, and proportional to a time factor e^{ipt}, if $2\pi/p$ be the frequency of alternation.

The electric force in the general case is axial, and will be denoted by R. There is no axial magnetic component. Let the magnetic components radial and transversal with respect to an origin in the axis of the first wire be $(\alpha'\beta')$.

Then in a medium of permeability μ,

$$\alpha' = \frac{1}{\mu r}\frac{\partial H}{\partial \theta}, \quad \beta' = -\frac{1}{\mu}\frac{\partial H}{\partial r} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

All vectors are independent of z. The current is parallel to the axis, and of magnitude

$$w = \frac{1}{4\pi r}\left\{\frac{\partial}{\partial r}(r\beta') - \frac{\partial \alpha'}{\partial \theta}\right\}$$
$$= -\frac{1}{4\pi\mu}\left\{\frac{\partial^2 H}{\partial r^2} + \frac{1}{r}\frac{\partial H}{\partial r} + \frac{1}{r^2}\frac{\partial^2 H}{\partial \theta^2}\right\}.$$

If the medium have specific inductive capacity K, and specific resistance σ,

$$w = \frac{R}{\sigma} + \frac{K}{4\pi}\frac{\partial R}{\partial t}$$
$$= \left(\frac{1}{\sigma} + \frac{\iota K p}{4\pi}\right)R$$

for the periodic vibration.

Moreover, so far as inductive action is concerned,

$$R = -\frac{\partial H}{\partial t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and thus, H being entirely due to inductive agency,

$$\frac{\partial^2 H}{\partial r^2}+\frac{1}{r}\frac{\partial H}{\partial r}+\frac{1}{r^2}\frac{\partial^2 H}{\partial \theta^2}-4\pi\mu\iota p\left\{\frac{1}{\sigma}+\frac{\iota K p}{4\pi}\right\}H=0. \quad (4)$$

In the wire, the current may be regarded as due to ohmic conductivity alone, and writing

$$k^2=-4\pi\mu\iota p/\sigma, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

then

$$\frac{\partial^2 H}{\partial r^2}+\frac{1}{r}\frac{\partial H}{\partial r}+\frac{1}{r^2}\frac{\partial^2 H}{\partial \theta^2}+k^2 H=0, \quad . \quad . \quad . \quad (6)$$

whose solution finite at the axis of the wire is

$$H=\Sigma_0^\infty A_n J_n(kr)\cos n\theta, \quad . \quad . \quad . \quad . \quad (7)$$

where

$$J_n(kr)=\frac{k^n r^n}{2^n n!}\left\{1-\frac{k^2r^2}{2^2 1!\,n+1}+\frac{k^4r^4}{2^4 2!\,n+1\,n+2}\cdots\right\}, \quad (8)$$

and is the ordinary Bessel function of the first kind. In the surrounding medium, $\frac{1}{\sigma}$ may be neglected, and writing $1/K\mu=C^2$, where C is the velocity of propagation in the medium,

$$\frac{\partial^2 H}{\partial r^2}+\frac{1}{r}\frac{\partial H}{\partial r}+\frac{1}{r^2}\frac{\partial^2 H}{\partial \theta^2}+k^2 H=0, \quad . \quad . \quad . \quad (9)$$

where $k=p/C$.

The solution proper to the space outside the wire, satisfying the condition of being finite at infinity, is

$$H=\Sigma_0^\infty B_n K_n(\iota kr)\cos n\theta, \quad . \quad . \quad (10)$$

where

$$K_n(\iota kr)=\int_0^\infty e^{-\iota kr\cosh\phi}\cosh n\phi\, d\phi. \quad . \quad . \quad (11)$$

This function is connected with the more usual Bessel function of the second kind, introduced by Hankel, by the formula *

$$K_n(\iota kr)=-\tfrac{1}{2}\iota^{-n}\{Y_n(kr)+\iota\pi J_n(kr)\}, \quad . \quad (12)$$

* *Vide e. g.* Somne, Math. Ann. 16, 1880.

h being real, and, if $\gamma = \cdot 577 \ldots$

$$Y_n(x) \equiv 2J_n(x)\left\{\log\frac{x}{2}+\gamma\right\}-\left(\frac{2}{x}\right)^n\left\{n-1!+\frac{n-2!}{1!}\left(\frac{x}{2}\right)^2\ldots\right\}$$
$$-\left(\frac{x}{2}\right)^n\left\{\frac{1}{n!}S_n-\frac{1}{1!\,n+1!}\left(\frac{1}{1}+S_n\right)\left(\frac{x}{2}\right)^2+\frac{1}{2!\,n+2!}\left(\frac{1}{1}+\frac{1}{2}+S_n\right)\left(\frac{x}{2}\right)^4+\ldots\right\}, \quad (13)$$

where
$$S_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}. \quad . \quad . \quad . \quad (14)$$

The Single Wire.

We consider first the distribution of vectors in and about a single wire conveying an alternating current, whose distribution is undisturbed by the presence of a second wire. All vectors are independent of orientation, and the impressed electric force may be supposed equal at all points of the wire, the resistivity (σ) of the latter being constant. Let Ee^{ipt} be the line integral of this force per unit length. Then the total force, inductive and non-inductive, in the wires is

$$R = E - \frac{\partial H}{\partial t}$$
$$= E - ip\,AJ_0(kr),$$

since H does not depend on θ. The time factor is ignored throughout.

The corresponding force outside is

$$R = -\frac{\partial H}{\partial t} = -ip\,BK_0(ihr),$$

and by continuity, if a be the radius of the wire,

$$E - ip\,AJ_0(ka) = -ip\,BK_0(iha).$$

The tangential magnetic force (β) at the surface is also continuous, and therefore, if μ be the permeability of the wire,

$$\frac{1}{\mu}Ak\,J_0'(ka) = ih\,Bk_0'(iha),$$

we deduce that

$$A = \frac{-E}{ip}\,K_0'(iha)/\left(K_0'(iha)\,J_0(ka) - \frac{k}{\mu hi}\,J_0'(ka)\,K_0(iha)\right), \quad (15)$$

$$B = \frac{kE}{p\,h\mu}\,J_0'(ka)/\left(K_0'(iha)\,J_0(ka) - \frac{k}{\mu hi}\,J_0'(ka)\,K_0(iha)\right), \quad (16)$$

and the vector potential in the wire is

$$H = A J_0(kr), \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and outside is

$$H = B K_0(\iota hr). \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The current crossing any section is the line integral of magnetic force round the section, divided by 4π, and becomes

$$w = \frac{1}{4\pi} 2\pi a(\beta)_{r=a}$$
$$= Aka J_0'(ka)/2\mu, \quad . \quad . \quad . \quad . \quad (19)$$

and with the above value of A,

$$\frac{E}{wip} = -\frac{2\mu}{k}\left\{\frac{J_0(ka)}{J_0'(ka)} - \frac{k}{\mu h \iota}\frac{K_0(\iota ha)}{K_0'(\iota ha)}\right\}, \quad . \quad (20)$$

and the effective self-induction is the real part of the expression on the right. The effect due to variability of current in the section, deduced from the first term, is in accord with the result obtained by Lord Kelvin * and otherwise by Lord Rayleigh †. Maxwell and Heaviside have also given other forms of this solution.

Transformation of Bessel Functions to a new origin.

We have by the integral formula for the function finite at infinity,

$$K_n\{\iota(c+r)\} = \int_0^\infty e^{-\iota c + r\cosh\phi}\cosh n\phi\, d\phi.$$

But

$$e^{\frac{1}{2}\iota\left(z-\frac{1}{z}\right)} = J_0(c) + \sum_1^\infty J_n(c)\left\{z^n + \frac{(-)^n}{z^n}\right\}, \quad . \quad (21)$$

by an ordinary definition of Bessel functions of the first kind whose order is an integer.

Writing $z = -\iota e^\phi$,

$$e^{-\iota c\cosh\phi} = J_0(c) + 2\sum_1^\infty (-\iota)^n J_n(c)\cosh n\phi. \quad (22)$$

if the series be convergent.

* Presidential Address to Inst. of Elect. Engineers, 1889. Math. and Phys. Papers, vol. iii. p. 492.

† *Vide* Phil. Mag. 1886, where a complete practical solution may be found.

Thus

$$K_n(\iota . \overline{c+r})$$

$$= \int_0^\infty d\phi \, . \, e^{-\iota r \cosh\phi} \cosh n\phi \, . \, \{J_0(c) + 2 \sum_1^\infty (-\iota)^s J_s(c) \cosh s\phi\}$$

$$= J_0(c) \int_0^\infty e^{-\iota r \cosh\phi} \cosh n\phi \, d\phi + \sum_1^\infty (-\iota)^s J_s(c) \int_0^\infty e^{-\iota r \cosh\phi} \left\{ \begin{matrix} \cosh \overline{n+s}\phi \\ + \cosh \overline{n-s}\phi \end{matrix} \right\} d\phi$$

$$= J_0(c) K_n(\iota r) + \sum_{s=1}^\infty (-\iota)^s J_s(c) \{K_{n+s}(\iota r) + K_{n-s}(\iota r)\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

where $K_{s-n}(\iota r)$ is identical with $K_{n-s}(\iota r)$.

This may also be written

$$K_n(\iota . \overline{c+r}) = J_0(r) K_n(\iota c) + \sum_{s=1}^\infty (-\iota)^s J_s(r) \{K_{n+s}(\iota c) + K_{n-s}(\iota c)\}$$
$$. \quad . \quad . \quad (24)$$

Now the series

$$K_n(\iota hc) J_0(hr) + \sum_{s=1}^\infty (-\iota)^s \{K_{n+s}(\iota hc) + K_{n-s}(\iota hc\} J_s(hr) \cos s\theta$$

is a solution of the equation (9) which, when $\theta = 0$, takes the form of the right-hand side of (24). Such a solution is also $\cos n\phi \, . \, K_n(\iota h\rho)$, where ρ and ϕ are connected with r and θ as in the figure.

Thus
$$\rho^2 = r^2 + c^2 + 2cr \cos\theta. \quad . \quad . \quad . \quad . \quad (25)$$

$$\tan\phi = \frac{r \sin\theta}{c + r\cos\theta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

and when $\theta = 0$, $\cos n\phi K_n(\iota h\rho) = K_n\{\iota h(c+r)\}$,

Thus the two solutions only differ by a function vanishing with θ, and therefore capable of expression in a series of sines. This function is at once seen to be zero, by an application of Fourier's theorem. The function $\cos n\phi K_n(\iota h\rho)$ and the series last written do not change sign with θ. Thus if (ρ, ϕ) are connected with (r, θ) as before,

$$\cos n\phi \, . \, K_n(\iota h\rho)$$

$$= K_n(\iota hc) J_0(hr) + \sum_{s=1}^\infty (-\iota)^s \{K_{n+s}(\iota hc) + K_{n-s}(\iota hc)\} J_s(hr) \cos s\theta \quad (27)$$

and similarly

$$\cos n\theta \, . \, K_n(\iota hr) = K_n(\iota hc) \, J_0(h\rho)$$

$$+ \sum_{s=1} \iota^s \{K_{n+s}(\iota hc) + K_{n-s}(\iota hc)\} J_s(h\rho) \cos s\phi. \quad (28)$$

Effect of the Return Wire.

It has been shown that if an impressed force **E** (ignoring a time-factor e^{pt}) acts per unit length in a single wire of radius a, the vector potential produced outside is

$$H = BK_0(\iota hr),$$

where $h = p/C$, C being the velocity of propagation of electromagnetic disturbances in the medium, and

$$B = \frac{kE}{ph\mu}\, J_0'(ka)/\{K_0'(\iota h\alpha)J_0(k\alpha) - \frac{k}{\mu h\iota}J_0'(k\alpha)\, K_0(\iota h\alpha)\},$$

where the origin is in the axis of the wire, and

$$k^2 = -4\pi\mu\rho/\sigma.$$

It will be convenient to write

$$\alpha = K_0'(\iota ha)\, J_0(ka) - \frac{k}{\mu h\iota}J_0'(ka)\, K_0(\iota ha). \quad . \quad (29)$$

where α depends entirely on the first wire.

The corresponding function of the second wire will be called β, and its radius b.

Thus

$$B = \frac{kE}{ph\mu\alpha}J_0'(ka). \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Writing $n=0$ in (28), it appears that

$$K_0(\iota hr) = K_0(\iota hc)\, J_0(h\rho) + 2\sum_{s=1}^{\infty} \iota^s K_s(\iota hc\, J_s(h\rho)\cos s\phi, \quad (31)$$

and if hb is small, as in all applications proposed in this paper, this series is rapidly convergent in the neighbourhood of the second wire.

We may write

$$\theta_0 = K_0(\iota hc), \quad \theta_s = 2\iota^s K_s(\iota hc) \quad s \neq 0, \quad . \quad (32)$$

and the vector potential H, which may be regarded as incident on the second wire, becomes

$$H = B\theta_0 J_0(h\rho) + \sum_1^{\infty} B\,.\,\theta_s\,.\,J_s(h\rho)\cos s\phi. \quad . \quad (33)$$

We neglect, in the present section, the terms dependent on orientation. This will afterwards appear to be equivalent to a neglect of terms of the order $(a/c)^2$, so that the wires are

not close together. Thus

$$H = B\theta_0 J_0(h\rho), \quad . \quad . \quad . \quad . \quad . \quad (34)$$

is incident on the second wire. A secondary disturbance will be thrown off, of the form

$$H = CK_0(\iota h\rho), \quad . \quad . \quad . \quad . \quad . \quad (35)$$

zero at infinity, and inside the wire, a disturbance

$$H = D J_0(k'\rho), \quad . \quad . \quad . \quad . \quad . \quad (36)$$

finite at the origin, will be introduced, where

$$k'^2 = -4\pi\nu\iota p/\sigma', \quad . \quad . \quad . \quad . \quad . \quad (37)$$

ν being the permeability of the second wire.

Since the electric force R is continuous at the surface, H is also continuous (for this disturbance due to induction), and therefore

$$DJ_0(k'b) = CK_0(\iota hb) + B\theta_0 J_0(hb),$$

and because the magnetic force is also continuous,

$$\frac{1}{\nu} k' D J_0'(k'b) = \iota h C K_0'(\iota hb) + B h \theta_0 J_0'(hb).$$

Adopting the notation

$$\left.\begin{aligned} \delta_1 &= J_0(ka)\, J_0'(ha) - \frac{k}{\mu h} J_0(ha)\, J_0'(ka) \\ \delta_2 &= J_0(k'b)\, J_0'(hb) - \frac{k'}{\nu h} J_0(hb)\, J_0'(kb) \end{aligned}\right\}, \quad . \quad (38)$$

it appears that

$$\left.\begin{aligned} C &= \iota B\theta_0\delta_2/\beta \\ D &= \iota B\theta_0/hb\beta \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad (39)$$

where the numerator of D has been simplified by the reduction

$$\begin{aligned} &\iota K_0'(\iota hb)\, J_0(hb) - K_0(\iota hb)\, J_0'(hb) \\ &= \tfrac{1}{2}\left(J_0'(hb)\, Y_0(hb) - J_0(hb)\, Y_0'(hb)\right) \\ &= -\frac{1}{hb}, \end{aligned}$$

by the usual property of the Hankel solution Y_0.

The second wire, therefore, throws out a disturbance of vector potential

$$H = \frac{\iota B\theta_0\delta_2}{\beta} K_0(\iota h\rho). \quad . \quad . \quad . \quad . \quad . \quad (40)$$

Potential independent of Orientation.

In the calculation just made, the terms of a transformed series of Bessel functions involving orientation were neglected. We now trace the series of compensating disturbances which may be regarded as thrown off by the wires, and form their sum under the same condition. This leads to a very accurate result if the wires are not very close together.

A vector potential $H = BK_0(\iota hr)$ leaving the first wire has, for its main term near the second, a magnitude $B\theta_0 J_0(h\rho)$, and causes (1) an inside potential

$$H = -\frac{B\theta_0}{\iota h\delta\beta} J_0(h\rho), \quad . \quad . \quad . \quad . \quad . \quad (41)$$

and (2) a reflexion of vector potential from that wire, of

$$H = \frac{\iota B\theta_0\delta_2}{\beta} K_0(\iota h\rho). \quad . \quad . \quad . \quad . \quad . \quad (42)$$

This is again incident on the first wire, having for a leading term there,

$$H = \frac{\iota B\theta_0\delta_2}{\beta} . \theta_0 J_0(hr), \quad . \quad . \quad . \quad . \quad (43)$$

and causing an inside potential

$$H = \frac{\iota B\theta_0^2\delta_2}{\beta} . \frac{\iota}{ha\alpha} . J_0(hr), \quad . \quad . \quad . \quad . \quad (44)$$

and a reflexion

$$\left.\begin{aligned} H &= \frac{\iota B\theta_0^2\delta_2}{\beta} . \frac{\iota\delta_1}{\alpha} . K_0(\iota hr) \\ &= -B\theta_0^2 . \frac{\delta_1\delta_2}{\alpha\beta} . K_0(\iota hr). \end{aligned}\right\} \quad . \quad . \quad . \quad (45)$$

This process may be continued. The successive waves of reflexion form a geometric series of common ratio

$$\theta_0^2\delta_1\delta_2/\alpha\beta.$$

The subsequent reduction indicates that in general, with a frequency of the magnitude dealt with in this paper, this series is convergent. The effect inside the second wire, of a potential

$$H = BK_0(\iota hr),$$

originally supposed to leave the first, is a potential

$$H=\frac{iB\theta_0}{hb\beta}J_0(k'\rho)\left\{1-\theta_0^2\frac{\delta_1\delta_2}{\alpha\beta}+\theta_0^4\frac{\delta_1^2\delta_2^2}{\alpha^2\beta^2}-\ldots\right\}$$

$$=\frac{i\theta_0}{hb\beta}\frac{B}{1+\theta_0^2\frac{\delta_1\delta_2}{\alpha\beta}}J_0(k'\rho). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Moreover, the effect on the potential in the first wire, due to the waves caused by the potential originally leaving that wire, is

$$H=-\frac{B\theta_0^2\delta_2}{ha\alpha\beta}\left\{1-\theta_0^2\frac{\delta_1\delta_2}{\alpha\beta}+\ldots\right\}J_0(hr)$$

$$=-\frac{B\theta_0^2\delta_2}{ha\alpha\beta}\frac{J_0(hr)}{1+\theta_0^2\frac{\delta_1\delta_2}{\alpha\beta}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

A similar potential $H=B'K_0(ih\rho)$ leaving the second wire produces secondary potentials

$$H=\frac{i\theta_0}{ha\alpha}\frac{B'}{1+\theta_0^2\frac{\delta_1\delta_2}{\alpha\beta}}J_0(kr) \quad . \quad . \quad . \quad . \quad (48)$$

$$H=-\frac{B'\theta_0^2\delta_1}{hb\alpha\beta}\frac{J_0'(k'r)}{1+\theta_0^2\frac{\delta_1\delta_2}{\alpha\beta}}, \quad . \quad . \quad . \quad . \quad (49)$$

in the first and second respectively, and as

$$B=\frac{k}{ph\mu}\frac{EJ_0'(ka)}{\alpha} \text{ by (16),}$$

so

$$B'=\frac{k'}{ph\nu}\frac{EJ_0'(k'b)}{\beta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

so that the secondaries may be exactly analogous to the first group.

The value of H in the first wire is therefore, to this order,

$$H=\left\{A-\frac{B\theta_0^2\delta_2}{ha\alpha\beta}\frac{1}{1+\theta_0^2\frac{\delta_1\delta_2}{\alpha\beta}}+\frac{i\theta_0B'}{ha\alpha}\frac{1}{1+\theta_0^2\frac{\delta_1\delta_2}{\alpha\beta}}\right\}J_0(kr) \quad (51)$$

with a corresponding total current

$$\varpi_1=\frac{ka}{2\mu}J_0'(ka).\frac{H}{J_0(kr)}. \quad . \quad . \quad . \quad (52)$$

As previously indicated, ha and hb are to be supposed very small, so that $ha\,J_0'(ha)$ is of order $(ha)^2$.

Neglecting this order of magnitude, a justification for the process appearing later,

$$\left.\begin{aligned}\delta_1 &= -\frac{k}{\mu h}\,J_0'(k\alpha)\\ \delta_2 &= -\frac{k'}{\nu h}\,J_0'(k'b)\end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad (53)$$

and therefore

$$B=\frac{-E}{p\alpha}\,\delta_1, \quad B'=\frac{-E}{p\beta}\,\delta_2; \quad . \quad . \quad . \quad . \quad (54)$$

and in the first wire, on reduction,

$$H=\frac{-EJ_0(kr)}{phz\alpha}\left(1+\frac{\iota\theta_0\delta_2}{\beta}\right)\Big/\left(1+\frac{\theta_0^2\delta_1\delta_2}{\alpha\beta}\right), \quad . \quad (55)$$

and in the second,

$$H=\frac{-EJ_0(k'\rho)}{phb\beta}\left(1+\frac{\iota\theta_0\delta_1}{\alpha}\right)\Big/\left(1+\frac{\theta_0^2\delta_1\delta_2}{\alpha\beta}\right). \quad (56)$$

The total currents are

$$w_1=\frac{E\delta_1}{2p\alpha}\left(1+\frac{\iota\theta_0\delta_2}{\beta}\right)\Big/\left(1+\frac{\theta_0^2\delta_1\delta_2}{\alpha\beta}\right), \quad . \quad . \quad (57)$$

$$w_2=\frac{E\delta_2}{2p\beta}\left(1+\frac{\iota\theta_0\delta_1}{\alpha}\right)\Big/\left(1+\frac{\theta_0^2\delta_1\delta_2}{\alpha\beta}\right), \quad . \quad . \quad (58)$$

by (52).

If L is the self-induction of the system per unit length, ιLp is the imaginary part of $\frac{E}{w_1}+\frac{E}{w_2}$, or of

$$2p\left(1+\frac{\theta_0^2\delta_1\delta_2}{\alpha\beta}\right)\left\{\frac{1}{\frac{\delta_1}{\alpha}\left(1+\frac{\iota\theta_0\delta_2}{\beta}\right)}+\frac{1}{\frac{\delta_2}{\beta}\left(1+\frac{\iota\theta_0\delta_1}{\alpha}\right)}\right\}. \quad (59)$$

In particular, if the wires be equal and of the same material, as will usually occur in practice, ιL is the imaginary part of

$$\frac{4\alpha}{\delta_1}\left(1+\frac{\theta_0^2\delta_1^2}{\alpha^2}\right)\Big/\left(1+\frac{\iota\theta_0\delta_1}{\alpha}\right), \text{ or of } \frac{4\alpha}{\delta_1}\left(1-\frac{\iota\theta_0\delta_1}{\alpha}\right). \quad (60)$$

Case of a Small Frequency.

When p is small, $ha = pa/C$ is exceedingly small for a wire not of excessive thickness, and may be safely ignored as in the above calculation, even in its first power, and thus

$$\begin{aligned} K_0(\iota ha) &= -\tfrac{1}{2}\left\{Y_0(ha) + \iota\pi J_0(ha)\right\} \\ &= -\left(\gamma + \frac{\iota\pi}{2} + \log\frac{ha}{2}\right) \\ &= -\log ha \quad \text{practically.} \quad . \quad . \quad . \quad (61) \end{aligned}$$

$$K_0'(\iota ha) = -\frac{1}{\iota ha}, \quad . \quad . \quad . \quad . \quad . \quad (62)$$

so that on reduction,

$$\alpha = \frac{k}{h\mu} J_0'(k\alpha)\left\{\log ha - \frac{\mu J_0(k\alpha)}{ka J_0'(k\alpha)}\right\}, \quad . \quad (63)$$

and

$$-\frac{\iota\delta_1}{\alpha} = 1 \Big/ \left\{\log(ha) - \frac{\mu J_0}{ka J_0'}\right\}. \quad . \quad . \quad (64)$$

$$\theta_0 = K_0(\iota hc) = -\log hc.$$

If hc is not too great

$$1 + \frac{\iota\theta_0\delta_1}{\alpha} = \frac{\log \overline{h^2ac} - \mu J_0/kaJ_0'}{\log ha - \mu J_0/kaJ_0'},$$

and has the value 2 for a small value of h.

$$1 + \frac{\theta_0^2\delta_1\delta_2}{\alpha\beta} = -\left\{\log\frac{c^2}{ab} + \epsilon_1 + \epsilon_2\right\} \Big/ \log h, \quad . \quad (65)$$

to the same order, where

$$\epsilon_1 = \mu J_0(k\alpha) / kaJ_0'(ka), \quad . \quad . \quad . \quad . \quad (66)$$

and ϵ_2 is the same function of (k', b),

$$\frac{\delta_1}{\alpha} = \frac{\iota}{\log h} = \frac{\delta_2}{\beta},$$

and ϵ_1 reduces practically, under the present circumstances, to

$$-2\mu / k^2a^2\left\{1 - \frac{k^2a^2}{4}\right\},$$

and quoting the value of k, the real part is $\frac{1}{2}\mu$, and the self-induction becomes

$$L = 2\log\frac{c^2}{ab} + \frac{1}{2}(\mu + \nu), \quad . \quad . \quad . \quad . \quad (67)$$

in accord with the results deduced from a distribution assumed uniform.

Case of two equal wires.

When the wires are equal, ιL is the imaginary part of

$$\frac{4\alpha}{\delta_1}\left(1-\iota\theta_0\frac{\delta_1}{\alpha}\right).$$

When h^3c^3 and h^3a^3 are both neglected, this leads to

$$L=4\log\frac{c}{a}+\frac{4\mu}{x}\frac{\operatorname{ber}x\operatorname{ber}'x+\operatorname{bei}x\operatorname{bei}'x}{(\operatorname{ber}'x)^2+(\operatorname{bei}'x)^2}, \quad . \quad (68)$$

where, if $\frac{n}{2\pi}$ is the frequency of alternation per second,

$$x=2a\sqrt{\frac{\pi\mu n}{\sigma}}. \quad . \quad . \quad . \quad . \quad . \quad (69)$$

The functions $\operatorname{ber}x$, $\operatorname{bei}x$, are those introduced by Lord Kelvin*, and subsequently tabulated.

To this order of approximation, the result might have been written down at once. For c/a has already been assumed so great that the distribution may be regarded as made up of cylindrical current sheets. The number of linkages of the lines of force outside the cylinders† is $4\log c/a$, and the linkages due to the lines of force wholly in the metal give rise to the second part of (68) ‡.

To the same order the approximation for two wires becomes

$$\left.\begin{aligned}L=2\log\frac{c^2}{ab}&+\frac{2\mu}{x}\frac{\operatorname{ber}x\operatorname{ber}'x+\operatorname{bei}x\operatorname{bei}'x}{(\operatorname{ber}'x)^2+(\operatorname{bei}'x)^2}\\&+\frac{2\nu}{y}\frac{\operatorname{ber}y\operatorname{ber}'y+\operatorname{bei}y\operatorname{bei}'y)}{(\operatorname{ber}'y)^2+(\operatorname{bei}'y)^2}\end{aligned}\right\}, \quad . \quad (70)$$

where

$$x=2a\sqrt{\frac{\pi\mu n}{\sigma}}, \quad y=2b\sqrt{\frac{\pi\nu n}{\sigma'}}, \quad . \quad . \quad (71)$$

and may also be written down at once.

When the vibration becomes more rapid, even though h^3a^3 may be neglected in comparison with unity, the latter cannot

* *Loc. cit. ante.*

† Russell, Alternating Currents, i. p. 56 *et seq.*

‡ *Vide* Gray and Matthews, Treatise on Bessel Functions.

itself be neglected in comparison with $\log(ka)$. Thus (61) requires modification, and it appears that

$$\frac{-\iota\delta_1}{a} = 1 \Big/ \left\{ \log\frac{(ha)}{2} + \gamma + \frac{\iota\pi}{2} \right\} - \mu \mathrm{J}_0/ka\mathrm{J}_0' \Big\},$$

and $\iota\mathrm{L}$ is the imaginary part of

$$\frac{4a}{\delta_1}\left(1 - \iota\theta_0\frac{\delta_1}{a}\right)$$

or of

$$-4\iota\left\{\log\frac{ha}{2} + \gamma + \frac{\iota\pi}{2} - \frac{\mu\mathrm{J}_0}{ka\mathrm{J}_0'}\right\} - 4\iota\,.\left\{-\log\frac{h\iota}{2} - \gamma - \frac{\iota\pi}{2}\right\}$$

leading to the same value as before.

But when the wires are different, the new approximation thus calculated differs from the old, and is very cumbrous. It is therefore to be desired that, in practical work involving a knowledge of the inductance of parallel leads, the leads should be of the same size and material. The formula for inductance is then of the same accuracy over a wider range of frequency.

Cases may arise in practice in which c is so great in comparison with a that the neglect of h^2a^2 may not involve that of h^2c^2. Tables of the Bessel functions of the second kind must then be employed. L is the real part of

$$-4\left\{\log\frac{ha}{2} + \gamma + \frac{\iota\pi}{2} - \frac{\mu\mathrm{J}_0}{ka\mathrm{J}_0'}\right\} - 4\mathrm{K}_0(\iota hc),$$

and becomes

$$\mathrm{L} = 2\mathrm{Y}_0(hc) - 4\left\{\log\frac{ha}{2} + \gamma\right\} + \frac{4\mu}{x}\frac{\mathrm{ber}\,x\,\mathrm{ber}'x + \mathrm{bei}\,x\,\mathrm{bei}'x}{(\mathrm{ber}'x)^2 + (\mathrm{bei}'x)^2} \quad (72)$$

where x has its previous value, and Y_0 represents the function introduced by Hankel. Tables of this function have been given by B. A. Smith *.

If $\frac{n}{2\pi}$ is the frequency,

$$h = \frac{n}{3}\,.\,10^{-10}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

A general idea of the range of frequency with which these formulæ may be used, can be obtained at once. The neglected terms are of order h^2a^2 in comparison with those retained.

* Phil. Mag. (5) xlv. p. 122 (1898).

The radius of a wire is rarely greater than two millimetres, and thus with a frequency of a hundred million per second, k^2a^2 becomes $1{\cdot}6 \cdot 10^{-5}$, which may be neglected at once in comparison with unity. Most frequencies dealt with in practice are therefore within the scope of these formulæ, when the wires are not of large cross-section. But another source of error is present, owing to the neglect of terms depending on orientation. Upon examination, the error is found to be of relative order a^2/c^2. Even if $c = 5a$, the formulæ are sufficiently accurate for most applications. The error is about ten per cent. if $c = 3a$, which is the limiting closeness for the majority of practical purposes. More accurate results suited to this case are obtained in the next section.

Higher Approximation.

The terms depending on orientation in the vector potential do not contribute to the total current across any section of a wire, which is proportional to a line integral round the boundary. In considering the supposed waves of potential, it is therefore only necessary to calculate the successive coefficients of the Bessel functions of zero order.

A potential

$$H = BK_0(\iota hr), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

leaving the first wire, reaches the second as

$$H = B\theta_0 J_0(h\rho) + B\Sigma_1^{\infty} \theta_s J_s(h\rho) \cos s\phi$$

where

$$\theta_s = 2\iota^s K_s(\iota hc).$$

If the reflected and internal potentials be respectively

$$\left.\begin{aligned} H &= c_0 K_0(\iota h\rho) + \Sigma_1^{\infty} c_s K_s(\iota h\rho) \cos s\phi \\ H &= d_\infty J_0(k'\rho) + \Sigma_1 d_s J\ (k'\rho) \cos s\phi \end{aligned}\right\}, \quad . \quad (75)$$

the surface conditions readily give

$$c_s = \iota B\theta_s \delta_s s/\beta_s, \qquad d_s = \iota B\theta_s/hb\beta_s, \quad . \quad . \quad (76)$$

where

$$\delta_1{}^s = J_s(ka)J_s'(ha) - \frac{k}{\mu h} J_s(ha)J_s'(ka), \quad . \quad . \quad (77)$$

$$\alpha_s = K_s'(\iota ha)J_s(ka) - \frac{k}{\mu h_\iota} J_s'(ka)K_s(\iota ha), \quad . \quad (78)$$

and (δ_s', β_s) are corresponding functions for the second wire. The potential

$$H = c_0 K_0(\iota h\rho) + \Sigma_1^\infty c_s K_s(\iota h\rho) \cos s\phi$$

leaves the second wire. But, if the argument of the functions K be ιhc in all cases,

$$K_1(\iota h\rho)\cos\phi = K_1 J_0(hr) + \Sigma_{s=1}^\infty (-\iota)^s \{K_{s+1} + K_{s-1}\} J_s(hr)\cos s\theta,$$

$$K_2(\iota h\rho)\cos 2\phi = K_2 J_0(hr) + \Sigma_{s=1}^\infty (-\iota)^s \{K_{s+2} + K_{s-2}\} J_s(hr)\cos s\theta,$$

and so on,

Thus the potential becomes

$$H = A_0 J_0(hr) + \Sigma_1^\infty A_s J_s(hr)\cos s\theta, \quad . \quad . \quad (79)$$

where

$$A_0 = C_0 K_0 + C_1 K_1 + C_2 K_2 + \ldots$$

$$A_1 = -\iota C_0 . 2K_1 - \iota C_1 (K_2 + K_0) - \iota C_2 (K_3 + K_1) \ldots$$

$$A_2 = -C_0 . 2K_2 - C_1 (K_3 + K_1) - C_2 (K_4 + K_0) \ldots . \quad (80)$$

The first term alone contributes to the current crossing any section of the first wire. It causes an internal potential

$$H = -\frac{A_0}{\iota hac} J_0(kr) \quad . \quad . \quad . \quad . \quad . \quad (81)$$

in accordance with previous calculation, and a reflected wave

$$H = \iota \frac{A_0 \delta_1}{\alpha} K_0(\iota hr) + \iota \Sigma_{s=1}^\infty \frac{A_s \delta_1'}{\alpha_s} K_s(\iota hr)\cos s\theta. \quad (82)$$

We now introduce the fact that the squares of ha, hb, hc may be ignored. Thus

$$K_0 = -\log(hc),$$

$$K_n = -\frac{\iota^{-n}}{2}\{Y_n(hc) + \iota\pi J_n(hc)\}$$

$$= \frac{1}{2} . n-1\,! \left(\frac{2}{\iota hc}\right)^n \quad n \neq 0,$$

$$J_s(ha) = \left(\frac{ha}{2}\right)^n . \frac{1}{n!},$$

$$K_s'(\iota ha) = -2^{n-1} n\,! / (\iota ha)^{n+1}.$$

If

$$\lambda_s = \left(1 - \frac{\mu s\, \mathrm{J}_s(ka)}{ka\, \mathrm{J}_s'(ka)}\right) \Big/ \left(1 + \frac{\mu s\, \mathrm{J}_s(ka)}{ka\, \mathrm{J}_s'(ka)}\right), \quad . \quad . \quad (83)$$

and μ_s be the corresponding quantity for the second wire, it appears on reduction that

$$\mathrm{C}_s = -\frac{\mathrm{B}\theta_s\mu_s(hb)^{2s}\epsilon^s}{2^{2s-1}\, s\,!\, s-1\,!}$$

$$= -\frac{\mathrm{B}\mu_s h^s b^{2s}\epsilon^s}{s\,!\, 2^{s-1}c^s}. \quad . \quad . \quad . \quad . \quad . \quad (84)$$

The coefficients of type c therefore rapidly become small, especially if c/b is large. If hc is large and hb small the convergence is still more rapid, but the first form of c_s is to be used. θ_s itself decreases rapidly as hc increases.

Accordingly, when c is so large that the approximate values above cannot be used for the functions $\mathrm{K}_n(ihc)$, the result will be correct to the extent of an error of order h^2a^2, and this case needs no further consideration.

When hc is small, on reduction

$$\left.\begin{aligned} \mathrm{A}_0 &= \mathrm{C}_0\mathrm{K}_0 - \mathrm{B}\Sigma_1^\infty \frac{\mu_s}{s}\left(\frac{b}{c}\right)^{2s}, \\ \mathrm{A}_1 &= -\frac{2\mathrm{C}_0}{hc} + \frac{2\mathrm{B}}{hc}\,\Sigma_1^\infty \mu_s \left(\frac{b}{c}\right)^{2s}, \\ \mathrm{A}_2 &= \frac{4\mathrm{C}_0}{h^2c^2} + \frac{4\mathrm{B}}{h^2c^2}\,\Sigma_1^\infty (s+1)\mu_s\left(\frac{b}{c}\right)^{2s}. \end{aligned}\right\}, \quad . \quad . \quad (85)$$

and so on, very convergent series being obtained if $\dfrac{b}{c}$ is small.

The reflected wave of (82) becomes, near the second wire,

$$\mathrm{H} = \mathrm{B}_0\mathrm{J}_0(h\rho) + \Sigma_1^\infty \mathrm{B}_s\mathrm{J}_s(h\rho)\cos s\phi, \quad . \quad . \quad (86)$$

where

$$\mathrm{B}_0 = \frac{\iota\mathrm{A}_0\delta_1}{\alpha}\,\mathrm{K}_0 + \iota\Sigma_1^\infty \frac{\mathrm{A}_s\delta_1^s}{\alpha_s}\,\mathrm{K}_s,$$

$$\mathrm{B}_1 = 2\iota\mathrm{K}_1 \cdot \frac{\iota\mathrm{A}_0\delta_1}{\alpha} - \Sigma_1^\infty \mathrm{A}_s\,(\mathrm{K}_{s+1} + \mathrm{K}_{s-1}).$$

The ratios a/c, b/c, will be retained to the second power only. To this order,

$$\mathrm{B}_0 = \frac{\iota\delta_1\mathrm{K}_0}{\alpha}\left\{\mathrm{C}_0\mathrm{K}_0 - \mathrm{B}\mu_1\frac{b^2}{c^2}\right\} + \lambda_1\mathrm{C}_0\frac{a^2}{c^2} \quad . \quad . \quad (87)$$

B_1 will only subsequently be required to the first order, and becomes

$$B_1 = \frac{\iota\delta_1}{\alpha} 2\iota K_1 . C_0 K_0. \quad . \quad . \quad . \quad . \quad . \quad (88)$$

We note that the change in B_0 due to taking account of $c_1 c_2 \ldots .$ is of order a^2/c^2 as stated in the previous section.

So far as the current is concerned, the internal potential produced in the second wire is

$$H = -\frac{B_0}{\iota hb\beta} J_0(k\rho), \quad . \quad . \quad . \quad . \quad . \quad (89)$$

and a potential

$$H = c_0 J_0(hr) + \Sigma_1^{\infty} c_s J_s(hr) \cos s\theta \quad . \quad . \quad . \quad (90)$$

is incident again on the first wire, where

$$\begin{aligned} c_0 &= \frac{\iota\delta_2}{B} . B_0 K_0 + \iota \Sigma_1^{\infty} B_s \frac{\delta_2'}{\beta_s} K_s \\ &= \frac{\iota\delta_2}{B} . B_0 K_0 + \frac{\iota\delta_2'}{\beta_1} B_1 \quad \text{to order} \quad \frac{a^2}{c^2} \\ &= \rho_1\rho_2\left(C_0 K_0 - B\mu_1 \frac{b^2}{c^2}\right) + \lambda_1\rho_2 C_0 \frac{a^2}{c^2} - \mu_1\rho_1 C_0 \frac{b^2}{c^2} \quad (91) \end{aligned}$$

where

$$\rho_1 = \frac{\iota\delta_1 K_0}{\alpha} \qquad \rho_2 = \frac{\iota\delta_2 K_0}{\beta} \quad . \quad . \quad . \quad (92)$$

and

$$c_1 = -2\iota K_1 \rho_1 \rho_2 C_0. \quad . \quad . \quad . \quad . \quad . \quad (93)$$

If the next wave meets the second wire in the form

$$H = D_0 J_0(h\rho) + \Sigma_1^{\infty} D_s J_s(h\rho) \cos s\phi, \quad . \quad . \quad (94)$$

then in a similar manner,

$$\begin{aligned} D_0 &= -\rho_1 C_0 - \lambda_1 \frac{a^2}{c^2} \rho_1\rho_2(-c_0) \\ &= \rho_1^2\rho_2 \left(C_0 K_0 - B\mu_1 \frac{b^2}{c^2}\right) + 2\lambda_1\rho_1\rho_2 C_0 \frac{a^2}{c^2} - \rho_1^2\mu_1 C_0 \frac{b^2}{c^2} \\ D_1 &= 2\iota K_1 \rho_1^2 \rho_2 c_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95) \end{aligned}$$

The leading coefficient in the potential next meeting the first wire is

$$E_0 = \rho_1^2\rho_2^2\left(C_0 K_0 - B\mu_1 \frac{b^2}{c^2}\right) + 2\lambda_1\rho_1\rho_2^2 C_0 - 2\rho_1^2\rho_2\mu_1 C_0 \frac{b^2}{c^2}, \quad (96)$$

and so on, the mode of formation being now obvious.

The potentials containing the coefficients A_0, C_0, E_0, $G_0 \ldots$ contribute to the current in the first wire, and those with d_0, B_0, D_0, $\ldots$ to that in the second.

Now

$$A_0+c_0+E_0+\ldots=\left(C_0K_0-B\mu_1\frac{b^2}{c^2}\right)(1+\rho_1\rho_2+\rho_1^2\rho_2^2+\ldots)$$

$$+\left(\lambda_1\rho_2\frac{a^2}{c^2}-\mu_1\rho_1\frac{b^2}{c^2}\right)C_0(1+2\rho_1\rho_2+3\rho_1^2\rho_2^2+\ldots)$$

$$=\left(\rho_2BK_0-B\mu_1\frac{b^2}{c^2}\right)\Big/(1-\rho_1\rho_2)$$

$$+\rho_2B\left\{\lambda_1\rho_2\frac{a^2}{c^2}-\mu_1\rho_1\frac{b^2}{c^2}\right\}\Big/(1-\rho_1\rho_2)^2, \quad . \quad (97)$$

since

$$C=\iota B\theta_0\frac{\delta_2}{\beta}=\rho_2B,$$

and

$$d_0+B_0+D_0+F_0+\ldots=\left(BK_0-\rho_1B\mu_1\frac{b^2}{c^2}+\lambda_1\frac{a^2}{c^2}B\rho_2\right)\Big/1-\rho_1\rho_2$$

$$+B\rho_1\rho_2\left(\lambda_1\rho_2\frac{a^2}{c^2}-\mu_1\rho_1\frac{b^2}{c^2}\right)\Big/(1-\rho_1\rho_2)^2 \quad . \quad (98)$$

on summation.

Similarly, due to $B'K_0(\iota h\rho)$ originally leaving the second wire, we have in the first an addition to $A_0+C_0+\ldots$ of magnitude

$$\left(B'K_0-\rho_2B'\lambda_1\frac{a^2}{c^2}+\mu_1\frac{b^2}{c^2}B'\rho_2\right)\Big/(1-\rho_1\rho_2)$$

$$+B'\rho_1\rho_2\left(\mu_1\rho_1\frac{b^2}{c^2}-\lambda_1\rho_2\frac{a^2}{c^2}\right)\Big/(1-\rho_1\rho_2)^2. \quad . \quad (99)$$

But

$$B=-\frac{E\delta_1}{p\alpha}=\frac{\iota E\rho_1}{p}, \quad B'=-\frac{E\delta_2}{p\beta}=\iota\frac{E\rho_2}{p} \text{ by (54).}$$

Adding the potential $H=AJ_0(kr)$ obtained in a previous section, and reducing, the total potential in the first wire, which contributes to the current, becomes

$$H=\frac{-EH_1}{pha\alpha}J_0(kr), \quad . \quad . \quad . \quad . \quad . \quad . \quad (100)$$

where

$$H_1 = 1 + \left(\rho_2 K_0 - \mu_1 \frac{b^2}{c^2}\right)\frac{\rho_1}{K_0} \cdot \Big/ (1-\rho_1\rho_2)$$
$$+ \frac{\rho_1\rho_2}{K_0}\left(\lambda_1\rho_2\frac{a^2}{c^2} - \mu_1\rho_1\frac{b^2}{c^2}\right)\Big/(1-\rho_1\rho_2)^2$$
$$+ \frac{\rho_2}{K_0}\left(K_0 - \rho_2\lambda_1\frac{a^2}{c^2} + \mu_1\rho_2\frac{b^2}{c^2}\right)\Big/(1-\rho_1\rho_2)$$
$$+ \frac{\rho_2^2}{K_0}\left(\mu_1\rho_1\frac{b^2}{c^2} - \lambda_1\rho_2\frac{a^2}{c^2}\right)\Big/(1-\rho_1\rho_2)^2, \quad . \quad . \quad (101)$$

and the current is

$$\varpi_1 = \frac{E\delta_1}{2p\alpha} \cdot H_1. \quad . \quad . \quad . \quad . \quad . \quad (102)$$

A similar expression may be written down at once for the current in the second wire, and a formula for the self-induction obtained. When the wires are similar in all respects,

$$H = 1 + \left(\rho_1 K_0 - \lambda_1\frac{a^2}{c^2}\right)\frac{\rho_1}{K_0}\Big/(1-\rho_1^2)$$
$$+ \frac{\rho_1}{K_0}\left(K_0 - \rho_1\lambda_1\frac{a^2}{c^2} + \rho_2\lambda_1\frac{a^2}{c^2}\right)\Big/(1-\rho_1^2)$$
$$= \frac{1+\rho_1}{1-\rho_1^2} - \frac{\lambda_1}{K_0}\cdot\frac{a^2}{c^2}\cdot\frac{\rho_1}{1-\rho_1^2}.$$

If L be the self-induction of the system, ιLp is the imaginary part of

$$\frac{4p\alpha}{\delta_1}\left\{\frac{1}{1-\rho_1} - \frac{\lambda_1 a^2}{K_0 c^2}\cdot\frac{\rho_1}{1-\rho_1^2}\right\}^{-1},$$

or of

$$\frac{4p\alpha}{\delta_1}\cdot(1-\rho_1)\cdot\left\{1 + \frac{\lambda_1 a^2}{K_0 c^2}\cdot\frac{\rho_1}{1+\rho_1}\right\} \quad . \quad . \quad (103)$$

to the same order.

The first term of this expression leads to the results already given. The second term, writing

$$\rho_1 = -K_0\Big/\left(\log h\alpha - \frac{\mu_0 J_0}{h\alpha J_0'}\right),$$
$$1+\rho_1 = 2 \text{ practically},$$

becomes

$$4ip\frac{a^2}{c^2}\lambda_1 \cdot \frac{\log\left(\frac{a}{c}\right) - \frac{\mu J_0}{ka J_0'}}{\log(h^2ac) - \frac{\mu J_0}{ka J_0'}}.$$

If R be the real part of

$$4\frac{a^2}{c^2}\lambda_1 \cdot \frac{\log\frac{a}{c} - \frac{\mu J_0}{ka J_0'}}{\log h^2ac - \frac{\mu J_0}{ka J_0'}} \quad . \quad . \quad . \quad . \quad (104)$$

where

$$\lambda_1 = \left(1 - \frac{\mu}{ka}\frac{J_1}{J_1'}\right) \Big/ \left(1 + \frac{\mu}{ka} \cdot \frac{J_1}{J_1'}\right), \quad . \quad . \quad (105)$$

and the argument of the Bessel functions is in all cases

$$ka = xi^{3/2} = 2a\sqrt{\frac{\pi\mu n}{\sigma}} \cdot i^{3/2} \text{ as before.} \quad . \quad (106)$$

Then the self-induction is

$$L = 4\log\frac{c}{a} + \frac{4\mu}{x} \cdot \frac{\text{ber}\, x\, \text{ber}'\, x + \text{bei}\, x\, \text{bei}'\, x}{(\text{ber}'\, x)^2 + (\text{bei}'\, x)^2} + R. \quad (107)$$

This formula may be used even when $c = 3a$, to obtain a result not in error by more than one per cent. For the application of this result, the functions analogous to ber x, and bei x derived from the Bessel function of order 1, require tabulation.

When the resistivity of the wires is not very great, and the frequency is large, the Bessel functions in these results may usually be replaced by their ordinary asymptotic expansions for large argument, and tables cease to be necessary. The solutions then approximate to that for an infinite frequency, which is well known*. The formula (72), applicable when the wires are very far apart in comparison with their radii, is chiefly useful in determining a differential effect, for the ends of the wires will have a great influence in most cases to which the formula might be applied if the wires were really infinite.

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* Russell, *l. c. ante.*

Discussion.

Mr A. Campbell congratulated the Author and said it was often desirable when measuring self-inductances to check the results by calculation. The Author's formulæ would be useful in finding the inductance of parallel leads, although this was best obtained experimentally by a separate experiment.

Dr Russell said that it required considerable mathematical skill and great patience to attack this problem, and so he thanked the Author for having obtained a solution sufficiently accurate for practical purposes. He pointed out that the formula would have to be used with caution at very high frequencies as the capacity current was then appreciable. He suggested to the Author that he should compute the kinetic energy of the ions or electrified particles by the motion of which modern theory explains the phenomena of the electric current. It has hitherto been the custom to assume that the self-inductance of a current flowing in a circuit can be calculated by finding the magnetic energy only, and this is what Dr Nicholson has done. If we accept modern theory, however, we must admit that part of the total energy of a current is due to the kinetic energy of the ions. To take this into account it is necessary to add the term $\Sigma mv^2/i^2$ to Dr Nicholson's solutions, where m is the mass of an ion, v the mean velocity given to it by the electromotive force, and i the current. In the case of metallic conduction this term is in general negligible compared with those found by the Author. In cases, however, where the current takes place through gases—as in vacuum tube lighting—the value of the correcting term is appreciable, owing to the high velocity of the electrified particles. In a paper to the Röntgen Society on the measurement of a current through an X-ray tube, Mr Duddell recently described some very interesting phenomena which he had noticed during his difficult but very successful experiments. The speaker thought that probably some of these effects might be explained by taking into account the kinetic energy of the moving ions.

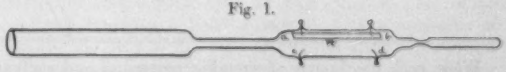
XXX. *A Note on the Photoelectric Properties of Potassium-Sodium Alloy.* By J. A. Fleming, *M.A., D.Sc., F.R.S., Professor of Electrical Engineering in University College, London* *.

In connexion with further researches on my oscillation valve or glow-lamp radiotelegraphic wave-detector, I was led to examine instances of electronic emission other than that due to the incandescence of metals or carbon *in vacuo*, to ascertain how far rectifying effects with high frequency currents could be obtained by them†. It is well known that under the action of ordinary and ultra-violet light the electropositive metals lose a negative charge of electricity, and it was shown by Elster and Geitel that this photoelectric effect is most pronounced in the case of rubidium, potassium, and the liquid alloy of potassium and sodium.

With the object of examining this effect, experiments were made with various forms of apparatus, and, as the outcome of these, it was found that a convenient mode of preparing a suitable specimen of highly photoelectric metal was as follows:—

A tube of lead glass about 50 cms. long, and 1·5 to 2 cms. internal diameter has a constriction made at one place, and on

Fig. 1.



one side of this a couple of platinum wires are sealed through the glass which are welded to a slip of platinum foil *a*, *b*, about 5 cms. long and 1 cm. wide (see fig. 1). This foil is fixed close to the inside surface of the tube and opposite to it two other platinum wires *c d* are sealed through the tube.

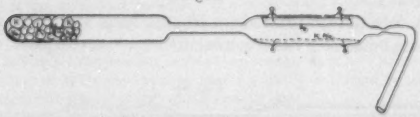
* Read November 13th, 1908.

† See J. A. Fleming, "On the Conversion of Electric Oscillations into Continuous Currents by means of a Vacuum Valve," Proc. Roy. Soc. Lond. vol. lxxiv. p. 476 (1905). Also "The Construction and Use of Oscillation Valves for rectifying High Frequency Currents," Phil. Mag. May 1906, p. 659.

The end of this part of the tube is then drawn off and sealed to a length of smaller tube with a constriction in it by which the tube can ultimately be closed. A quantity of metallic potassium and sodium is then cut up into small cubes under naphtha, so as to give a number of clean morsels of the metal.

The operation of filling the glass tube is then as follows :— The fine quill tube at one end is first bent over at an angle of about 75° (see fig. 2), and by means of an indiarubber tube, coal-gas, which has been dried by bubbling through strong sulphuric acid, is led into it. The glass tube as a whole must have previously been well cleansed and carefully dried. When all the air is expelled, the fragments of sodium and potassium are dropped in at the wide open end of the tube, and about equal volumes of the two metals introduced. The open end is then loosely stopped with a plug of asbestos, and the glass tube heated in the blowpipe, and the open end so plugged is melted and sealed. If this is done skilfully, the tube can be sealed off with a rounded end by utilizing the gas-pressure to blow out the softened glass.

Fig. 2.



When cool, the lumps of metallic potassium and sodium are tilted over and should fill up about 10 to 15 cms. of the length of the so sealed off tube (see fig. 2). The tube is then attached to a mercury pump for the purpose of making a high vacuum in it, and should be supported in a slanting position on a sheet of asbestos cardboard and covered by an iron trough. The tube should then be exhausted, and at the same time the metallic potassium and sodium heated so as to melt the lumps into a mass of liquid alloy which, however, will be covered with crusts of oxide.

When a good vacuum has been produced, the whole tube being hot and dry, the constriction in the quill tube may be

melted with the blowpipe and the glass tube sealed off from the pump. If this is properly done, the part of the tube having in it the platinum plate will remain quite clean, and the molten metal or liquid alloy will be contained in the other portion of the tube. It is then easy to tilt the tube and transfer the clean mercury-like alloy of K and Na into the part of the tube containing the platinum plate, so as to make a pool of liquid alloy on the bottom of the tube having an electrical connexion with the outside by means of the platinum wires sealed through the tube, and having over it, and not far from its surface, a platinum plate also in connexion with the outside by sealed-in wires. In so doing, care should be taken that the liquid alloy is not splashed upon the platinum plate, but that the latter is kept quite clean and free from adherent drops of potassium-sodium.

It is better not to attempt to seal off that part of the tube in which the alloy is melted from that part to which it is decanted over, because the glass nearly always cracks and spoils the apparatus in so doing if there is the slightest particle of alloy smeared on it at the part heated in the flame.

A tube so prepared should be carefully handled, as if it is broken the liquid K and Na alloy is spontaneously inflammable and deflagrates violently on coming in contact with moisture. It is convenient to mount it on a wooden stand, which should be placed in an iron tray in case of an accidental breakage. By the aid of such a specimen of photoelectric alloy we can show many interesting experiments. If the tube is supported in a horizontal position, an electric arc contained in a projection lantern, equipped with the ordinary condenser-lens, can be so placed and tilted downwards as to converge on to the brilliant mercury-like surface of the pool of alloy a very concentrated beam of light. If, then, we connect the platinum plate and the alloy by means of wires with a sensitive mirror galvanometer, we find that the impact of the light upon the surface of the alloy not merely facilitates the escape of negative electricity from it, but actually creates an electromotive force and current in the galvanometer circuit, which is in such a direction as to indicate that negative electricity is caused to move, by the

action of the light, through the vacuum from the surface of the alloy to the platinum plate *.

It has been pointed out by Professor Sir J. J. Thomson in his book 'The Discharge of Electricity through Gases' (see p. 73) that air traversed by ultra-violet light behaves like an electrolyte; and it is also stated that Stoletow has shown by direct experiment that two different metals immersed in such illuminated air are brought down to the same potential just as when they are immersed in a liquid electrolyte. Again, it is said that when plates of two different metals are opposed to each other and ultra-violet light allowed to fall on one of them, whilst the terminals of a battery of electromotive force E are connected to them so as to force a current across the space between them, the total E.M.F. in the circuit is not E but is $E+M/M'$, where M/M' is the contact difference of potential between the metals. There is, however, here an unexplained discrepancy. If metallic potassium or sodium were placed with a platinum plate in a liquid aqueous electrolyte, and the two metals connected by a wire outside so as to construct a voltaic cell, the current through the cell, that is the movement of positive electricity, would be from the potassium or sodium to the platinum through the electrolyte, and therefore the movements of negative electricity would be in the opposite direction. In the case of the photoelectric cell, the movement of negative electricity in the cell is from the K-Na alloy to the platinum. Hence the physical operation of the photoelectric cell is not identical with that of an ordinary voltaic cell. Stoletow has described (*Physikalische Revue*, i. p. 765, 1892) experiments in which two plates of different metals, one perforated with holes, were placed parallel and at a little distance from each other, and ultra-violet light allowed to fall through the openings of the perforated plate on to the other plate. The plates, when connected together, were found to produce a current of electricity in the connecting circuit. Prof. Sir J. J. Thomson,

* In the case of one tube of alloy prepared as above, my assistant, Mr. G. B. Dyke, noticed that the galvanometer deflexion was largely increased for a time by tilting the tube so as to make momentary contact between the alloy and the platinum, but the increased effect is not permanent.

however, remarks (*loc. cit.* p. 73) that for this to happen in accordance with voltaic principles, the perforated plate must be made of the more electropositive metal, for then only would the negative electricity move towards it across the interspace between the plates. As a matter of fact, the best photoelectric metals are the most electropositive metals. Accordingly, if we imagine a sheet of potassium opposed to a sheet of zinc, then to make the photoelectric current agree in direction with the volta-electric current, we should have to illuminate the zinc plate with ultraviolet light, but keep the potassium plate in the dark. In the present case, the photo-electric current is in the opposite direction to the volta-electric current, assuming the ionized gas replaced by an aqueous electrolyte.

It seems therefore that in the case considered there are two separate sources of electromotive force, viz.: the volta contact-difference of potential of the metals in the cell and a photoelectromotive force due to the illumination; and to these may be added an external electromotive force due to any battery inserted in the circuit. In two tubes, made as above, I found that the electromotive force produced by the incident light was equal to 0·45 volt in one, and in the other tube to 0·6 volt. This E.M.F. was measured by the counter E.M.F. (produced by a shunted cell) which had to be introduced into the photoelectric cell circuit to reduce the galvanometer deflexion to zero. These differences are no doubt due to small differences in composition of the alloy in the two cases, to differences in the intensity of the light used at the time of making the measurements, or to differences in the pressure of the residual gas in the tube. The current produced by the tube of E.M.F. equal to 0·6 volt through a galvanometer of resistance 180 ohms, was found to be 5·4 microamperes. This indicates that the equivalent resistance of the tube is 74,000 ohms on the assumption that the effective E.M.F. is the same when the circuit is open and closed. The current increases very rapidly at first with the intensity of the incident light, but the author has not yet been able to find time to make measurements of the relation between the illumination per square cm. of the surface and the E.M.F. created in the circuit.

By the use of two such cells placed in series, I have found that the separate photoelectromotive forces are additive, and that if such photoelectric cells are joined in series like voltaic cells and separately illuminated, the individual E.M.F.'s are added together in an external circuit connecting the first and last plate. In the case of the two tubes mentioned above, giving separately 0·45 and 0·6 volt E.M.F., the E.M.F. when in series was found to be 1·0 volt.

At one time I contemplated constructing a photoelectric battery of such cells which should give a high E.M.F. by the mere impact of light upon its most electropositive elements.

With the above described cells we can also verify easily a number of interesting observations made by previous experimentalists.

As regards the nature of the radiation which is chiefly effective, we find that for potassium alone, or for the potassium-sodium alloy, the effective rays are the most refrangible ones of the visible spectrum. The ultra-violet light in the arc is considerably filtered out, if not altogether stopped, by the thick glass condenser-lens, and by the glass walls of the tube. We can, however, effect a further separation of visible rays by screens of coloured glass, or solutions or stained gelatine films.

If we interpose in the path of the incident light a very thin film of gelatine stained with a yellow dye or a sheet of ordinary yellow glass, the deflexion of the galvanometer drops almost to zero. The same reduction is effected by a sheet of ruby glass or gelatine film stained red. A green glass cuts off a good deal of the deflexion, but a sheet of cobalt glass reduced it only to about two-thirds, showing that the blue glass is fairly transparent to those rays which can produce this electromotive force.

The galvanometer deflexions increase very rapidly with the intensity of the incident light, and unless an extremely sensitive galvanometer is used it is necessary to throw a very intense beam of light upon the surface of the metal to obtain any marked evidence of the production of electromotive force by the mere action of light upon the surface of the alloy.

If the energy required to produce this photoelectric current comes from the incident light, then the latter must in some degree be absorbed, and the active rays must therefore be those which are absorbed by the photoelectric metal or conversely emitted by it when it is heated. We know well that the light due to the violet line in the flame spectrum of potassium salts passes easily through cobalt glass. Hence the fact that light which has passed through cobalt glass is still exceedingly active in producing the photoelectric effect with potassium, may indicate that it is this ray which is absorbed in its production. In the case of rubidium, Elster and Geitel showed that the yellow and orange rays were relatively more active in the production of the photoelectric effect, and a glance at the flame spectrum of rubidium salts shows that the lines are most numerous in the red, yellow, and green region.

Also, we can confirm easily another observation of the effect of the plane of polarization of the incident light. If we polarize the incident light by a Nicol's prism, and connect the metals of the photoelectric cell with a galvanometer either with or without a battery of 1 or 2 secondary cells inserted in the circuit, then we find at once that the galvanometer deflexion is much greater when the plane of polarization is at right angles to the plane of incidence, than when these planes coincide.

On the electromagnetic theory of light, this may be stated by saying that the effect is a maximum when the electric vector of the plane-polarized light is normal to the surface of the photoelectric metal, and a minimum when it is parallel to it. The theory that fits in best with the above facts, is that the light facilitates or causes an escape of negative corpuscles or electrons from the surface of the photoelectric metal, and to do this it must of course impart to them energy sufficient to give them a velocity enough to carry them beyond the range of attraction of the positive charge which remains behind on the metal.

Since the electropositive metals are those which most easily lose electrons from their atoms, this is in accordance with the observed fact that the most electropositive metals

are the most highly photoelectric substances. Again, if there is an emission under the action of light of electrons from the surface, we have an explanation of another fact easily proved with these tubes, viz. that the formation of a magnetic field parallel to the surface of the alloy greatly reduces the photoelectric current. We can show this easily by the great reduction which occurs in the galvanometer deflexion when even an ordinary horseshoe magnet is placed with its poles across the tube. Again, we can explain on this hypothesis the unilateral conductivity of the vacuous space over the illuminated alloy surface. If we connect the negative pole of a secondary cell to the external terminal of the alloy and the positive to the platinum plate, and insert a galvanometer as usual in the circuit, we greatly increase the galvanometer deflexion, which occurs when no cell is inserted. On the other hand, if the cell is reversed, then, provided the platinum is quite free from splashes of alloy, we cannot reverse the deflexion of the galvanometer.

It is obvious that if a magnetic field is created parallel to the surface of the alloy, a force will be exerted on the negative ion as it moves normally away from the surface, deflecting it from its path, and this will reduce the number of ions which in any time reach the platinum plate, and therefore will diminish the current. Again, the negative electrification of the photoelectric metal will increase the force of propulsion on the ion, and hence increase the emission per unit of time, and therefore increase the observed current.

It is obvious, then, that the ionized gas over the illuminated alloy has a unilateral conductivity, and will conduct the current from a voltaic cell in one direction, but not in the opposite. Negative electricity can be conveyed from the alloy to the platinum across the rarefied air-space, but not in the opposite direction. Hence, an alternating current whether of low or of high frequency, can be rectified, and by interposing such a light-cell in the circuit of a galvanometer, in which circuit high-frequency oscillations are also created by the inductive action of a discharging condenser, I have been able to rectify these oscillations. The action, however, proved to be much more feeble and irregular than the similar

rectifying effect which can be produced by a glow-lamp, and as at that time I had succeeded in finding a far more efficient glow-lamp rectifier or oscillation detector, the experiment with the rectifying effects of potassium-sodium alloy were not continued.

The interesting question, however, still remains as to the source and nature of this photoelectromotive force which is produced by the absorption of light by the surface of highly electropositive metals.

It is much affected by temperature, being increased by heating the alloy and by exposing it to powerful radiation for some time. If a galvanometer is used as described to detect the photoelectric effect, the current is found to be dependent upon the pressure and nature of the residual gas in the tube.

This was confirmed as follows:—A tube was prepared as above described with potassium-sodium alloy. Before sealing it off the pump, dry hydrogen gas was admitted and pumped out and then admitted again, and the tube exhausted to a pressure of about 0·01 mm. This tube was compared with others, in which the residual gas was air, but the exhaustion carried below 0·001 mm. The former tube exhibited only the very smallest evidence of photoelectric effect as measured by the galvanometer current, whereas the highly vacuous tube with residual air under very small pressure exhibited the effects well. The case of other tubes made as described, in which water vapour was accidentally present, which in a short time liberated hydrogen under the action of the metal; these after an interval ceased to give any current, even under the action of the strongest illumination, when the platinum plate was connected with the potassium-sodium alloy through a galvanometer. These experiments and others show that to obtain the effect well a very high vacuum is necessary. In the process of preparing the tube the potassium and sodium should be melted, and the alloy and tube well heated for some time whilst the pump is going to drive off all traces of water vapour, also of hydrocarbon derived from the naphtha in which the metallic potassium is commonly preserved, and especially to drive off the hydrogen which

seems to be occluded in considerable quantity by the alkaline metals, no doubt derived from the decomposition of water vapour.

The effect of variation of pressure of the gas upon the photoelectric effect was first investigated by Stoletow, and an account of his researches is given in Prof. Sir J. J. Thomson's treatise on the "Conduction of Electricity through Gases" (p. 224). He found that the current increased rapidly as the pressure diminished, which continued until the current reached a maximum value, after which it began to decline, but had a finite value at the lowest attainable pressures.

Stoletow's experiments appear to have been made with zinc plates and ultra-violet light. Owing to the greater manipulative difficulty when dealing with the more oxidisable and electropositive metals no attempt has been made to push these investigations with potassium-sodium alloy very far. The primary object in view in conducting them was to ascertain if the photoelectric effects could be utilized as an oscillation detector in radiotelegraphy; but as a type of glow-lamp detector has now been found by the writer far more efficient than that originally proposed by him, involving the employment of a carbon filament, these photoelectric experiments have not been pursued. They are put on record here merely for the sake of aiding any who may wish to show them as interesting lecture or class experiments, or pursue the purely physical investigation of the effect itself still further.

The question of photoelectric effects is not without interest in connexion with long distance radiotelegraphy. It has been shown that perfectly dust-free air is not ionized by ultra-violet light. If, therefore, the absorption of long radiotelegraphic electric waves which is found to exist when they pass through considerable distances of sunlit air is due to the presence of free ions in the air, these may arise from the photoelectric action of the light upon the dust particles. This suggests the question whether these particles may not be the same that create the blue colour of the sky. We know that whenever photoelectric effects take place, light must be absorbed, and light of the same kind as that emitted by the

photoelectric substance if it radiates. Thus zinc is rendered photoelectric under the action of ultra-violet light, but zinc when heated, whether by being used as spark-balls for an electric spark or as arc terminals for an electric arc, radiates much ultra-violet light.

In the same manner, the photoelectric effect of potassium appears to be due to the absorption of that violet ray which potassium itself emits if heated. If, then, these atmospheric particles absorb ultra-violet light, that would account for the relatively small percentage of ultra-violet light found in sunlight at the earth's surface, and also for the ionization found to exist in the atmosphere. This suggests the need for further observation on the number of ions present in the terrestrial atmosphere at various heights above the sea-level.

In conclusion, I have pleasure in mentioning the aid rendered in these experiments by my assistant, Mr. G. B. Dyke.

Discussion.

Prof. S. P. Thompson asked the Author whether Sodium was more sensitve to yellow light than to light of other wave-lengths.

Dr R. S. Willows remarked that the E.M.F. was in the direction required by theory: the alloy emits negative ions, and hence the direction of flow of negative electricity in the cell is from the alloy to the platinum or the alloy is the positive pole. The action was similar in results to those produced by solution pressure when metals were immersed in acids except that the ions produced by light were of the other sign. Prof. Fleming's idea that metals respond to those rays only which they absorb was interesting in view of the fact that the weight of evidence was in favour of the emission of ions being due to an explosion in the atom arising from resonance, but Ladenburg's experiments did not support Dr Fleming's theory. He measured the intensity of the light with a thermocouple, and found for the same intensity that the shorter the wave-length the more efficient was the light in producing emission of ions.

Dr Fleming, in reply to Prof. Thompson, said Sodium was most sensitive to yellow light.

XXXI. *A Graphic Method of dealing with Combinations of Co-axial Refracting Surfaces.* *By* H. STANLEY ALLEN, *M.A., B.Sc. Senior Lecturer in Physics at King's College, London* *.

[ABSTRACT.]

In this paper a graphic method is given that may be applied to the case of two thin lenses not in contact, or to the case of two co-axial refracting surfaces (as in the case of the thick lens). Further the method may be extended to the combination of any two co-axial systems. It affords a way of finding, by a simple geometric construction, the focal length of the combination, and also the positions of the focal planes and principal planes (planes of unit magnification) of Gauss.

Although in actual experiments we are restricted by the formulæ of the first order to rays making small angles with the axis, it is legitimate for purposes of construction to consider rays making finite angles with the axis. By the choice of two particular rays passing through a definite point in the plane of the geometric centre of two refracting systems, the cardinal points of the combination may easily be determined.

In this abstract the method is applied to the general case of the combination of two refracting systems in any media †. We assume that the positions of the focal points and principal planes of each system are given. We can then deduce the existence and positions of the nodal points, defined by the property that an in-ray through one nodal point is parallel to an out-ray through the other nodal point.

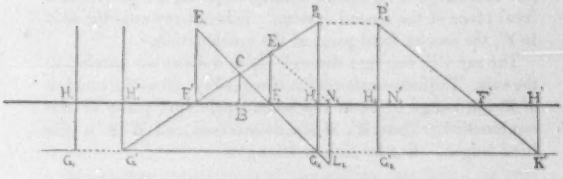
Through F_1', the second focal point of the first system (fig. 1), draw a line $F_1' E_1$ at right angles to the axis, equal to the second focal length of that system. Draw $F_2 E_2$ similarly through the first focal point of the second system, equal to the first focal length of that system. Let $F_1' E_2$ and $F_2 E_1$ intersect in C. Draw CB at right angles to the

* Read November 27, 1908.

† Drude, The Theory of Optics, p. 28.

axis. The point B may be called the "geometric centre" or the "optic centre" of the system. It divides the distance between the components in the ratio of their focal lengths. Hence the image of CB formed by the first system is of the

Fig. 1.



same size as the image of CB formed by the second system. It follows that these images are planes of unit magnification with regard to the combination, in other words they are the principal planes *.

The construction proceeds as follows. Let CE_1, produced if necessary, meet the first principal plane, H_2, of the second system in G_2, and the first nodal plane, N_2, of that system in L_2. Let CE_2, produced if necessary, meet the second principal plane, H_1, of the first system in G'_1, and the first principal plane of the second system in P_2. Then $G_1'G_2$ is parallel to the axis. Through P_2', a point in the second principal plane of the second system at the same distance from the axis as P_2, draw a line parallel to E_2N_2 meeting the axis in F' and $G_1'G_2$ in K'. Then F' is the second focal point of the combination, and K', the image of C formed by the second system, lies in the second principal plane H' of the combination. The second focal length of the combination is given by $H'F'$ or by N_2L_2.

* "If we imagine a circular aperture or pupil to be placed at the geometric centre then the image of it formed in the first principal plane by L_1 (the first lens) will be exactly equal to the other image of it formed in the second principal plane by L_2 (the second lens). The two principal planes are in this way shown to be planes of unit magnification relatively to one another."—H. Dennis Taylor, "A System of Applied Optics."

I am indebted to my former colleague, Mr. J. Nicol, for drawing my attention to the importance of the optic centre in the general case.

To verify these results trace the course of the rays CE_2, CF_2 through the system from left to right.

The in-ray corresponding to $F_1'C$ passes through G_1 and is parallel to the axis. The out-ray passes through P_2' and is parallel to E_2N_2, for it must pass through the image of E_2, and this image is formed at infinity since E_2 lies in the first focal plane of the second system. This out-ray cuts the axis in $\mathbf{F}'$, the second focal point of the combination.

The ray CF_2 emerges through G_2' in a direction parallel to the axis. The intersection of the two out-rays gives the position of $\mathbf{K}'$ the image of C, in the second principal plane of the combination. Thus $\mathbf{H}'$, $\mathbf{F}'$ are determined and $\mathbf{H}'\,\mathbf{F}'$ is the focal length. But by similar triangles

$$\frac{\mathbf{F}'\mathbf{H}'}{\mathbf{H}'\mathbf{K}'} = \frac{F_2N_2}{F_2E_2} = \frac{F_2N_2}{F_2H_2}.$$

Therefore

$$\mathbf{F}'\mathbf{H}' = \frac{F_2N_2}{F_2H_2} \times H_2G_2 = N_2L_2.$$

The first focal point, the first principal plane, and the first focal length can be determined in a similar way.

In the more common case in which both systems are in the same medium, as for two lens systems in air, the construction is still simpler, since the nodal points and principal points coincide. In this case $F_2N_2 = F_2H_2 = F_2E_2$ so that E_2N_2 makes an angle of 45° with the axis *.

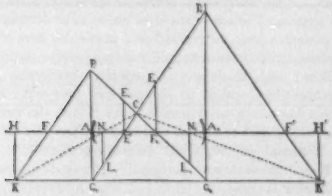
In fig. 2 we have the application of the method to the case of two co-axial refracting surfaces a finite distance apart, as in the thick lens. The principal planes of the first refracting surface coincide at A_1, the nodal points coincide at N_1 the centre of curvature.

The case of two thin lenses separated by a finite interval is shown in fig. 3. It is to be noticed that $P_1\mathbf{K}$, $P_2\mathbf{K}'$ make

* We may note that if a stop is put at B_1 the geometric centre of two lens systems in air, the images of this point formed by the two systems are orthoscopic points, for in this case the images coincide with the nodal points of the combination. (Drude, 'The Theory of Optics,' pp. 64–65).

angles of 45° with the axis, so that the focal length, **HF**, of the combination, is equal to **HK** or to A_1G_1.

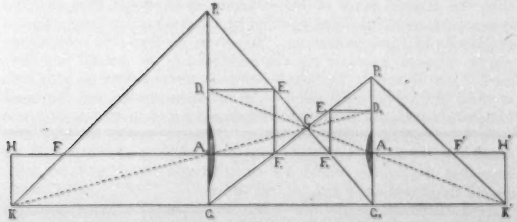
Fig. 2.



Interesting illustrations of the method are found in eye-pieces formed of two lenses, such as those of Ramsden or Huyghens.

It appears to the writer that the method here given may be of value for investigating the way in which the properties of a combination are affected by making given alterations in

Fig. 3.



the character or separation of the components. It may also be of use in enabling non-mathematical students to realise the existence of the cardinal points for a complicated system of refracting surfaces.

Discussion.

Dr Eccles remarked that the Author's treatment of refracting surfaces was limited to spherical surfaces and to the direct incidence of

pencils of infinitesimal aperture. If the treatment were extended to the discussion of oblique pencils and the construction of caustics it would be very valuable. The Author's method, as developed to the present, consisted of applying the well-known graphical construction for a harmonic mean, to the successive lenses or spherical surfaces met with in the coaxial system of lenses or surfaces under consideration. Now the harmonic construction just mentioned is a particular case of the general graphical connection between two collinear projective (or homographic) ranges of points; and therefore the speaker preferred, in his own teaching of geometrical optics, to use general projective constructions rather than this harmonic construction. As a fact, the general projective method would be felt to be the easier of the two, by anyone familiar with the fundamental principles of projective geometry, or, as it is sometimes called, "geometry of position." From the point of view of projective geometry the object-points and the image-points on the axis of a refracting (or reflecting surface), since they have a true one-one correspondence (as shown by their algebraic homographic connection), constitute two collinear projective ranges of points. Two such ranges can be graphically correlated, it is known, by at most three projections. The rays used in the first and last of these projections may be interpreted optically as rays of light through a source of light and through its image. The intermediate projection gives the central point on which the Author lays stress. If this point be taken at infinity the planes of unit magnification become defined. Since any number of successive projections at successive surfaces leave the ultimate range of image-points homographic with the original range of object-points, it is proved that any lens system whatever of the kind specified by the Author can be summarised graphically by three projections. Moreover, no graphical construction can be obtained which is not comprehended in the general projective method just sketched. In treatises on projective geometry the properties of these projective ranges and pencils are demonstrated, and they need merely be translated into optical language to give many beautiful and powerful graphical optical theorems. There is reason to think that caustics can be approached along these lines. The speaker expressed the hope that the Author would devote still further effort to these much desired simplifications of geometrical optics.

Dr A. Russell congratulated the Author, and indicated a simple graphical method of determining the focal length of two thin lenses in contact when the focal length of each was known.

Prof. C. H. Lees thought the graphical method described by the Author was an improvement upon algebraic methods.

XXXII. *The Diffusion of Actinium and Thorium Emanations.*

By SIDNEY RUSS, *B.Sc., Demonstrator in Physics, Manchester University* *.

Introduction.

No direct determinations of the molecular weights of the radioactive emanations have yet been made. Owing to the very small quantities of these substances available, experimental work on this subject has been almost completely restricted to a study of the way in which the emanations diffuse, and by a comparison of the diffusion coefficients so obtained with those of gases of known molecular weights, values of the molecular weights for the emanations themselves may be inferred by means of Graham's law.

The work of Curie and Danne †, Rutherford ‡, Rutherford and Miss Brooks §, and Makower ‖ on these lines has indicated, for the emanations of radium and thorium, a molecular weight of the order one hundred. Bumstead and Wheeler ¶, however, give a rather higher value in the case of radium, while for the emanation of actinium, Debierne ** obtained a molecular weight of seventy.

On the disintegration theory radium and thorium emanations should have very approximately the same molecular weight, namely, 222. There is thus a very large discrepancy between the theoretical and experimental numbers.

In order to account for this it has been suggested that the ordinary laws of diffusion do not perhaps hold when very minute quantities of the substances in question are being dealt with. There being no obvious remedy in this direction, it was thought that by a study of the diffusion of the emanations in gases differing considerably in molecular weight, any existing discrepancies would be brought to light. With

* Read November 27, 1908.

† *Comptes Rendus*, 1903, p. 114.

‡ 'Radioactivity,' p. 276.

§ Trans. Roy. Soc. of Canada, 1901-2.

‖ Phil. Mag. 1905, p. 56.

¶ *Le Radium*, June 1907.

** American Journal of Science, Feb. 1904.

this object in view, the diffusion coefficients of actinium emanation in air, hydrogen, carbon dioxide, and sulphur dioxide were determined. While the work was in progress there appeared in the American Journal of Science, June 1908, a paper by P. B. Perkins, in which the diffusion of radium emanation was compared directly with that of mercury vapour, the experiments giving a molecular weight of 235 for the emanation, a number approximating closely to that predicted by theory. The results of these experiments suggested that it might be advisable to determine the coefficient of diffusion of actinium emanation, itself presumably monatomic, into a monatomic gas. For this purpose a quantity of argon was prepared, and the coefficient obtained by two methods

Whatever the nature of the supposed deviations from the ordinary laws of diffusion referred to above, it might reasonably be expected that they would be exhibited to an equal extent by the different emanations. A direct comparison was therefore carried out over a fairly wide range of pressure between the diffusion of actinium and thorium emanations, a comparison of their coefficients leading directly to the ratio of their molecular weights.

Method of Experiment.

The theory of the method here adopted for obtaining the diffusion coefficient has been given by Rutherford*. If a layer of radioactive material be placed at the bottom of a cylindrical vessel a gradient of the emanation is set up, the partial pressure at any layer distant x from the base being given by $p = p_0 e^{-\sqrt{\frac{\lambda}{K}} \cdot x}$, where λ is the radioactive constant of the emanation and K its diffusion coefficient. It is not convenient in practice to measure the partial pressure of the emanation, but if a metal rod be placed along the axis of the cylinder in question and kept at a high negative potential, the active deposit produced by the emanation is directed to the rod soon after formation. We have thus a layer of active matter on the rod, the gradient of which is that of

* 'Radioactivity,' p. 275.

the emanation within the cylinder. The distribution of active matter along the rod can be found by measuring the ionization produced by the α rays emitted by successive small segments of the rod. This gives us the variation of p with x in the above equation, and knowing the value of λ, that of K is determined.

Debierne* has shown that the above theory is also applicable when the emanation is allowed to diffuse up into the space between two parallel plates. In an exactly analogous way a deposit of active matter is obtained on the plates, the gradient of which is that of the emanation itself.

Both of these experimental arrangements have been used in the course of this work.

Actinium Emanation in Different Gases.

For determining the coefficient of diffusion of the emanation in different gases the following experimental arrangement was used. A cylindrical vessel 12·5 cms. long and 7·5 cms. diameter was fitted with a base on which was placed a tray containing a preparation of actinium; about 1 cm. above the surface of the tray which was covered with tissue paper, were suspended vertically two glass plates 10 cms. long, 2·2 cms. broad, and ·2 cm. thick, which were kept separated ·2 cm. from one another. The plates were silvered and metallic connexions supplied, so that an electric field might be applied between the two plates if necessary. It may be stated here that the gradients obtained on the plates with, and without, an electric field were identical, the only difference being that nearly all of the activity was concentrated on the negative plate when the field was applied, while it was shared equally between the two plates with no field.

The gases before entering the diffusion vessel were passed through calcium chloride and cotton-wool tubes to render them dry and dust free. The vessel was placed inside a large tank of water, which prevented any appreciable change of temperature occurring during an experiment. During the whole course of the work the temperature varied between 10° and 18° C.

* *Le Radium*, June 1907

The glass plates were usually exposed for about twenty-four hours, and then they were removed in order to measure the gradient of active deposit on their surfaces. One of the plates was placed on a small platform which could be moved along so that small portions (·4 cm. lengths) of it came beneath a slit in a lead sheet, just above which was an ionization vessel. The saturation current produced in this vessel by the α rays emitted from the active deposit on the plate was measured by means of an electrometer, and readings were usually taken for about 4 cms. length of the plate. Corrections were applied for the decay of the active deposit with time, for change in sensitiveness of the electrometer during the measurements, and for the natural leak of the ionization vessel.

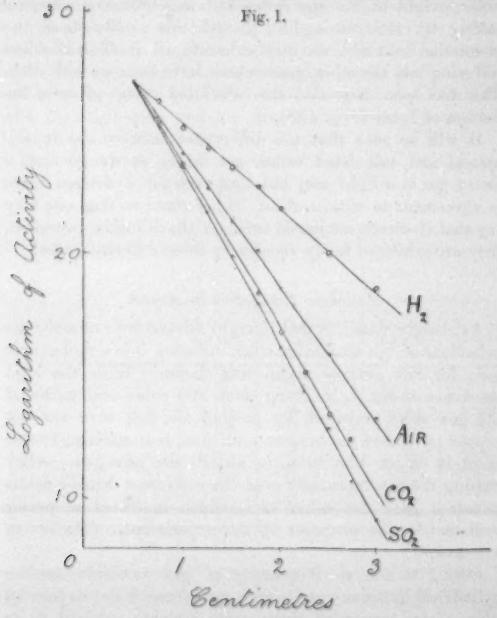
A selection of the results for the different gases will be found in Table I. and seen graphically in fig. 1, where the abscissæ represent distances along the plate and the ordinates the logarithm of the corresponding activity. For convenience in representing the results the initial ordinate has always been taken through the same point. The mean of three

TABLE I.—Diffusion of Actinium Emanation.

Distance in cms.	*Air.* Log. of Activity.	*Hydrogen.* Log. of Activity.	*Carbon Dioxide.* Log. of Activity.	*Sulphur Dioxide.* Log. of Activity.
·5	2·341	2·194	2·581	2·709
·75	2·182	2·115	2·416	2·526
1·0	2·036	1·997	2·233	2·358
1·25	1·901	1·919	2·026	2·173
1·5	1·768	1·837	1·846	1·932
1·75	1·648	1·761	1·710	1·798
2·0	1·494	1·668	1·531	1·616
2·25	...	...	1·391	1·439
2·5	1·224	1·496	1·212	1·281
2·75	...	...	1·112	1·068
3·0	·904	1·351	...	·912
Diffusion coefficient.	Mean of three	Mean of two	Mean of two	Mean of two
K Experimental...	·096	·330	·073	·062
K Calculated	...	·363	·078	·064

values of the coefficient for air was ·096, which is somewhat lower than ·112, the value found by Debierne. In the light of subsequent work with different diffusion vessels, which

Fig. 1.



gave higher values, it is thought that the rather low value obtained is due to the fact that the preparation of actinium did not completely cover the base of the diffusion vessel, this would cause some of the emanation to diffuse to the sides of the vessel, and consequently make the vertical gradient too

steep. This, however, does not interfere with a comparison being made in the different gases.

According to Graham's law the diffusion coefficient of a gas is inversely proportional to the square root of the molecular weight of the gas into which it is diffusing. Hence, taking the value obtained for the diffusion coefficient of the emanation into air, we may calculate its coefficient when diffusing into the other gases which have been worked with. This has been done and the calculated values given at the bottom of Table I. (p. 488).

It will be seen that the differences between the experimental and calculated values get larger as we go from a heavy gas to a light one; but that even for hydrogen there is agreement to within about 10 per cent., so that one may say that if discrepancies do exist in the diffusion processes, they are exhibited fairly equally by these different gases.

Actinium Emanation in Argon.

As already stated, it was thought advisable to observe the behaviour of the emanation when diffusing into a monatomic gas; for this purpose argon was chosen. With the kind assistance of Mr. J. N. Pring about 400 cubic centimetres of the gas were prepared, by passing air, first over red hot copper to remove the oxygen, and then over calcium, heated strongly in an iron tube, to absorb the nitrogen. After passing the gas repeatedly over the calcium a density determination gave the value 19·7, which indicated a purity sufficient for the purposes of the experiment. (Density of argon = 20.)

Owing to the small quantity of gas available smaller cylindrical diffusion vessels were made; one with two parallel glass plates suspended above some actinium, spread so as to completely cover the base of the vessel, and the other fitted with an ebonite plug 3 cms. long resting on the base of the vessel. A uniform layer of actinium was spread over the plug, a hole through the centre of which served to keep a metal rod along the axis of the cylinder, which was made of brass. The metal rod consisted of small seg-

ments of brass sliding on a central steel rod. On applying an electric field between the rod and containing vessel, the active deposit was directed to the rod and the gradient determined by placing each of the small brass segments successively into an α ray electroscope and measuring the ionization thereby produced. The gradient on the plates was determined in the manner already described. The results of the two methods of measurement are given in Table II. and seen in fig. 2 (p. 492), where the gradients obtained with the segmented rod are drawn above those measured from the glass plates.

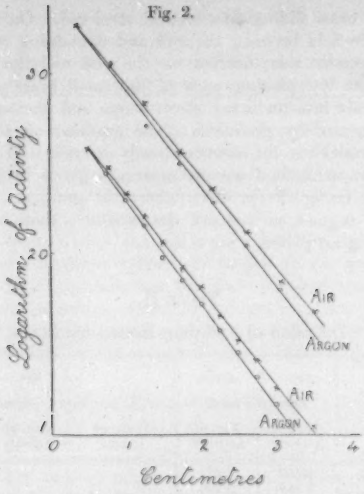
TABLE II.

Diffusion of Actinium Emanation in :—

	Air.	*Argon.*		*Air.*	*Argon.*
	Parallel Plates.			*Segmented Rod.*	
Distance in cms.	Log. of Activity.	Log. of Activity.	Distance in cms.	Log. of Activity.	Log. of Activity.
·5	2·550	2·574			
·75	2·454	2·438			
1·0	2·301	2·297	1·37	2·609	2·472
1·25	2·178	2·143	1·82	2·380	2·260
1·5	2·033	1·983	2·27	2·153	1·897
1·75	1·885	1·841	2·73	1·885	1·686
2·0	1·781	1·713	3·18	1·682	1·470
2·25	1·623	1·582	3·62	1·438	1·239
2·5	1·489	1·455	4·07	1·163	·990
2·75	1·382	1·305	4·53	1·014	
3·0	1·231	1·175			
3·5	1·020	·956			
4·0	·896	·612			
Diff. coeff. K	Mean of three			Mean of two	
Experimental	·118	·106		·122	·109
Calculated	...	·100		...	·104

The coefficients obtained by the two different methods agree well with one another, also with the numbers calculated from Graham's law, making use of the numerical values obtained for air.

Fig. 2.



Variation of the Coefficient with Pressure.

According to the kinetic theory of gases the coefficient of diffusion is inversely proportional to the total pressure of the two diffusing gases; the product of the coefficient and the pressure should therefore remain constant.

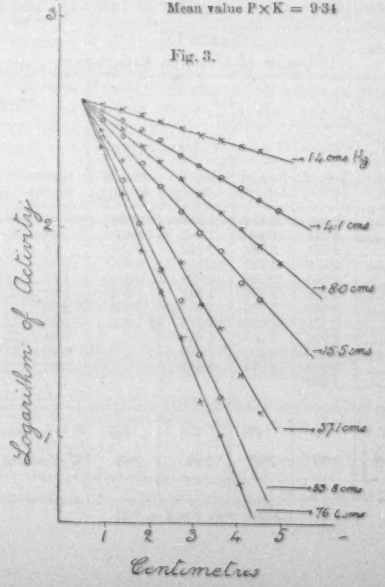
The variation of the coefficient has been determined for pressures between 76·4 cms. and 1·4 cm.; the small vessel containing the segmented rod was used. The results are recorded in Table III. & fig. 3. It will be seen that over this range of pressure, with the exception of 1·4 cm. pressure, the product $P \times K$ is approximately constant, its mean value being 9·34. It is difficult to say whether the high value 10·9 at the lowest pressure indicates a deviation from the above law as the gradient obtained is a very small one, and relatively large errors may be made in estimating it.

TABLE III.—Actinium Emanation in Air.
Variation of Diffusion Coefficient with Pressure.

Distance in cms.	1·4 cms. Hg. Log. of Activity.	4·1 cms. Log. of Activity.	8·0 cms. Log. of Activity.	15·5 cms. Log. of Activity.	37·1 cms. Log. of Activity.	55·8 cms. Log. of Activity.	76·4 cms. Log. of Activity.
·46	1·618	1·797	1·850	1·254	1·518	2·246	2·609
·92	1·601	1·744	1·759	1·157	1·367	2·057	2·380
1·37	1·564	1·689	1·704	1·074	1·230	1·860	2·153
1·83	1·548	1·646	1·625	·946	1·044	1·650	1·885
2·28	1·509	1·596	1·548	·829	·909	1·434	1·682
2·72	1·485	1·523	1·469	·712	·745	1·286	1·458
3·17	1·445	1·477	1·391	·611	·539	1·024	1·163
3·63	1·425	1·421	1·313	·528	·428	·829	1·014
4·09	1·389	1·380	1·232	·374	·204	·734	·784
4·55	1·375	1·316	1·147	·208	·020		
5·01	...	1·268	1·071				
Diff. coeff. K	7·81	2·37	1·16	·582	·248	·167	·125
Pressure × K	10·9	9·7	9·3	9·0	9·2	9·3	9·5

Mean value P × K = 9·34

Fig. 3.



Diffusion of Thorium Emanation in Air and Argon.

Since the time-periods of thorium and actinium emanations are of the same order of magnitude, and previous work (*loc. cit.*) has shown their diffusion coefficients to be similar, the opportunity presented itself of comparing these two emanations under precisely the same conditions of diffusion.

The small diffusion vessels which had been used for actinium were emptied of their preparations and thoria substituted. The measurements were conducted in a similar way, except that the α ray electroscope could not now be used, as the thoria being considerably weaker in emanating power than the actinium, the amount of active deposit collected was smaller. The gradient along the metal rod was consequently measured by means of a rather sensitive electrometer in the manner already described for the glass plates.

Observations were made in air between 76 cms. and 8·25 cms. pressure, and in argon at 76 cms.

The results are placed together in Table IV. and may be

TABLE IV.

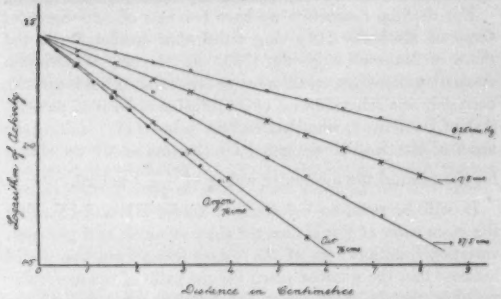
Diffusion of Thorium Emanation.

	Air.				Argon.	
Pressure ...	8·25 cms.	17·5 cms.	37·52 cms.	76·12 cms.	76 cms.	
Distance in cms.	Log. of Activity.	Log. of Activity.	Log. of Activity.	Log. of Activity.	Distance in cms.	Log. of Activity.
0	2·401	2·472	2·373	2·368	0	1·875
·8	2·375	2·406	2·282	2·453	·38	1·813
1·6	2·327	2·354	2·192	2·329	·76	1·741
2·4	2·289	2·285	2·134	2·203	1·14	1·672
3·2	2·255	2·239	2·028	2·082	1·52	1·608
4·0	2·206	2·176	1·947	1·962	1·90	1·563
4·8	2·169	2·116	1·853	1·833	2·28	1·490
5·6	2·131	2·058	1·775	1·713	2·66	1·434
6·4	2·094	1·989	1·685		3·04	1·369
7·2	2·053	1·937	1·602		3·42	1·327
8·0	2·014	1·874	1·517		3·80	1·254
8·8	1·969					
Diff. coeff. K	·966	·436	·211	·103	K experimental	·084
Pressure × diff. coeff. (P × K) ..	7·97	7·63	7·94	7·88	K calculated ...	·087

Mean value P × K = 7·85

seen in fig. 4, where the abscissæ represent the distances along the rod and the ordinates the logarithm of the corresponding activity.

Fig. 4.



It will be seen from the last row of Table IV. that the product, pressure × diffusion coefficient (P × K) remains sensibly constant, the mean value being 7·85.

As in the case of actinium emanation no large deviation from ordinary gas laws is observable when thorium emanation diffuses into a monatomic gas of comparatively low molecular weight like argon; the experimental value obtained for the diffusion coefficient in this case being very nearly that calculated by means of Graham's law.

Comparison of Actinium and Thorium Emanation.

A direct comparison may now be made between the diffusion coefficients of these two emanations, which leads to a ratio of their molecular weights.

Taking the mean value of P × K for actinium emanation in air as 9·34, and for thorium as 7·85, and dividing each by 76 to reduce to atmospheric pressure, we have:—

$$\frac{\text{Diffusion coefficient of actinium emanation in air}}{\text{Diffusion coefficient of thorium emanation in air}} = \frac{\cdot 123}{\cdot 103} = 1\cdot 19$$

which gives the ratio:—

$$\frac{\text{Molecular weight of thorium emanation}}{\text{Molecular weight of actinium emanation}} = (1\cdot 19)^2 = 1\cdot 42.$$

It is interesting to compare this ratio with that which is obtained by taking the values of the diffusion coefficients of thorium and actinium emanations which have been determined by other observers under different experimental conditions.

For thorium emanation we have two sets of experiments; those of Rutherford* giving a diffusion coefficient ·09 and those of Makower† giving ·109. In the case of actinium emanation there has, up till now (as far as the author is aware), been only one determination of its diffusion coefficient, namely, that of Debierne‡, who obtained the value ·112. Taking the mean of the two values quoted for thorium as ·99 we obtain for the ratio of the molecular weights $\left(\frac{\cdot 112}{\cdot 99}\right)^2 = 1\cdot 28$.

It will be seen, on reference to Tables III. and IV., that the *mean value* of P × K does not show as much as 5 per cent. variation from any one of the values themselves, this would indicate that the number given for the ratio of the molecular weights, namely, 1·42 is not subject to as much as a 10 per cent. error.

The conclusion arrived at that actinium emanation is of considerably lower molecular weight than thorium (and therefore also than radium) emanation, is quite in agreement with much of the recent work which has been done, showing that actinium is not one of the products in the *direct* line of descent from uranium to radium §.

Summary.

1. The diffusion of actinium emanation in gases such as air, hydrogen, carbon dioxide, sulphur dioxide, and argon shows no considerable deviations from the ordinary laws of diffusion.

2. The variation with pressure of the coefficients of diffusion of the actinium and thorium emanations appears to be quite regular down to pressures of a few centimetres.

* 'Radioactivity,' p. 276.

† Makower, Phil. Mag. 1905, p. 56.

‡ Debierne, *Le Radium*, 1907, p. 213.

§ Rutherford, Phil. Mag. 1907, vol. xiv. p. 733; Boltwood, 'Nature,' 1907, p. 544.

3. A comparison of the diffusion coefficients of the actinium and thorium emanations in air under similar experimental conditions gives the ratio of their molecular weights as 1·42, that of thorium being the heavier.

In conclusion I have much pleasure in thanking Professor Rutherford for several suggestions during the course of the work, and Mr. J. N. Pring for assistance in preparing the argon.

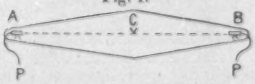
DISCUSSION.

Prof. C. H. LEES pointed out that the low values for the molecular weights of the emanations had all been obtained by allowing the emanations to diffuse into light gases. Prof. Pickering had allowed them to diffuse into mercury vapour, and obtained a value about 230.

XXXIII. *An Accurate Method of Measuring Moments of Inertia.* *By the late* Professor W. R. CASSIE *.

IF ACB (fig. 1) is the beam of a balance with moment of inertia I about C,

Fig. 1.



length of arm AC = BC = a,
coefficient of directive coupie L,
and scale pans, each of mass P, hung from the knife edges,

then the time of a small oscillation is

$$t_1 = 2\pi\sqrt{\frac{I + 2Pa^2}{L}}.$$

Here Pa^2 is a quantity of the nature of a moment of inertia involving only a mass and a length, quantities which can be accurately measured. Since any body suspended from the knife-edge does not rotate, the distribution of mass in it does not matter.

The method consists in taking as standard moment of

* Read November 27, 1908.

inertia a known mass hung from the knife-edge of a balance, and comparing others with it.

For instance, if t_2 were the period with a mass Q in each scale-pan,

$$Qa^2 = (I_L/8\pi^2)(t_2^2 - t_1^2).$$

Now if without altering L we could attach to the beam a bar of moment of inertia I′ (unknown), and the period with I′ attached to the beam, and the scale-pans only on the knife-edges, were t_3, then

$$I' = (I_L/8\pi^2)(t_3^2 - t_1^2),$$

so that

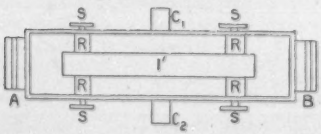
$$I' = Qa^2(t_2^2 - t_1^2)/(t_2^2 - t_1^2).$$

The value of I′ is thus obtained in terms of quantities which can be accurately measured.

There remains the practical problem of attaching the bar to the beam without altering L. This can be done as follows.

Make the beam in the form of a trough (for the plan see fig. 2) so that the bar I′ may be laid inside it. A and B

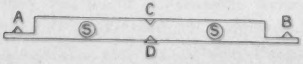
Fig. 2.—Plan of beam.



are the knife-edges for the scale-pans and C_1C_2 is the fulcrum, consisting of two portions projecting from the two sides of the trough.

Now we have to make the centre of gravity of I′ coincide with the knife-edge of the fulcrum C_1C_2. To this end

Fig. 3.—Elevation of beam.



ACB being in one plane (see fig. 3), make the centre of

gravity of the beam come on the knife-edge C (by adjusting a gravity bob), and apply the directive couple by hanging a known weight Wg from the knife-edge D parallel to but below C. Then with the weight Wg removed and the beam in neutral equilibrium, lay I′ in the trough and adjust its position until the beam is again in neutral equilibrium. Then the centre of gravity of I′ is also in the knife-edge C, and when Wg is replaced the directive couple L is unchanged.

This adjustment of the position of I′ must be made without moving the centre of gravity of the beam or changing I, the moment of inertia of the beam about C. This may be done as follows (see fig. 4):—

Let the bar I′ lie on two rests each consisting of two wedge-shaped pieces RR forming a V, and let these wedges be moved simultaneously and equally in opposite directions by the right- and left-hand screw-threads cut in the bar whose ends SS appear outside the trough of the beam. By this movement of RR the height of I′ may be altered until the equilibrium is neutral. Now these movements are entirely parallel to the fulcrum C_1C_2, and therefore do not alter either I or the position of the C.G. of the beam.

Fig. 4.
Section of beam.

Thus the process for measuring the moment of inertia of a bar would be as follows:—

1. Bring the balance with weight Wg removed to neutral equilibrium.
2. Hang on Wg and take the period, t_1, of a small oscillation.
3. Put equal weights Q in the scale-pans and take the period, t_2.
4. Remove the weights Q and Wg and put the bar I′ in the trough, and adjust again to neutral equilibrium by means of the screwheads SS.
5. Hang on Wg and take the period, t_2.
6. Measure a, the length of the arm,

then $$I' = 2Qa^2(t_2^2 - t_1^2)/(t_2^2 - t_s^2).$$

Discussion.

The Secretary read a letter from Prof. W. Stroud describing a method of determining moments of inertia in use in the University of Leeds. In this method two vibrators perpendicular to one another are used, and they can be arranged to produce a Lissajou figure. The horizontal vibrator carries a vertical slit and *vice versa*; and as the two slits are placed one behind the other, a small spot of light can be seen through the combination. The vertical vibrator consists of a stout metal framework supported by a torsion wire, and carrying a platform upon which the body whose moment of inertia is required can be placed. The time of swing of this vibrator is then adjusted (by the addition of rings of known moment of inertia) to be equal to that of the horizontal vibrator, so that the Lissajou figure is invariable. The body is then placed upon the platform, and inertia rings are removed till the Lissajou figure is reproduced or nearly so. Then, accurate observations are taken of the number of oscillations required to complete a cycle with inertia rings slightly below and slightly above the correct value, which is got by interpolation. The vibrators are kept in continuous oscillation by an electromagnetic make and break.

Prof. C. H. Lees said that Prof. Stroud's method did not seem capable of giving very accurate results on account of want of homogeneity of the rings.

Dr. A. Russell asked if it was necessary to take account of changes in barometric pressure in accurate work on moments of inertia.

The Chairman remarked that in delicate work attention must be paid to the effects of changes in barometric pressure.

XXXIV. *On the Lateral Deflexion and Vibration of "Clamped-Directed" Bars.* By John Morrow, *M.Sc.* (*Vict.*), *D. Eng.* (*L'pool*), *Lecturer in Engineering, Armstrong College* (*University of Durham*) *.

Section I.—*Introduction and Contents.*

§ 1. The vibrations of a bar under the terminal conditions which are probably most frequent in engineering practice, appear to have hitherto attracted but little attention. These conditions occur when one end of the bar is clamped, and the other is constrained to retain its original direction. Such conditions may be realized by having two initially parallel

* Read June 12, 1908.

bars, CA and DB (fig. 1), each clamped at one end, with the otherwise free ends connected by a rigid bar AB.

Figs. 1 and 2 show two distinct ways in which such a

Fig. 1.

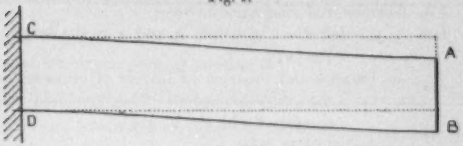
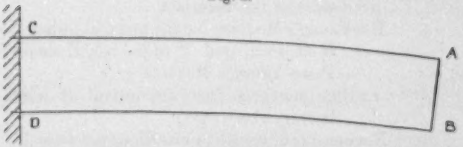


Fig. 2.



system may vibrate. In the former case we have pure lateral vibration, whereas in the latter there is a necessary accompaniment of longitudinal vibrations in each bar. The former case is the more important since in it the frequency of the fundamental type will, in general, be the lower.

I propose to refer to a bar under the end conditions of fig. 1 as a "Clamped-Directed" bar.

Although there exists a considerable literature on the theory of the lateral vibration of thin rods, the practical application of the results is extremely limited. The explanation of this is to be sought, not in any lack of enterprise on the part of the technologist, but rather in the fact that the problems that have been solved are not necessarily those of which the solutions are most required in practice.

§ 2. As an important example, in which the terminal conditions here considered occur, one might mention the case of the cylinder of a steam-engine supported on two or more mild steel standards. It will be recognized that the upper ends of the standards must be treated as "directed."

It is worthy of notice that these terminal conditions are mentioned by Lord Rayleigh in his 'Theory of Sound'*

* See Rayleigh's 'Sound,' vol. i. p. 259, 1894 edition.

but that the directed end is at once dismissed from consideration with the remark that "there are no experimental means by which the contemplated constraint could be realized."

§ 3. *Notation and End Conditions.*—

Let y, y_z = deflexions at points x and z in the length of the bar;

ω, I = area and moment of inertia of cross-section, the cross-dimensions being supposed small compared with the length;

l = length of bar;

ρ = density of the material;

E = Young's Modulus for the material (when there is an axial pull P in the bar, E stands for P/ω + Young's Modulus);

t = time, measured from any instant at which y is everywhere zero;

N = number of complete vibrations per second.

The symbol $\ddot{y}$ is written for d^2y/dt^2; and if y_1 is the instantaneous deflexion at some particular point in the length of the bar, we have, for simple harmonic vibrations,

$$-\ddot{y}/y = -\ddot{y}_1/y_1 = k^2 \text{ (say)}, \quad . \quad . \quad . \quad . \quad (1)$$

where

$$N = k/2\pi.$$

The curve assumed by the elastic central line at any instant is given in terms of y_1. If a is the amplitude at the point in the length at which y_1 is the instantaneous deflexion, the value of y_1 is given by

$$y_1 = a \sin kt. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The origin will always be taken at the clamped end. At $x=0$ we have, therefore, $y=dy/dx=0$; whilst at the directed end, $x=l$, $dy/dx=0$, and, if there be no concentrated load there, $d^3y/dx^3=0$.

Section II.—*Unloaded Massive Bar.*

§ 4. In the ordinary case of a bar vibrating under the effect of its own mass only, the form of the elastic central line and the frequency of the lateral vibrations are to be

determined from the well-known equation

$$\frac{d^4y}{dx^4} = \mu^4 y,$$

in which

$$\mu^4 = -\frac{\rho\omega^2}{EI}\frac{y_1}{y_1},$$

and of which the general solution is

$$y = A \sin \mu x + B \cos \mu x + C \sinh \mu x + D \cosh \mu x.$$

The end conditions give four equations whose compatibility requires that

$$\tan \mu l = -\tanh \mu l, \quad \ldots \quad (3)$$

of which the least root is

$$\mu l = 2{\cdot}36502 \ldots \quad \ldots \quad (4)$$

The frequency of the fundamental is therefore

$$N = \frac{5{\cdot}5933}{2\pi}\sqrt{\frac{EI}{\rho\omega l^4}}.$$

§ 5. It can be shown that the values of θ satisfying

$$\cos\theta \cosh\theta = 1, \quad \ldots \quad (5)$$

are the same as those which satisfy

$$\tan\frac{\theta}{2} = \pm \tanh\frac{\theta}{2},$$

and hence that the admissible roots of equation (3) are each one half of the alternate roots of (5). Accurate solutions of (5) have been provided by Rayleigh *, and from these the values of μl which satisfy (3) are

$$\mu_1 l = 2{\cdot}365020$$
$$\mu_2 l = 5{\cdot}497804$$
$$\mu_3 l = 8{\cdot}639380,$$

after which

$$\mu_n l = (n - \tfrac{1}{4})\pi,$$

to more than six decimal places, n being an integer.

§ 6. The values of μl given above for the lower harmonics may be readily calculated by the method explained below for $\mu_2 l$.

* 'Theory of Sound,' Rayleigh, vol. i. p. 278, 1894 edition.

We know that $\mu_2 l = (2 - \frac{1}{4})\pi$ approximately. Hence

$$\tanh \mu_2 l = \cdot 999967, \quad . \quad . \quad . \quad . \quad . \quad (5a)$$

to the limits of accuracy of the tables at my disposal. By equation (3) therefore

$$\tan \mu_2 l = -\cdot 999967 \text{ approximately.}$$
$$\therefore \ \mu_2 l = 2\pi - \cdot 7853815 = 5 \cdot 4978038,$$

which is more than sufficiently accurate*. Even when this method is used for the fundamental, the error is less than 0·01 per cent.

§ 7. To examine the curve assumed by the centre line of the bar, we obtain a further relation by putting $y = y_1$ when $x = l$; whence it can easily be shown that

$$y = \tfrac{1}{2} y_1 [(\sinh \mu l + \sin \mu l)(\sin \mu x - \sinh \mu x) - (\cosh \mu l - \cos \mu l)(\cos \mu x - \cosh \mu x)] \div (1 - \cos \mu l \cosh \mu l). \quad (6)$$

The values of μl to be used in this equation are those given in § 5.

§ 8. When dealing with harmonics the positions of the nodes are to be found from (6) by putting $y = 0$. We thus get, at each node,

$$\frac{\sin \mu x - \sinh \mu x}{\cos \mu x - \cosh \mu x} = \frac{\cosh \mu l - \cos \mu l}{\sinh \mu l + \sin \mu l}. \quad . \quad . \quad (7)$$

The fundamental is free from nodes. For the first harmonic the value of the right-hand side of (7) is 1·000033. For higher tones it may be taken as unity.

Solving (7) by trial, we find in the case of the first harmonic that the distance of the node from the clamped

* This method depends on the fact that a slight difference between two numbers of the magnitudes involved makes no appreciable difference in their hyperbolic tangents. If h is small

$$\tanh (x + h) - \tanh x = h(1 - \tanh^2 x)$$

approximately. In the example in the text the result shows that

$$\tanh \mu_2 l = \tanh (2 - \tfrac{1}{4})\pi$$

to at least eight decimal places. There can be no hesitation, therefore, in accepting equation (5 a). The method appears more natural than the normal one, which would be to put $\mu_2 l = (2 - \frac{1}{4})\pi + x$ in $\tan \mu_2 l$ and $\tanh \mu_2 l$ (where x is small), and to proceed by approximations.

end is given by

$$x = \cdot 7166\, l.$$

For higher harmonics, the ith tone has $i-1$ nodes; and of these the first node, that is the one nearest the clamped end, is at

$$\frac{x}{l} = \frac{5 \cdot 0175}{4i-1};$$

the second occurs at

$$\frac{x}{l} = \frac{8 \cdot 9993}{4i-1},$$

and beyond this the jth node is given with sufficient accuracy by

$$\frac{x}{l} = \frac{4j+1}{4i-1}.$$

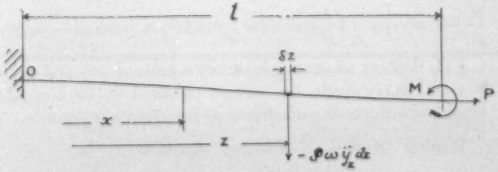
Section III.—*Unloaded Massive Bar, Longitudinal Tension.*

§ 9. When the bar is subjected to an axial tensile force P as shown in figure 3, the differential equation, based on the ordinary Bernoulli-Eulerian theory, may be written

$$\mathrm{EI}\frac{d^2y}{dx^2} + \rho\omega \int_x^l \ddot{y}_z\,(z-x)dz + \mathrm{P}(y_1 - y) + \mathrm{M} = 0. \quad . \quad (8)$$

in which M is the couple required to direct the end and y_1 is the deflexion there.

Fig. 3.



Differentiating twice with respect to x,

$$\mathrm{EI}\frac{d^4y}{dx^4} + \rho\omega\ddot{y} - \mathrm{P}\frac{d^2y}{dx^2} = 0.$$

which is the differential equation common to all unloaded bars of constant flexural rigidity. It may be written

$$\frac{d^4y}{dx^4} + \frac{\rho\omega}{\mathrm{EI}}\frac{y_1}{y_1}y - \frac{\mathrm{P}}{\mathrm{EI}}\frac{d^2y}{dx^2} = 0, \quad . \quad . \quad . \quad . \quad (9)$$

for the case of stationary simple harmonic vibrations. The general solution is then of the form

$$y = A\cosh \alpha x + B\sinh \alpha x + C\cos \beta x + D\sin \beta x,$$

in which, whatever the end conditions,

$$\left.\begin{aligned} -\frac{\ddot{y}_1}{y_1} &= \frac{\alpha^2}{\rho\omega}(EI\alpha^2 - P) \\ -\frac{\ddot{y}_1}{y_1} &= \frac{\beta^2}{\rho\omega}(EI\beta^2 + P) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (10)^*$$

The terminal conditions give four equations for which it is necessary that

$$\alpha \tanh \alpha l = -\beta \tan \beta l. \quad . \quad . \quad . \quad . \quad (11)$$

Elimination of α and β from the equations (10) and (11) would give an expression for $-\frac{\ddot{y}_1}{y_1}$ and enable the frequency to be calculated.

§ 10. The form of the centre line of the bar at any instant is then given by

$$y = y_1(\tanh \alpha l \sinh \alpha x - \cosh \alpha x - \frac{\alpha}{\beta}\tanh \alpha l \sin \beta x + \cos \beta x) \div \left(\frac{1}{\cos \beta l} - \frac{1}{\cosh \alpha l}\right),$$

whilst the couple M at the directed end can be obtained from

$$\frac{M}{y_1} + P = -\rho\omega\frac{\ddot{y}_1}{y_1}\left(\frac{1}{\beta^2}\cosh \alpha l + \frac{1}{\alpha^2}\cos \beta l\right) \div (\cosh \alpha l - \cos \beta l).$$

§ 11. Failing an exact solution of equations (10) and (11), a useful approximate one can be obtained by limiting the longitudinal force to such values as may occur in practice.

Writing $\phi = \alpha l$, $\theta = \beta l$, equation (11) is

$$\phi \tanh \phi = -\theta \tan \theta,$$

$$\therefore \quad \frac{d\phi}{d\theta} = -\frac{\theta + \tan \theta + \theta \tan^2 \theta}{\phi + \tanh \phi - \phi \tanh^2 \phi}.$$

Also (10) gives

$$\theta^2 = \phi^2 - \frac{Pl^2}{EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

* *Cf.* Rayleigh's 'Theory of Sound,' vol. i. 1894, p. 299.

and when P is zero (equation (4))

$$\phi = \theta = 2{\cdot}36502$$

for vibrations of the fundamental type.

Now, when Pl^2/EI is small, θ differs little from 2·36502, and we have, to the first order in this difference

$$\phi = \left[\phi\right]_{\theta=2{\cdot}365} + \left[\frac{d\phi}{d\theta}\right]_{\theta=2{\cdot}365} (\theta - 2{\cdot}365),$$

that is

$$\alpha l = 10{\cdot}5083 - 3{\cdot}4432\,\beta l. \quad . \quad . \quad . \quad . \quad (13)$$

Eliminating α from (12) and (13), we find, after an expansion by the Binomial as far as the second term

$$\beta^2 l^2 = 5{\cdot}5933 - {\cdot}2251\,\frac{Pl^2}{EI},$$

by means of which the second of (10) gives

$$k^2 = 31{\cdot}285\,\frac{EI}{\rho\omega l^4}\left\{1 + {\cdot}09831\,\frac{Pl^2}{EI} - {\cdot}00557\left(\frac{Pl^2}{EI}\right)^2\right\}. \quad (14)$$

§ 12. For harmonics we may in all cases take

$$\tanh\phi = 1;$$

thus, reasoning as in the last paragraph,

$$\alpha l = (1-\eta)\beta l + \tfrac{1}{2}\eta^2,$$

where

$$\eta \equiv (2i - \tfrac{1}{2})\pi,$$

whence

$$\beta^2 l^2 = \tfrac{1}{4}\eta^2 - \frac{1}{\eta}\,\frac{Pl^2}{EI}$$

and

$$k^2 = \left(\frac{\eta}{2}\right)^4 \frac{EI}{\rho\omega l^4} + \frac{\eta}{2}\left(\frac{\eta}{2} - 1\right)\frac{P}{\rho\omega l^2} - \frac{\eta - 1}{\eta^3}\,\frac{P^2}{\rho\omega EI} \quad . \quad (15)$$

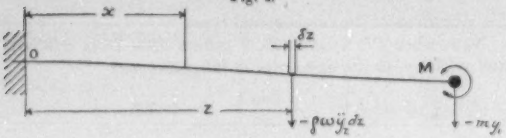
Section IV.—*Massive Bar, Load at Directed End.*

§ 13. When a massive clamped-directed bar carries a load concentrated at its directed end, the frequency and type of lateral vibration can be found to any required degree of accuracy by the method of continuous approximation *.

* See Phil. Mag., July 1905, pp. 113–125, and March 1906, pp. 354–374.

In figure 4, let the origin be at the clamped end and

Fig. 4.



assume for the first approximation to the vibration curve

$$y = y_1\{3(x/l)^2 - 2(x/l)^3\}, \quad . \quad . \quad . \quad . \quad (16)$$

which, at the clamped end, satisfies $y = \frac{dy}{dx} = 0$, and, at the directed end, $y = y_1$, $\frac{dy}{dx} = 0$.

Let m = the mass concentrated at the end,
M = the couple required to maintain the direction at that end.

Then the ordinary approximate theory leads to

$$EI\frac{d^2y}{dx^2} = -\rho\omega\frac{\ddot{y}_1}{y_1}\int_x^l y_z(z-x)dz - m\ddot{y}_1(l-x) + M. \quad (17)$$

Inserting the value of y_z from (16) and performing the integrations,

$$-EIy = \rho\omega\ddot{y}_1\left(\frac{13}{120}l^2x^2 - \frac{1}{12}lx^3 + \frac{1}{120}\frac{x^6}{l^2} - \frac{1}{420}\frac{x^7}{l^3}\right) + m\ddot{y}_1\left(\frac{lx^2}{4} - \frac{x^3}{6}\right), \quad . \quad (18)$$

the constants of integration and the value of M having been determined by the end conditions. This expression is the second approximation to the vibration type, and is often sufficiently accurate.

Putting $x = l$ we have

$$k^2 = \frac{EI}{\frac{13}{420}\rho\omega l^4 + \frac{1}{12}ml^3}. \quad . \quad . \quad . \quad . \quad (19)$$

§ 14. Proceeding to a still closer approximation we can insert in (17) the value of y_z given by (18) and again

integrate. In the resulting expression for y, we can again put $x=l$ and get

$$k^2 = \frac{\text{EI}}{\left(\begin{matrix}\cdot 3832\,\rho\omega l + m \\ \cdot 3714\,\rho\omega l + m\end{matrix}\right)\frac{13}{420}\rho\omega l^4 + \frac{ml^3}{12}} \quad . \quad . \quad . \quad (20)$$

The portion shown in brackets is the correction which this result gives to that first obtained. When the mass of the load is equal to that of the bar, (20) reduces to

$$k^2 = \frac{8{\cdot}730\,\text{EI}}{\rho\omega l^4}.$$

Section V.—*Loaded Bars; Mass of Bar Neglected.*

§ 15. When the mass of the bar is neglected and vibrations occur under the action of a concentrated load only, the exact solutions can be determined directly.

If there be no longitudinal force and the mass m be at the directed end, the equation of equilibrium is

$$\text{EI}\frac{d^2y}{dx^2} = -m\ddot{y}_1(l-x) - \text{M}.$$

$$\therefore \quad \text{EI}y = -m\ddot{y}_1(\tfrac{1}{4}lx^2 - \tfrac{1}{6}x^3),$$

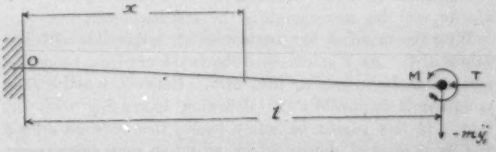
and

$$k^2 = \frac{12\,\text{EI}}{ml^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

§ 16. *Longitudinal Thrust.*—If in addition to the load m there be a longitudinal thrust T acting on the massless bar as represented in figure 5,

$$\text{EI}\frac{d^2y}{dx^2} = -m\ddot{y}_1(l-x) + \text{T}(y_1-y) - \text{M}.$$

Fig. 5.



By virtue of the conditions that at $x=0$, $y=\dfrac{dy}{dx}=0$,

and, at $x = l$, $\dfrac{dy}{dx} = 0$, the solution may be written

$$y = \frac{m\ddot{y}_1}{nT}\left\{\frac{\cos nl - 1}{\sin nl}(1 - \cos nx) + nx - \sin nx\right\}$$

in which $n^2 = T/EI$; and putting $x = l$,

$$k^2 = \frac{T}{ml}\,\frac{nl \sin nl}{2 - 2\cos nl - nl \sin nl}. \qquad . \qquad . \qquad (22)$$

This result might have been deduced from that for a bar clamped at each end with a load at the centre *.

Equation (22) shows that $k^2 = 0$ and vibration ceases when $nl = \pi$, that is, when

$$T = EI\,\pi^2/l^2.$$

After expansion, (22) becomes

$$k^2 = \frac{12EI}{ml^3}\left(1 - \frac{4}{15}n^2l^2 + \ldots.\right),$$

agreeing with (21) when T, and therefore n, is zero.

When $nl = 2i\pi$ (i being an integer) the right-hand side of (22) is indeterminate but has the limit $-T/ml$, vibration being impossible if T is positive.

§ 17. The result may also be written

$$k^2 = \frac{T}{ml}\,\frac{\frac{1}{2}nl}{\tan\frac{1}{2}nl - \frac{1}{2}nl},$$

from which (nl being positive) the expression for the frequency is real when $\tan\frac{1}{2}nl > \frac{1}{2}nl$, that is, for values of nl between 0 and π, and for decreasing intervals in the neighbourhood of 3π, 5π, &c.; vibration always ceasing when $nl = (2i-1)\pi$.

The first of these intervals is from $nl = 8{\cdot}9868$ ... to $nl = 3\pi$, and this corresponds to the first harmonic.

With the massless bar harmonics are impossible with low values of T. As T is increased the frequency falls, becoming zero when $nl = \pi$, $\ddot{y}$ being then zero. Between π and $8{\cdot}9868$ vibration is impossible, the deflexion increasing with the time. If this region be safely passed that between $8{\cdot}9868$ and 3π is reached, during which vibration may again occur,

* Phil. Mag. September 1906, p. 243.

T being now sufficient to enable the bar to assume the curve of the first harmonic type.

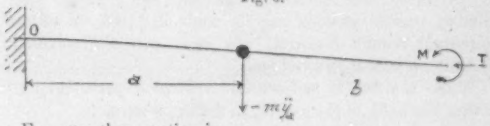
§ 18. If the mass be situated at any point in the length, let it divide the bar into segments a and b, as indicated in fig. 6. Then for $x<a$ the differential equation becomes

$$EI\frac{d^2y}{dx^2} = -m\ddot{y}_a(a-x) + T(y_1-y) - M,$$

of which the solution is

$$y = A_1 \sin nx + B_1 \cos nx + \frac{m\ddot{y}_a}{T}(x-a) + y_1 - \frac{M}{T}.$$

Fig. 6.



For $x>a$ the equation is

$$EI\frac{d^2y'}{dx^2} = T(y_1-y') - M,$$

and the solution

$$y' = A_2 \sin nx + B_2 \cos nx + y_1 - \frac{M}{T},$$

where the dashed symbols refer to values of y in the region b.

The conditions at $x=0$ and $x=l$ lead respectively to

$$\left.\begin{aligned} y &= -\frac{c}{n^3}\sin nx + \left(\frac{m\ddot{y}_a}{T}a + \frac{M}{T} - y_1\right)\left(\cos nx - 1\right) + \frac{m\ddot{y}_a}{T}x \\ y' &= \frac{M}{T}\left(\sin nl \sin nx + \cos nl \cos nx\right) + y_1 - \frac{M}{T} \end{aligned}\right\} \quad (23)$$

The conditions of continuity of y and $\frac{dy}{dx}$ when $x=a$ give an expression for the couple at the directed end, namely

$$M = m\ddot{y}_a \frac{\cos na - 1}{n \sin nl}.$$

Putting $x=a$ in the expressions for y and y' we find

$$k^{-2} = \frac{m}{nT}\left\{\sin na(2-\cos na) - \cot nl(1-\cos na)^2 - na\right\}. \quad (24)$$

When $a=l$, this reduces to the result of § 16.

It can be shown, from equations (23), that the amplitude is a maximum at the directed end.

Section VI.—*Deduction of the Results of some Statical Problems.*

§ 19. The calculations in Section V. are similar to those required for the corresponding statical problems. Thus if a clamped-directed bar be subjected to a force W at and perpendicular to the directed end, the centre-line assumes the form (see § 15)

$$y = \frac{W}{EI}\left(\frac{1}{4}lx^2 - \frac{1}{6}x^3\right).$$

Simi'ar transformations may be made in §§ 16 and 18, and important results obtained. In each case the maximum deflexion is at the directed end.

When the bar is subjected to a force w per unit length acting normally to the x-axis, its deflexion curve is

$$y = \frac{w}{EI}\left(\frac{1}{6}l^2x^2 - \frac{1}{6}lx^3 + \frac{1}{24}x^4\right).$$

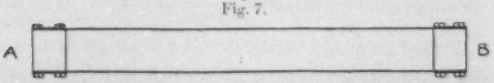
We notice that, for a force concentrated at the end, the maximum deflexion of a clamped-directed bar is one-quarter of that for a clamped-free bar; whilst when the force is uniformly distributed the ratio is one-third.

Section VII.—*Some Experimental Results.*

§ 20. The experiments described in this section were made merely to compare, in a few cases, the frequency calculated by the methods given in this paper with that obtained by direct observation. Care was taken that the accuracy should reach certain limits.

The composite beams used for the experiments consisted of two flat wrought-iron bars, each 3·00 cms. broad by 0·315 cm. thick, the latter dimension being in the plane of

Fig. 7.



vibration. The bars were rigidly connected together at their ends by being securely screwed to massive cast-iron blocks,

as shown in fig. 7. The effective length was measured between the blocks, and this, as well as the distance between the bars, was varied.

A striking feature of the apparatus is the certainty with which the end conditions can be relied upon. When this is the case, the distance between the bars is immaterial; some of the experiments may, in this respect, be taken as a verification of the absolute rigidity of the ends.

§ 21. The flexural rigidity of a piece of one of the bars was determined by supporting it on knife-edges at each end of a span of 58·42 cms. (23 inches), and applying loads at the centre. The observed deflexions are given in the following table.

Load in lbs.	Deflexion. Inches.	Difference. Inches.
0	·627	·047
1	·580	·048
2	·532	·049
3	·483	·050
4	·433	·050
3	·483	·048
2	·531	·049
1	·580	·049
0	·629	
Average difference per pound		·0488

The average deflexion at the centre was thus found to be 0·0488 inch per pound, corresponding to a flexural rigidity of $15{\cdot}20 \times 10^6$ grammes weight-cms.^{2} The value of E here involved is the static modulus, but the difference between it and the kinetic modulus would be too small to affect the calculations.

§ 22. The results of the experiments on the compound bars are given in the following table. In each case vibrations took place in a horizontal plane, that is, in the plane of fig. 7.

The clamped end was very securely bolted to a large lathe-bed in such a manner as to leave no room for doubt as to the applicability of the terminal conditions at that end.

No. of Experiment.	Effective Length. cms.	½ Weight at Directed end. gms.	Distance apart of bars. cms.	Observed Frequency.	Calculated Frequency.
1	114·3	1604	16·5	1·27	1·26
2	114·3	780	16·5	1·66	1·67
3	91·5	1604	16·5	1·78	1·78
4	91·5	780	16·5	2·40	2·40
5	114·3	1004	6·0	1·26	1·26
6	114·3	802	6·0	1·65	1·65

The last column is calculated from equation (20); the values are, however, practically identical with those given by (19). It will be seen that the agreement between the observed and calculated frequencies is very satisfactory.

I should like to take this opportunity of making a slight correction in my paper "On the Lateral Vibration of Bars subjected to Forces in the Direction of their Axes." On page 236 of the Philosophical Magazine for September 1906, and on page 227 of the Proceedings of the Physical Society, vol. xx., instead of $\tanh\phi$ I have used its reciprocal. The error makes very little difference to the final equation, the correct expression being

$$-\frac{\ddot{y}_1}{y_1} = 500{\cdot}56\,\frac{EI}{\rho\omega l^4} + 12{\cdot}91\,\frac{P}{\rho\omega l^2} - {\cdot}167\,\frac{P^2}{\rho\omega EI}.$$

Armstrong College,
January 1908.

XXXV. *An Experimental Investigation of Gibbs's Theory of Surface-Concentration, regarded as the basis of Adsorption.* (Second Paper.)

By Wm. C. McC. Lewis, *M.A.**

[From the Muspratt Laboratory of Physical and Electrochemistry, University of Liverpool.]

In a previous paper† upon this subject a number of determinations were recorded, which had been carried out with a view to investigate experimentally the expression deduced by Gibbs for the surface condensation of one of the components of a two-phase system at the boundary separating the phases. Two immiscible liquids—namely a hydrocarbon oil and distilled water—were brought into contact, there being a definite interfacial tension existing at the boundary of separation. To the water there were added various substances soluble in water, but insoluble in oil, viz. sodium glycocholate, methyl-orange, and Congo-red. These substances all produced a lowering effect on the interfacial tension—such lowering effect increasing with corresponding increase in concentration of the solute.

Gibbs's expression connecting the surface-concentrating of the solute with the lowering effect on the tension and the concentration of the solute in the bulk of the (aqueous) phase is

$$\Gamma = -\frac{c}{RT}\frac{d\sigma}{dc}$$

where Γ = the mass (in grams) of the solute per square centimetre of the dividing surface in excess of what would be there supposing the concentration to remain the same as in the bulk of the solution;

c = the bulk concentration of the solute in the aqueous phase;

R = the gas constant;

T = the absolute temperature;

σ = the interfacial tension; and

$\frac{d\sigma}{dc}$ = the rate of change of the tension with the concentration of the solute.

* Read November 27, 1908.

† Lewis, Phil. Mag. April 1908; also Proc. Phys. Soc. vol. xxi. p. 150 (1908).

The negative sign on the right-hand side indicates that $\frac{d\sigma}{dc}$ must be negative in order that Γ may be positive, *i. e.* those substances which lower the interfacial tension will increase in concentration at the interface.

It was shown how this expression might be applied to the phenomenon of adsorption, which is considered to be more or less a surface condensation. The determinations referred to consisted in measurements of the various quantities occurring in the above expression, and in comparing the values obtained on the two sides of the equation. The general results showed that the actually determined values for Γ far exceeded the calculated values—those found being of the order 10^{-6} grms./cm.^{2}, while the calculated ranged from 10^{-7} to 10^{-8} grms./cm.^{2} This discrepancy was observed in the case of all the substances mentioned in the previous paper. The adsorption coefficient (Γ) was measured experimentally under two quite different conditions—namely, at an approximately *plane* surface; and secondly at a very curved surface, with concordant results in both cases.

With the view of throwing some light on the cause of this discrepancy, further determinations of a similar character were carried out and are recorded in the present paper.

It may be pointed out that the substances mentioned are electrolytes (sodium salts of organic acids) and would therefore suffer considerable dissociation in aqueous solution. To account for the discrepancy referred to one possibility suggested itself, namely, the presence of some kind of "double layer" electrical effect. Of course this may or may not be the cause, and it will be shown that for these bodies this is probably *not* the cause, but evidently the first step was to examine other electrolytes—ordinary inorganic salts of simple constitution and comparatively low molecular weight, and compare the effects obtained in these cases with that obtained for sodium glycocholate and the dyestuffs.

Finally, to carry out adsorption determinations if possible with typical non-electrolytes and compare the results obtained in this case also.

The Interfacial Tension between Oil and Water measured by the Drop-pipette.

The pipette was identical with that shown in fig. 1 (previous paper). An expression giving the tension directly in absolute units may be derived as follows :—

Consider the equilibrium of a drop of oil formed in water and just about to break away from the nozzle of the pipette.

Let σ = the interfacial tension ;
a = the radius of the orifice ;
r = the mean radius of the drop ;
ρ_w = the density of water ;
ρ_0 = the density of oil.

The forces tending to draw the drop off are (1) gravitational, due to the difference in densities of the oil and water

$$= \frac{4}{3}\pi r^3(\rho_w - \rho_0),$$

and (2) the difference of pressure due to curvature,

$$= 2\pi a^2 \frac{\sigma}{r}.$$

The opposing force is due to the tension and $= 2\pi a\sigma$.

Hence

$$2\pi a\sigma - 2\pi a^2 \frac{\sigma}{r} = \frac{4}{3}\pi r^3(\rho_w - \rho_0).$$

Applying this to the particular case of the oil in which

$\rho_w = 1{\cdot}0$,
$\rho_0 = 0{\cdot}899$,
$a = 0{\cdot}175$ cm.,
$r = 0{\cdot}4125$ cm.,

we obtain $\sigma = 0{\cdot}0467$ gram per cm.

or 45·8 dynes per cm.

Employing the empiric formula of Lohnstein * and Kohlrausch † the value 40 dynes per cm. is obtained.

The above expression had been already obtained when it was found that an almost identical one had been deduced by

* Lohnstein. See previous paper.
† Kohlrausch. See previous paper.

G. Guglielmo * for liquid-air surfaces. He has applied it to the case of the water-air tension with the following results :—

Radius of orifice of pipette.	Tension in milligrams/mm.	
0·8 m'm.	7·78	mean 7·69 at 25° C.
1·1 „	7·60	
2·15 „	7·69	

The value obtained by the capillary-tube method is 7·485 at 25° C. The results are therefore in fair agreement.

As in previous determinations, the drop-pipette was employed for the various cases of adsorption dealt with in the present paper, the only difference being that the absolute values for the tension are here calculated by means of the expression deduced above.

The Interfacial Tension between a Hydrocarbon Oil and Aqueous Solution of Electrolyte.

The oil employed was similar to that previously experimented with. About twenty electrolytes in all were examined and all showed a *lowering* effect on the oil-water tension—in general monovalent ions of small atomic weight having least effect while copper and barium salts had a marked influence. This lowering of the tension is remarkable in view of the fact that inorganic salts (with the exception of lithium) *raise* the air-water tension †. Further, it is a general property of salts to be adsorbed by charcoal ; and if this is regarded as indicative of a lowering of tension at the charcoal surface, one may be perhaps justified in saying that for aqueous solutions of an electrolyte at a liquid or solid ‡ interface, one may expect in all cases a lowering of tension accompanied by adsorption, while on the other hand at the solution-air surface no such generalization can be made.

A certain number of these salts were selected for adsorption determinations—in the case of caustic soda the emulsion

* G. Guglielmo, *Accad. Lincei*, Atti, xv. p. 287 (1906).

† Whatmough, *Zeitschr. Phys. Chem.* vol. xxxix. p. 129 (1901).

‡ Using the word in its ordinary sense.

method * being employed; while with the rest the large-dropping apparatus was used—this being rendered necessary as the adsorption determinations were carried out for each *ion* of the salt separately, the concentration changes being estimated by ordinary volumetric and gravimetric means.

The Adsorption of Caustic Soda on Hydrocarbon Oil.

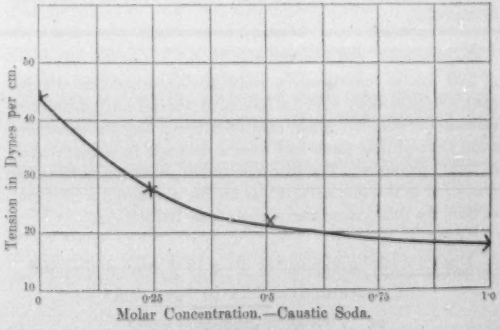
The values of the interfacial tension for different concentrations of the base (corrected for density) are summarized in the following table.

TABLE I.

Concentration		Interfacial Tension in dynes/cm.
per cent.	in gram-moles/litre.	
0	0	45·81
1·0	0·25	26·93
2·0	0·5	22·49
4·0	1·0	16·76
8·0	2·0	streaming from orifice of pipette – tension too small to be measured.

The values of the tension and concentration are plotted in fig. 1.

Fig. 1.



* See former paper.

Calculation of Γ *by means of Gibbs's expression* :—

Take as a particular case the concentration 0·3 per cent.

$$c = 0{\cdot}003 \text{ grm./cc.}$$

$$R = \frac{2 \times 4{\cdot}2 \times 10^7 \text{ ergs}}{\text{molecular weight}} = \frac{2 \times 4{\cdot}2 \times 10^7}{20},$$

assuming complete dissociation at this dilution.

$$T = 288° \text{ abs.}$$

$\frac{d\sigma}{dc}$, i. e. the tangent to the curve (fig. 1) at the concentration 0·3 per cent.,

$$= -\frac{9 \text{ dynes/cm.}}{0{\cdot}003 \text{ grm./c.c.}} = 3000.$$

Hence

$$-\frac{c}{RT}\frac{d\sigma}{dc} = \underline{7{\cdot}5 \times 10^{-9} \text{ grams per cm.}^2}$$

$$\text{or } \underline{1{\cdot}9 \times 10^{-10} \text{ gram-moles per cm.}^2}$$

It has been assumed here in using the term gram-moles that the substance has been adsorbed either as undissociated base or in ionically equivalent proportions. Gibbs's expression for surface concentration does not contain any term allowing for the preferential adsorption of a single ion if any should occur. Also the measurement of concentration changes by means of changes in tension embodies the same assumption.

Experimental Determination of Γ :—

500 c.c. of caustic-soda solution (concentration 0·3 per cent.) were shaken with 1·075 c.c. of the oil until a uniform emulsion resulted. After emulsification a tension determination showed that there had been a decrease in concentration of 0·003 per cent. The diameter of the emulsion particles as measured in the microscope was on the average 3×10^{-5} cm., so that the total adsorbing surface was 107500 cm.^{2}

Hence

$$\underline{\Gamma = 1{\cdot}5 \times 10^{-7}} \text{ grm./cm.}^2 \text{ or } 3{\cdot}7 \times 10^{-9} \text{ gram-moles/cm.}^2$$

$$\Gamma \text{ (calculated)} = \underline{7{\cdot}5 \times 10^{-9}} \text{ grm./cm.}^2$$

$$\text{or } 1{\cdot}9 \times 10^{-10} \text{ gram-moles/cm.}^2$$

The probable error in the experimental result may amount to 25 per cent. A comparison of the observed with calculated values shows once more the considerable discrepancy which exists between them. It should be observed, however, that the absolute value of the experimental determination has fallen from the order 10^{-6} (obtained in previous cases) to 10^{-7} grm./cm.^{2} It should also be mentioned that if any slight trace of acid were present in the oil the adsorption of a base would be considerably altered.

Re-determination of the Adsorption of Sodium Glycocholate.

Having obtained a purer specimen of the material than that used in former cases, the values for the tension between the oil and solutions of different concentrations were again determined. There were only small changes noted in the result —the calculated value for the adsorption (at a concentration 0·25 per cent.) being 7×10^{-8} grm./cm.^{2}, while in the former case the value was about 5×10^{-8}. The actual value determined by experiment was from $3{\cdot}5$ to $4{\cdot}7 \times 10^{-6}$ grm./cm.^{2} while in the previous case it had lain between the limits $3{\cdot}1$ and $5{\cdot}4$ grm./cm.^{2}

The Adsorption of Silver Nitrate.

In the measurements of the adsorption of this substance (and in all subsequent cases) the "large drop" * apparatus was employed. One litre of $\frac{1}{50}$ molar silver nitrate was prepared and about 750 c.c. poured into the tube through which the oil passed in the form of drops (of about 0·3 cm. diameter), and on the surface of these the solute was adsorbed. When the oil-reservoir (of 1 litre capacity) had been exhausted and the total number of drops estimated from the time taken in emptying the reservoir, the solution which had been thus treated was withdrawn and analysed. For this particular substance the silver only was estimated before and after the experiment by precipitation with hydrochloric acid, the Gooch crucible method being employed. An extremely small difference in weight was observed,

* See previous paper.

namely 1·6 milligrams of silver chloride. The total surface area of the oil was 7000 cm.^{2}

Hence the adsorption of the silver is

$$\Gamma_{Ag} = \underline{1{\cdot}7 \times 10^{-8} \text{ grm./cm.}^2}$$

A repetition of the determination gave

$$\Gamma_{Ag} = 3{\cdot}4 \times 10^{-8} \text{ grm./cm.}^2$$

The mean value is therefore

$$\Gamma_{Ag} = \underline{2{\cdot}5 \times 10^{-8} \text{ grm./cm.}^2}$$

It may be mentioned that the measuring vessels used in these and subsequent determinations were recalibrated for 15° C., and all solutions were cooled to this temperature in a bath before measuring out the volume.

It will be noted that the value obtained above is much smaller than for the previously mentioned substances (except caustic soda). One might therefore expect a closer approximation to the value calculated on Gibbs's expression.

Calculation of the Adsorption of Silver Nitrate.

As in previous cases the oil-solution tension curve was obtained by means of the drop-pipette. The oil employed for these measurements was not quite identical with that used with caustic soda. The values obtained (corrected for density) are given in the following table.

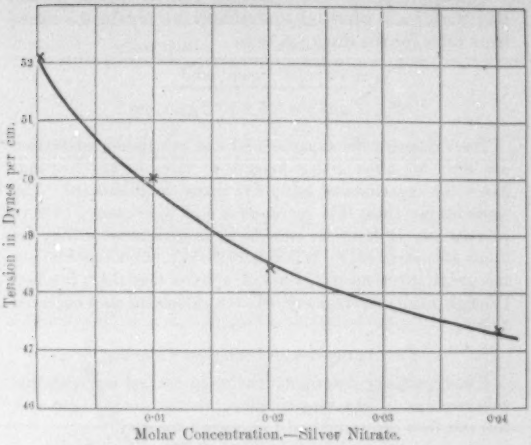
TABLE II.

Oil—Silver-Nitrate Solutions.

Concentration. in gram-moles per litre.	Pipette Drop-number.	Tension dynes cm.
0	135	52
0·01	140	50
0·02	145	48·4
0·04	149	47·3

The values of the concentration are plotted against tension in fig. 2.

Fig. 2.



The calculated adsorption, namely

$$\frac{c}{RT}\frac{d\sigma}{dc},$$

has the following value for a concentration $\frac{1}{50}$ molar :—

$$c = 0{\cdot}0034 \text{ grm. per c.c.}$$

$$R = \frac{8{\cdot}4 \times 10^7 \text{ ergs}}{\text{molecular weight}}.$$

The molecular weight at this dilution has been found by Smits * to be 94. $T = 289°$ abs.

$\frac{d\sigma}{dc}$ is the tangent to the curve at $\frac{1}{50}$ molar concentration and is equal to

$$\frac{1{\cdot}9 \text{ dyne/cm.}}{0{\cdot}0034 \text{ grm./c.c.}} = 556.$$

Hence

$$\frac{c}{RT}\frac{d\sigma}{dc} = 7{\cdot}3 \times 10^{-3} \text{ gram per cm.}^2$$

* Smits, *Zeitschr. Phys. Chem.* vol. xxxix. p. 418 (1902).

This refers to the salt $AgNO_3$, so that on the assumption that each ion is adsorbed equivalently we obtain the calculated value for the silver ion to be

$$\Gamma_{Ag} = \underline{4{\cdot}5 \times 10^{-6} \text{ grm./cm.}^2}$$

$$\Gamma_{Ag} \text{ (found)} = \underline{2{\cdot}5 \times 10^{-5} \text{ grm./cm.}^2}$$

The values for the experimental and calculated adsorption are thus in much closer agreement than in any previous case—the experimental being five times the calculated. The question is: Does this represent a real discrepancy even in this case, or is it to be considered as agreement within the limits of experiment? It is difficult to say, but without laying too great stress upon it, I am of opinion that there *is* a real (though small) discrepancy between calculated and observed values.

The Adsorption of Potassium Chloride.

Exactly similar determinations were carried out with this substance as in the case of silver nitrate, except that with this salt both the metal and acid were estimated.

The Adsorption of the Potassium of Potassium Chloride.—A very large quantity of oil was allowed to pass through the KCl solution in the large-drop apparatus. The resulting oil-surface area was 16,000 cm.^{2} The quantity of potassium present in the solution before and after adsorption was estimated as potassium sulphate. The difference in weight of the K_2SO_4 precipitate was 3·2 milligrams, whence the adsorption of the potassium is

$$\Gamma_K = 8 \times 10^{-6} \text{ grm./cm.}^2$$

A repetition of the experiment gave

$$\Gamma_K = 2 \times 10^{-5} \text{ grm./cm.}^2$$

Thus the mean value is

$$\Gamma_K = \underline{5 \times 10^{-6} \text{ grm./cm.}^2}$$

Calculation of Γ_K *on Gibbs's formula*:—

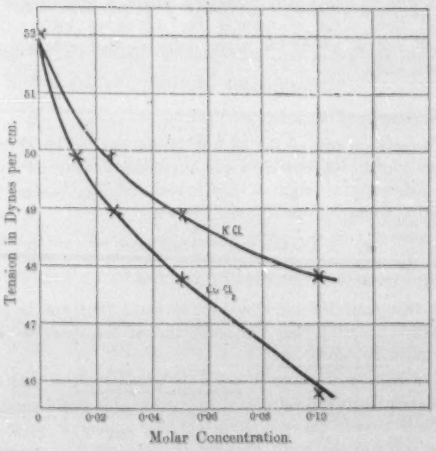
The values of the tension corresponding to different concentration (corrected for density) are given as follows:—

TABLE III.—Hydrocarbon Oil—Potassium Chloride.

Concentration in gram-moles per litre.	Pipette Drop-number.	Tension dynes/cm.
0	135	52
0·0125	139	50·4
0·025	140	49·9
0·05	143	48·9
0·1	147	47·8

The tension-concentration curve is given in fig. 3.

Fig. 3.



As before, the calculated adsorption for $\frac{M}{20}$ KCl is

$$c = 0{\cdot}0037 \text{ grm./c.c.}$$

$$R = \frac{8{\cdot}4 \times 10^7 \text{ ergs.}}{37}$$

where 37 is the osmotic molecular weight.

$$T = 289° \text{ abs.}$$

$$\frac{d\sigma}{dc} = 478.$$

Hence

$$\frac{c}{RT}\frac{d\sigma}{dc} = \underline{2{\cdot}3 \times 10^{-9}} \text{ gram KCl per cm.}^2 \text{ surface,}$$

which corresponds to $\underline{1{\cdot}7 \times 10^{-9}}$ grm. of potassium,

while Γ_x (found) is $\underline{5 \times 10^{-8}}$ grm./cm.^{2}

It will be noted that potassium shows a greater discrepancy than silver as regards the calculated value. The results are of about the same order as that obtained for caustic soda.

Estimation of the Adsorbed Chlorine :—

The surface area of the oil was 10,000 cm.^{2} The change in the chlorine concentration was determined as silver chloride, the difference in weight of AgCl precipitate due to adsorbed chlorine being 0·35 milligram. Hence

$$\Gamma_{Cl} = 8 \times 10^{-9} \text{ grm./cm.}^2$$

Repetition gave $\Gamma_{Cl} = 3 \times 10^{-9}$ grm./cm.^{2}

These quantities are, however, so small that one is only justified in stating that the adsorption of the chlorine is of the order $\underline{10^{-9}}$ grm./cm.^{2}

The calculated value is $\underline{1{\cdot}6 \times 10^{-9}}$ grm./cm.^{2}, so that the agreement between calculated and observed is fairly good.

These results for potassium chloride appear to point (at least as far as the small changes in concentration observed can be trusted) to the possibility of a certain amount of separation of the ions in the process of adsorption, *i. e.* a preferential cation adsorption. If this is so, it at once suggests

some electrical effect taking place which is not taken account of in Gibbs's capillary theory.

The Adsorption of Barium Chloride.

In the case of this substance the chlorine only was estimated, this being carried out volumetrically with $AgNO_3$ solution. The surface area of the oil was 4000 cm.^{2}. In the titration the change in concentration of chlorine due to adsorption was less than 0·1 c.c. $AgNO_3$ solution. From this one can only say, therefore, that the adsorption Γ_{Cl} is not of greater magnitude than 10^{-8} grm./cm.^{2}

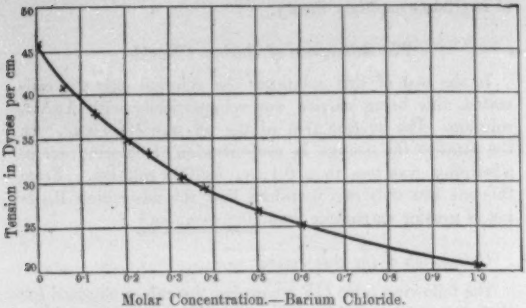
Calculation of the Adsorption :—

The following table (IV.) contains the values obtained for the tension (corrected for density) at different concentrations. The values of the tension are plotted against concentration in fig. 4. By taking the tangent to this curve the value of $\frac{d\sigma}{dc}$ is obtained for a particular concentration.

TABLE IV.

Concentration of the Barium Chloride		Density of the Solutions.	Drop-number.	Relative Tension.	Tension in dynes/cm.
Gram-moles per litre.	Per cent. anhydrous.				
0	0	1·000	153	1·000	45·81
0·062	1·29	1·009	171	0·895	40·99
0·125	2·59	1·019	188	0·829	37·98
0·2	4·14	1·038	210	0·757	34·68
0·25	5·18	1·048	222	0·722	33·07
0·33	6·90	1·066	246	0·663	30·37
0·375	7·76	1·073	255	0·644	29·50
0·5	10·35	1·099	290	0·580	26·57
1·0	20·70	1·216	419	0·444	20·34
(1·7) satd. soln.	34·00	1·289	585	0·338	15·48

Fig. 4.



Molar Concentration.—Barium Chloride.

For a solution of strength $\frac{M}{50}$ we have

$c = 0{\cdot}0041$ grm./cc.

$R = \frac{8{\cdot}4 \times 10^7}{75}$,

where 75 = osmotic molar weight,

$T = 289°$ abs.

$\frac{d\sigma}{dc} = 427.$

Hence

$$\frac{c}{RT}\frac{d\sigma}{dc} = \underline{5{\cdot}4 \times 10^{-9} \text{ gram } BaCl_2 \text{ per cm.}^2},$$

or $\underline{1{\cdot}6 \times 10^{-9}}$ grm. chlorine,

Γ_{Cl} (found) $> 10^{-8}$ grm./cm.^{2}

The result obtained here is of too qualitative a nature to show whether there is agreement or not.

Adsorption of Copper Chloride.

With this substance both copper and chlorine were separately estimated.

The Adsorption of the Copper :—

The oil-surface area was 8000 cm.^{2} The copper in the solution was estimated electrolytically, both before and after the adsorption. The change in weight of the electrode was only 0·2 milligram.

Hence

$$\Gamma_{Cu} = 2{\cdot}5 \times 10^{-8} \text{ grm./cm.}^2$$

A repetition of the determination gave

$$\Gamma_{Cu} = 4{\cdot}5 \times 10^{-8},$$

or mean value

$$\underline{\Gamma_{Cu} = 3{\cdot}5 \times 10^{-8} \text{ gram per cm.}^2}$$

Estimation of the Chlorine :—

As in previous cases, this was estimated as silver chloride. The difference in weight of the AgCl precipitate was 6 milligrams. The surface-area = 8000 cm.^{2}, and hence

$$\Gamma_{Cl} = 1{\cdot}8 \times 10^{-8} \text{ grm./cm.}^2$$

A repetition of the determination gave

$$\Gamma_{Cl} = 2{\cdot}3 \times 10^{-8};$$

so that the mean value is

$$\underline{\Gamma_{Cl} = 2 \times 10^{-8} \text{ grm./cm.}^2}$$

Hence

$$\underline{\Gamma_{Cu} + \Gamma_{Cl} = 5{\cdot}5 \times 10^{-8} \text{ grm./cm.}^2}$$

Calculation of the Adsorption of $CuCl_2$:—

The following table gives the value of the tension (corrected for density of the solution) at different concentrations :—

Hydrocarbon-Oil and $CuCl_2$.

Concentration in gram-moles per litre.	Pipette Drop-number.	Tension in dynes/cm.
0	135	52
0·0125	140	49·9
0·025	143	48·9
0·05	147	47·8
0·1	154	45·8

These values are plotted in fig. 3.

For a solution of concentration M/50

$$c = 0{\cdot}0027 \text{ grm./cm.}^2,$$

$$\frac{d\sigma}{dc} = 518,$$

$$R = \frac{8{\cdot}4 \times 10^7}{51}.$$

Where 51 is the molecular weight at this concentration as determined by Biltz *.

$$T = 289 \text{ abs.}$$

Hence

$$\frac{c}{RT}\frac{d\sigma}{dc} = 4{\cdot}0 \times 10^{-9} \text{ gram } CuCl_2 \text{ per cm.}^2,$$

$$\Gamma_{Cu} + \Gamma_{Cl} \text{ (found)} = 5{\cdot}5 \times 10^{-8} \text{ grm./cm.}^2$$

We have here once more inequality between observed and calculated values. It should be noted that it is the experimental value of Γ which is always the greater, and there are indications that the cation is adsorbed in somewhat greater quantity than the chemical equivalent of the anion.

Further determinations with other electrolytes must be undertaken before one could definitely state whether this important fact—the separation of the ions—takes place or not.

Before going on to discuss in more detail, however, the data already given, the experiments carried out with non-electrolytes will first be briefly recorded.

The Adsorption of Non-Electrolytes.

The chief difficulty at the outset with regard to almost all non-electrolytes is their small solubility in water. It is not easy, therefore, to draw a comparison between the tension effects produced by these bodies and the effects produced by electrolytes. Altogether seventeen non-electrolytes were examined as regards their influence on the interfacial tension. Two of these—aniline and p-nitrosodimethylaniline—were

* Biltz, *Zeit. Phys. Chem.* vol. xl. p. 199 (1901).

found to be soluble in the oil, and to this is probably due their comparatively marked effects on the tension *.

The following table (p. 532) contains the values obtained from the various substances examined.

The value obtained for cane-sugar is worthy of note. The lowering effect is small, although a more concentrated solution was employed than in the case of any other substance. The result is in agreement with that observed at the air-surface, in which case the presence of cane-sugar has no appreciable effect.

In the case of all these substances (except iodine which was practically without effect) there is a lowering of tension at the interface which on Gibbs's theory must be accompanied by adsorption. The same general rule, therefore, as regards the effect at the oil-surface seems to hold for non-electrolytes as for electrolytes.

The Anomalous Behaviour of Saponin.

Saponin is a non-electrolyte whose aqueous solutions are characterized by foaming, thus pointing to considerable

* *The Tension between two partially miscible phases.*

The Laplace theory of capillarity shows that the work required to form unit area of interface between two liquids A and B is given by the expression

$$\tfrac{1}{2}(\rho_A-\rho_B)^2\Big]_0^\infty z\psi(z)dz=\sigma_{AB},$$

where ρ_A and ρ_B = the densities of the liquids A and B respectively and the integral is supposed constant for all substances. This expression leads to the following relationship:—

$$\sqrt{\sigma_{AB}}=\sqrt{\sigma_{BC}}+\sqrt{\sigma_{CA}},$$

which has not been verified by experiment, but shows in a general way what takes place at the surface. Lord Rayleigh has on this hypothesis calculated the effect of making the transition gradual between A and B by the interposition of n liquids whose densities are in arithmetical series, in which case it is shown that the energy due to surface-tension is reduced to $(n+1)$ of its original value. Hence any diminution in the abruptness will diminish the energy due to surface-tension. The solubility of the aniline in the oil causes the diminution in abruptness, and consequently we find very marked lowering of tension.

TABLE IV.

Interfacial Tension: oil-solution.

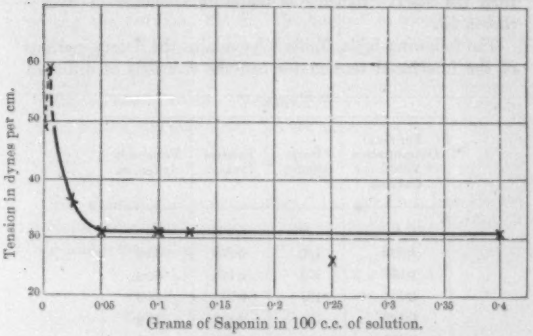
Substance.	Per cent. Concentration.	Molar Concentration.	Relative Tension.
Water	.....	.....	1·00
Iodine	0·035	$\frac{1}{360}$	slightly > 1·00
Hydroquinone	1·0	$\frac{1}{11}$	" < 1·00
Boric acid	0·1	$\frac{1}{62}$	0·97
Urea	0·8	$\frac{1}{7}$	0·97
Diphenylthiourea	< 0·1	...	0·96
Milk sugar	1·0	$\frac{1}{36}$	0·96
Succinimide	0·2	$\frac{1}{48}$	0·94
Phenol	0·1	$\frac{1}{94}$	0·94
Mercuric cyanide	1·0	$\frac{1}{25}$	0·91
p-nitrosodimethylaniline	< 0·1	$\frac{1}{<150}$	0·89
Cane-sugar	6·0	$\frac{1}{6}$	0·83
Amygdalin	1·0	$\frac{1}{51}$	0·82
Ethyl acetate	3·6	$\frac{2}{5}$	0·70
Caffeine	1·0	$\frac{1}{21}$	0·71
Aniline	3·0	$\frac{1}{31}$	0·63

lowering of tension *. It has also been noticed that bubbles blown with saponin solution appear to gelatinize, so that when once formed they show a crinkled appearance on contraction. Solutions of this substance showed the following effects at the oil-water interface. On increasing the concentration, there was first of all a rise of tension reaching a maximum at about 0·005 per cent. This was followed by a rapid fall and at 0·025 per cent. a slower fall to 0·05 per cent., beyond which the tension remained constant. Measurements were taken up to 0·4 per cent., at which point streaming

* See a paper bearing upon this subject by S. A. Shorter, Phil. Mag. vol. xi. p. 317 (1906).

effects began to be observable, so that no further readings could be taken. These peculiarities are shown in fig. 5.

Fig. 5.



It seems probable that the gelatinizing effects which have been observed at the air-surface are also effective at the oil-interface. Experiments were carried out to test whether the substance was soluble in the oil. It was found to be insoluble.

Of the substances enumerated in Table III., it will be seen that caffeine shows the most marked effects in proportion to its concentration. It was therefore selected as a suitable substance wherewith to carry out adsorption determinations.

Determination of $\frac{d\sigma}{dc}$ *for Caffeine.*

The material was obtained from Kahlbaum. Its molecular weight was determined by lowering of the freezing-point of water, and was found to be in agreement with the formula (weight 194). The substance employed is the monohydrate and crystallizes from water in long needles. That caffeine is a good example of a non-electrolyte follows from the value for its dissociation constant determined by Wood *, namely, 4×10^{-14}, which is of the same order as the constant for water.

* Wood, Journ. Chem. Soc. vol. lxxxix. p. 1844 (1906).

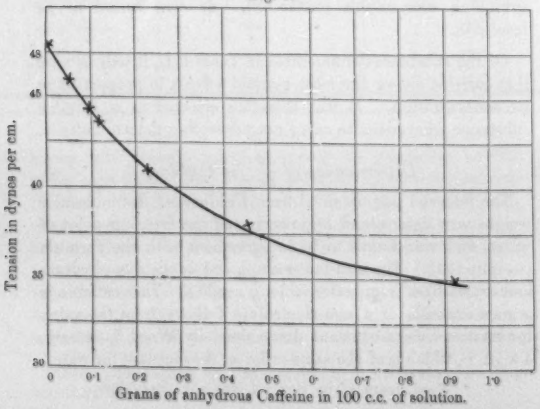
The sample of caffeine was first tested, as in previous cases, for any solubility in the oil. The same drop-number was obtained with the oil filtered from caffeine as for the oil fresh from the stock. Caffeine is therefore insoluble in hydrocarbon oil.

The following table (Table V.) contains the determinations of the interfacial tension for caffeine solutions of different

TABLE V.

Per cent. Concentration of Anhydrous Caffeine.	Drop-number.	Relative Tension.	Tension in dynes/cm.
0	146	1·000	48
0·046	153	0·954	45·8
0·092	158	0·924	44·4
0·114	161	0·907	43·5
0·229	171	0·853	40·9
0·457	184	0·793	38·1
0·915	201	0·726	34·8

Fig. 6.



concentrations, the tension being measured by the drop-pipette. It will be noted that this oil is not quite identical with that previously employed.

The interfacial tension is plotted against concentration, giving the curve on fig. 6. The tangent to the curve gives the value of $\frac{d\sigma}{dc}$ at the desired concentration.

Calculation of the Absorption Coefficient :—

As before, the expression for the adsorption is

$$\Gamma = -\frac{c}{RT}\frac{d\sigma}{dc}.$$

For a solution of concentration 0·114 per cent. (anhydrous) caffeine the right-hand side of this expression has the following numerical value :—

$$c = 0{\cdot}114 \text{ per cent.} = 0{\cdot}00114 \text{ grm./c.c.},$$

$$R = \frac{2 \times 4{\cdot}2 \times 10^7}{194} \text{ ergs},$$

$$T = 288° \text{ abs.},$$

$$\frac{d\sigma}{dc} = \frac{3 \text{ dynes/cm.}}{0{\cdot}00114 \text{ grm./c.c.}}.$$

$$\therefore \frac{c}{RT}\frac{d\sigma}{dc} = 2{\cdot}4 \times 10^{-8} \text{ grm./cm.}^2$$

It will be noted that this value is approximately of the same order as the similarly calculated values for previous substances. The substitution of a non-electrolyte for a simple inorganic salt does not seem therefore to have any marked effect on the theoretical value for the adsorption coefficient. It would seem, therefore, that in both cases we are dealing with a factor of the same magnitude as regards its effect on the tension and the consequent value of $\frac{d\sigma}{dc}$. It will be remembered, however, that the results obtained in previous cases pointed to the existence of certain discrepancies between observed and calculated values of Γ. It is of particular interest, therefore, to attempt to measure Γ directly for caffeine, since in this case there is no possibility of separation of ions.

Determination of Γ for Caffeine in Aqueous Solution.

The emulsion method * was employed in this determination.

Solutions of caffeine approximately 0·125 per cent. were prepared, and 500 c.c. shaken in steamed-out glass vessels for three days with about 1-2 c.c. of oil. The resulting emulsions were very uniform under the microscope, the average diameter of the particles being 6×10^{-5} cm. It is worthy of note that this is of the same order as the average diameter of the particles in the previous cases.

The results of two emulsion experiments are given below :—

Caffeine solution (approx. 0·125 per cent.).

Experiment no.	Drop-number before emulsification.	Drop-number after emulsification.	Area of oil-surface.
1	160	160, 160¼	87,130 cm.²
2	160, 160¼	160¼, 160¼	127,310 cm.²

The results show that there was *no readable difference in the concentration of the solution before and after emulsification—the difference being a small fraction of a drop.* Special experiments were undertaken to determine the accuracy of the readings made with the pipette, and it was found that for a small drop-number such as 160 the readings were reproducable to within about ¼th of a drop.

No very reliable quantitative relations can therefore be obtained from these determinations, as we are evidently dealing with a very small change in concentration, *i. e.*, a very small adsorption. The important feature is this—that *the calculated value of* Γ (viz. 10^{-8} grm./cm.²) *would also correspond to a small fraction of a drop* as indicating concentration change; while a value for Γ of the order obtained in the case of sod. glycocholate and the dyes (viz. 10^{-6} grm./cm.²) would have been indicated by a change of 4–5 drops, which would of course have been

* See previous paper.

perfectly readable. These experiments therefore show that *caffeine is very much less adsorbed than sodium glycocholate, Congo red, or methyl orange.*

It is perhaps going beyond the limits of accuracy warranted by the drop-pipette method to make an approximate calculation of the Γ from the data given above. Taking the second experiment, the change in concentration corresponds to about $\frac{1}{3}$th of a drop. This is only approximate, and in all probability would give a *maximum* value for Γ. From a curve * obtained by plotting the drop-numbers against concentrations given in Table IV. it is found that $\frac{1}{3}$th of a drop corresponds to a change in concentration of 0·00094 per cent., and hence the total amount of caffeine removed from the 500 c.c. emulsified is 0·0047 gram. The total adsorbing surface is 127310 cm.^{2}, hence

$$\Gamma = 3{\cdot}7 \times 10^{-8} \text{ grm./cm.}^2,$$

$$\Gamma \text{ calculated} = 2{\cdot}4 \times 10^{-8} \text{ grm./cm.}^2$$

An attempt was at once made to increase the adsorbing area in order to get more trustworthy data. It was found, however, that the limit had almost been reached in the case of the second emulsion—only a little more than 2 c.c. of oil can be emulsified by 500 c.c. of caffeine solution of 0·1 per cent. concentration. The method is therefore limited in this direction. A further attempt at more accurate measurement was made by employing a pipette of much larger oil-capacity (about 500 c.c). The drop-number against water was 2042, in place of 146 in the case of the smaller pipette. The drop-number against a solution of 0·1 per cent. concentration was 2383. One result was obtained in which the change of concentration before and after emulsification amounted to 5 drops, which corresponded to $3{\cdot}5 \times 10^{-8}$ grm./cm., but on attempting to reproduce these values quite variable results were obtained, owing to the change in wetting of the pipette during the long course (over two hours) of a single determination.

It appears probable from the foregoing attempts to measure

* The curve is a straight line between drop-numbers 155–200. A fraction of a drop can therefore be read fairly accurately.

Γ in the case of caffeine, that we are dealing with a quantity too small to be determined with accuracy by the method hitherto pursued. It may be possible by employing, for example, colloidal mercury instead of the oil to determine the caffeine adsorption at a mercury surface with accuracy.

Behaviour of Sodium Glycocholate and Caffeine in contact with Charcoal.

In view of the considerable difference between the values for the adsorption of these two substances at the oil-surface, it was of interest to compare the results obtained with charcoal. The material was the same as that employed by Freundlich—namely, Merck's blood-charcoal purified by acid. Freundlich has observed that the further washing of this material does not affect the result of the adsorption determinations, and this was confirmed in the present instance by the fact that there was not the slightest change in the oil-water tension in the case of distilled water which had been shaken up with a quantity of charcoal.

Adsorption of Sodium Glycocholate:—

500 c.c. of a solution 0·3 per cent. concentration were treated with one gram of charcoal, and the decrease in bulk-concentration of the solute amounted to 0·125 per cent. The change in concentration was measured by means of the change in value of the oil-solution tension as given by the drop-pipette. The drop-difference (before and after) was 47.

Hence 1 gram charcoal adsorbs from a 0·3 per cent. solution 0·625 gram sodium glycocholate.

Adsorption of Caffeine:—

500 c.c. of solution 0·218 per cent. concentration (*i. e.*, approximately equimolar with the sodium glycocholate solution) brought into contact with one gram of charcoal gives a drop-difference of 10, which corresponds on the caffeine curve to a change in concentration of 0·108 per cent.

Hence 1 gram charcoal adsorbs from a 0·218 per cent. solution 0·324 gram of caffeine.

It will be noted that the behaviour of caffeine and sodium glycocholate towards charcoal and oil is very different. In contact with charcoal there appears to be no very marked difference as regards the magnitude of the adsorption for both substances. Possibly this may be due to oxidation of the caffeine by means of the oxygen adsorbed on the charcoal surface.

Calculation of the Range of the Concentration *Effects at the Oil-surface.*

Consider the case of an infinite mass of solution (of a non-electrolyte in water) whose bulk-concentration is c_1 and

surface-concentration c_2. Suppose one gram-mole of solute transferred from the bulk into the surface-layer. The osmotic work done is

$$RT \log \frac{c_2}{c_1}.$$

There is a corresponding decrease in the surface energy due to adsorption of the solute given by

$$-\frac{d\sigma}{d\Gamma};$$

and equating these two expressions,

$$RT \log \frac{c_2}{c_1} = -\frac{d\sigma}{d\Gamma}.$$

Experimental case :—

If we assume that the solution of caffeine obeys in all probability Gibbs's expression for surface-*condensation*, from the data previously given we may construct the following table :—

Aqueous solution of Caffeine at 15° C.

Per cent. Bulk-concentration (anhydrous).	Tension in dynes/cm.	Γ calculated from Gibbs's equation in grm./cm.²
0·0	48	0·0
0·046	45·8	$1·2 \times 10^{-8}$
0·092	44·4	1·8 „
0·114	43·5	2·0 „
0·229	40·9	2·3 „
0·457	38·1	2·8 „
0·915	34·8	3·0 „

The tension was plotted against Γ, and the tangent to the curve taken at the point corresponding to a bulk-concentration 0·114 per cent. The tangent gives

$$\frac{d\sigma}{d\Gamma} = \frac{7·25 \text{ ergs/cm.}^2}{1 \times 10^{-10} \text{ gram-moles/cm.}^2} = 7·25 \times 10^{10}.$$

At this point

$$\Gamma = 2 \times 10^{-8} \text{ grm./cm.}^2 = 1 \times 10^{-10} \text{ gram-moles/cm.}^2$$

The equation therefore becomes

$$\log_e c_2 = \log_e c_1 + \frac{7{\cdot}25 \times 10^{10}}{RT}.$$

$$\therefore \quad \log_{10} c_2 = \log_{10} 0{\cdot}0064 + 0{\cdot}044$$
$$= -2{\cdot}1498.$$

$$\therefore \quad c_2 = 0{\cdot}0071 \text{ gram-moles per litre}$$
$$= 0{\cdot}00138 \text{ grm./c.c.}$$

Now if D is the range through which the surface-concentration effects are practically complete, it follows that the average concentration in the surface-layer is given by the expression.

$$c_2 = \frac{\Gamma}{D},$$

or $$D = \frac{\Gamma}{c_2};$$

and on substituting the values obtained above it is found that

$$D = 14 \times 10^{-6} \text{ cm.}$$

It is evident that we have here obtained a value for the "range of molecular action," as may be seen by comparing it with other estimates of the same quantity.

		Method.
Quincke	5×10^{-6} cm.	"Wedge method."
Reinold & Rücker.	$1{\cdot}2 \times 10^{-6}$	Limiting thickness of soap film.
Wiener	$1{\cdot}2 \times 10^{-6}$	Effect on phase of reflected light.
Plateau	$11{\cdot}4 \times 10^{-6}$	Limiting thickness of glycerine film.
Drude	$1{\cdot}7 \times 10^{-6}$	" " soap film.
Parks	$13{\cdot}4 \times 10^{-6}$	Wetting of powders.
Johonnot	from 10^{-6} to 10^{-8}	soap film.

Discussion of Results.

The conclusions to which the foregoing experiments point are:—

(1) Caffeine in aqueous solution, in all probability obeys Gibbs's law quantitatively.

(2) Ordinary inorganic salts—Potassium chloride, Silver nitrate, Barium chloride, and Copper chloride—are

adsorbed in quantities which are of the same order as the Gibbs's calculated effect, though the experimentally found values are in all cases *greater* than the calculated. Caustic soda shows a more marked discrepancy between calculated and observed values.

(3) Complex organic salts—Sodium glycocholate, Sodium oleate *, Congo red, and Methyl orange—show a very large discrepancy between observed and calculated values. The first thing is therefore to consider the various possibilities of accounting for this large discrepancy.

The Adsorption of Sodium glycocholate, &c.

First can the excess exhibited by these substances be explained by an extension of the capillary theory?

The expression

$$\Gamma = -\frac{c}{RT}\frac{d\sigma}{dc}$$

is deduced on the assumption that we are dealing with *one* component only which suffers surface-concentration, *i. e.*, the solute. Since the above-mentioned substances are salts, they will be associated, and almost entirely so, at the dilution worked with. Milner †, in deducing the Gibbs's expression, has allowed for dissociation by the introduction of van 't Hoff's factor (i) thus:

$$\Gamma = \frac{c}{iRT}\frac{d\sigma}{dc}$$

This is, however, really the expression employed in the foregoing experiments, since in all cases for the evaluation of R (the gas constant) the molecular weight of the solute in *solution* has been employed. By simply making this allowance for dissociation, therefore, the discrepancy between observed and calculated values for adsorption is still very marked—as a matter of fact, such a correction acts in the opposite direction.

The next possible explanation may lie in incorrectness in

* Reference will be made later to this substance.

† Milner, Phil. Mag., Jan. 1907.

the assumption (which has been made throughout) that the water suffers no surface density-change.

It is very probable that water has not quite the same density in the surface-film as in the bulk of the liquid. Evidence for this is to be found in the phenomenon of heat evolution observed by Parks and others, when water is poured upon finely-divided powders of silica, quartz, glass, &c., where chemical action is of course excluded. Also one must not overlook the fact that in the adsorption of sodium glycocholate and the dyes we are dealing with surface-concentrations far exceeding the ordinarily accepted values for their solubility. Of course in using the term "solubility" we assume that the surface-layer water possesses the same properties as water in bulk. For a solution of sodium glycocholate of 0·25 per cent. bulk concentration, Γ was found to be 5×10^{-6} grm./cm.^{2} The value of the range through which the surface-concentration is sensibly different from the bulk has been shown to be approximately 14×10^{-6} cm. Hence the *average* surface-concentration is 0·37 gram per c.c. or 37 per cent., while the ordinarily accepted value for the solubility of sodium glycocholate is 3·9 per cent. There was, however, no visible colour change on the oil surface due to any precipitation.

Again, methyl orange* showed an adsorption for which $\Gamma = 5{\cdot}5 \times 10^{-6}$, whence the average concentration is 39 per cent. The actual solubility is only 0·078 per cent.

The substances therefore which have shown very great discrepancies as regards Gibbs's theory are *those whose surface concentration greatly exceeds their solubility in the solvent*. For the other substances examined, viz. caffeine and the inorganic salts, *in no case is the ordinary solubility exceeded*, and the adsorption of these substances is in much closer agreement with theory. Thus the surface-concentration of caffeine was found to be 0·26 per cent., while its solubility at ordinary temperature is 1·35 per cent. For silver nitrate the surface-concentration was approximately 0·28 per cent., while its solubility is 122 per cent. at 0° C. Similarly for potassium chloride, barium chloride, and copper chloride.

* See previous paper, *l. c.*

Another possibility however presents itself—namely, the existence of some electrical effect, such as an electrostatic attraction, causing the discrepancy, since sodium glycocholate and the dyes are electrolytes. If this were so, one would also expect somewhat of the same order of discrepancy manifesting itself in the case of the ordinary inorganic salts. Such does not take place. Caustic soda alone has marked effects, and shows an adsorption 20 times greater than the calculated. The other substances generally show a discrepancy of about 5 to 8 times. To whatever cause this smaller discrepancy is due, it is unlikely that the same reason is to be assigned to the very much larger discrepancies observed for sodium glycocholate.

The only other possible means, apparently, of explaining the want of agreement is on the assumption that we are dealing with some irreversible phenomenon of the nature of gelatinization upon the oil surface. The fact of the surface-concentration being much greater than the ordinarily accepted value for the solubility, is in agreement with this view. It should be pointed out, however, that the overstepping of the solubility limit does not seem to be essential to gelatinization, as will be seen by referring to the determinations carried out with saponin. The solubility of this substance is great; according to Beilstein it is completely miscible with water, and although this cannot actually be the case, the constancy of the tension at comparatively dilute solutions cannot be explained by the solubility having been reached. This is a very striking case showing the sensitiveness of the tension to gelatinizing effects. The hypothesis of gelatinization has been put forward by Milner* to explain the behaviour of sodium oleate, although in this case the tension curve was continuous and quite similar to the sodium-glycocholate curve. From some unpublished experiments by Mr. H. E. Potts in this Laboratory, it was found that sodium oleate showed an adsorption at the oil surface, the magnitude of which was of the order 10^{-6} grm./cm.^{2}, while the calculated was of the order 10^{-8} grm./cm.^{2}, that is to say a discrepancy very similar to sodium glycocholate.

* *Loc. cit.*

Mention may also be made that Freundlich and Losev* have found that certain dyestuffs produced amorphous precipitates on the charcoal surface during adsorption determinations. This was an irreversible process (the precipitates being insoluble in water), and as such might be considered as analogous to gelatinization.

It is possible, therefore, that some irreversible effect of the nature of gelatinization is the real cause of the discrepancy between observed and calculated values in the adsorption of sodium glycocholate, Congo red, and methyl-orange.

The Adsorption of Inorganic Salts from Aqueous Solution.

It has already been pointed out that for certain purely inorganic salts, discrepancies are found between observed and calculated values of the adsorption, but that the existence of these discrepancies might be considered as not quite proved owing to possibility of great experimental error. On the whole, however, it is believed that there really does exist a small discrepancy which requires explanation. Attention might be drawn to the fact that in the cases measured the metal, *i. e.* the cation, shows an excess, while in the case of potassium chloride, and possibly also in barium chloride, the chlorine (the anion) is much nearer to the calculated value. This at once suggests some separation of ions, that is partial selective adsorption due to some electrical effect. The value, however, for the adsorption of the chlorine in copper chloride is scarcely in accord with the preceding statement. Granting that there is a tendency for selective ionic adsorption, it is evident that the phenomenon cannot be completely covered by Gibbs's capillary expression, in the deduction of which no allowance is made for any such ionic separation. Of course for non-electrolytes, where no ionic separation is possible, Gibbs's expression ought to hold quite readily, and experiment confirms this, although the determinations are unfortunately of a more qualitative character than one could wish. In electrolytes, the separation of cation from anion may possibly be regarded as

* Freundlich and Losev, *Zeit. Phys. Chem.* vol. lix. p. 284 (1907).

local electrolysis at the surface of the oil—that in fact we are dealing with a cell of small dimensions the poles of which are the two homogeneous masses, and the heterogeneous layer separating the masses functions as the "electrolytic medium." If this is so, we must treat the question as one of electrolytic conduction, but first of all it must be shown that there exists an electrical potential-difference between the electrodes, *i. e.* between the oil and the water. Owing to the non-conducting nature of the oil one cannot employ the usual methods, but by emulsifying the oil (*i. e.* converting it into small particles suspended in water) and placing the emulsion in an electrostatic field, it can be observed whether any motion of the particles takes place or not. This was done experimentally as follows :—

A simple Nernst form of U-tube, identical with that employed by Burton *, having a vertical capillary tube affixed at the bend through which water slowly runs so as to form a definite surface of separation (in both limbs) between the water and the oil-water emulsion, is fitted with platinum electrodes dipping into the emulsion. A potential-difference of 230 volts was maintained between the electrodes, and the movement of the emulsion-water boundary down in one limb and up in the other was read off on a scale attached to the vessel. It was found that the particles moved towards the anode, being therefore negatively charged. The velocity of the particles was $4{\cdot}3 \times 10^{-4}$ cm./sec. under a gradient of 1 volt/cm.

Burton † has deduced an expression for the potential existing between the particles of colloidal metals and the water in which they are suspended, viz.:

$$V = \frac{4\pi}{K}\frac{\eta v}{X},$$

where V is the p.d. required (in electrostatic units);
K is the specific inductive capacity of the medium;

* Burton, Phil. Mag. ii. p. 434 (1906).
† Burton, *loc. cit.*

η is the viscosity of the medium;

$\frac{v}{X}$ is the velocity of the particles under (electrostatic) unit potential gradient,

X being the p.d. maintained between the poles.

Applying this to the present case of oil suspended in water, we obtain

$$V = 0{\cdot}00048 \text{ electrostatic unit,}$$

or $\underline{0{\cdot}15 \text{ volt.}}$

There is thus quite a measurable potential existing between the oil and water; and in considering the adsorption of electrolytes we must not overlook this fact. It may be mentioned, in passing, that the above value for the potential is quite of the same order as that obtained by Burton for various colloidal metals.

Since the oil has been shown to be negatively charged, one would not be surprised, speaking generally, that the cation should be selectively adsorbed.

Summary.

(1) The adsorption of various substances on a hydrocarbon-oil surface has been measured. The results are collected in the following table:—

Substance.	Adsorption per cm.^{2} of the substance assumed to be in the form of *undissociated salt* or in chemically equivalent ionic proportions.	
	Found value.	Calculated value.
Sodium glycocholate. .	5×10^{-6} grm./cm.^{2}	7×10^{-8} grm./cm.^{2}
Congo-red	$3{\cdot}7\times10^{-6}$	$1{\cdot}1\times10^{-7}$
Methyl-orange	$5{\cdot}5\times10^{-6}$	$1{\cdot}2\times10^{-7}$
Sodium oleate	10^{-6}	10^{-8}
Caustic soda	$1{\cdot}5\times10^{-7}$	$7{\cdot}5\times10^{-9}$
Caffeine	$3{\cdot}7\times10^{-8}$	$2{\cdot}4\times10^{-8}$

Substance.	Adsorption of cation per cm.^{2}		Adsorption of anion per cm.^{2}	
	Found value.	Calculated value.	Found value.	Calculated value.
	grm./cm.^{2}	grm./cm.^{2}	grm./cm.^{2}	
Silver nitrate	$2{\cdot}5\times10^{-8}$	$4{\cdot}5\times10^{-9}$	.....	$2{\cdot}3\times10^{-9}$
Potassium chloride .	5×10^{-8}	$1{\cdot}7\times10^{-9}$	10^{-9}	$1{\cdot}6\times10^{-9}$
Barium chloride ...	.....	3×10^{-9}	not $> 10^{-8}$	$1{\cdot}6\times10^{-9}$
Copper chloride ...	$3{\cdot}5\times10^{-8}$	2×10^{-9}	2×10^{-8}	2×10^{-9}

(2) The large discrepancy between observed and calculated values for the first four substances is possibly due to gelatinization upon the oil-surface.

(3) Caffeine obeys Gibbs's law within the limits of experimental error.

(4) There is evidence of slight *selective* adsorption, the cation more than the anion, probably due to electrical effects, since the oil is shown to be negatively charged, the potential-difference between the oil and water being approximately 0·15 volt.

In conclusion, I gladly take this opportunity of expressing my gratitude to Professor F. G. Donnan for his advice and assistance in this work.

XXXVI. *On the Elliptic Polarization produced by the Direct Transmission of a Plane Polarized Stream through a Plate of Quartz, cut in a Direction oblique to the Optic Axis, with a Method of Determining the Error of a Plate supposed to be Perpendicular to the Axis.* By JAMES WALKER, *M.A., Oxford* *.

1. LET the primitive stream with polarization-vector $\mathrm{Exp}(\iota nt)$ be polarized in an azimuth α with respect to the principal section of the quartz plate. This may be replaced by the elliptically polarized stream represented by the vector

$$(\xi_1, \eta_1) = c_1(\cos\beta, -\iota\sin\beta)\,\mathrm{Exp}\{\iota(nt+\epsilon_1)\},$$

with its plane of maximum polarization (β being numerically less than $\pi/4$ and positive or negative according as the quartz is left- or right-handed) in the principal section, together with the oppositely polarized stream defined by the vector

$$(\xi_2, \eta_2) = c_2(-\sin\beta, -\iota\cos\beta)\,\mathrm{Exp}\{\iota(nt+\epsilon_2)\},$$

where

$$c_1\,\mathrm{Exp}(\iota\epsilon_1) = \cos\alpha\cos\beta + \iota\sin\alpha\sin\beta,$$
$$c_2\,\mathrm{Exp}(\iota\epsilon_2) = -\cos\alpha\sin\beta + \iota\sin\alpha\cos\beta.$$

These oppositely polarized streams traverse the plate with different speeds, and on emergence the second is retarded in phase relatively to the first by an amount δ, so that the emergent light will be in general elliptically polarized and may be denoted by

$$(u, v) = c(\cos\gamma, -\iota\sin\gamma)\,\mathrm{Exp}\{\iota(nt+\epsilon)\},$$

where

$$c(\cos\gamma\cos\theta + \iota\sin\gamma\sin\theta)\,\mathrm{Exp}\{\iota(\epsilon+\delta/2)$$
$$= c_1\cos\beta\,\mathrm{Exp}\{\iota(\epsilon_1+\delta/2)\} - c_2\sin\beta\,\mathrm{Exp}\{\iota(\epsilon_2-\delta/2)\},$$
$$c(\cos\gamma\sin\theta - \iota\sin\gamma\cos\theta)\,\mathrm{Exp}\{\iota(\epsilon+\delta/2)\}$$
$$= -\iota c_1\sin\beta\,\mathrm{Exp}\{\iota(\epsilon_1+\delta/2)\} - \iota c_2\cos\beta\,\mathrm{Exp}\{\iota(\epsilon_2-\delta/2)\},$$

θ being the angle between the principal section of the plate and the plane of maximum or minimum polarization of the resultant stream, according as γ is numerically less or greater than $\pi/4$.

* Read November 27, 1908.

Substituting for $c_1 \operatorname{Exp}(\iota\epsilon_1)$ and $c_2 \operatorname{Exp}(\iota\epsilon_2)$ and writing

$$\tan \mathrm{R} = \sin 2\beta \tan \delta/2, \quad \tan \Delta/2 = \cot 2\beta \sin \mathrm{R}, \quad . \quad (1)$$

we obtain

$$c(\cos\gamma\cos\theta + \iota\sin\gamma\sin\theta)\operatorname{Exp}\{\iota(\epsilon+\delta/2)\} = \cos(\alpha+\mathrm{R})\cos\Delta/2 + \iota\cos\alpha\sin\Delta/2,$$

$$c(\cos\gamma\sin\theta - \iota\sin\gamma\cos\theta)\operatorname{Exp}\{\iota(\epsilon+\delta/2)\} = \sin(\alpha+\mathrm{R})\cos\Delta/2 - \iota\sin\alpha\sin\Delta/2,$$

or

$$c\left\{\cos\gamma\cos\left(\theta-\frac{\mathrm{R}}{2}\right)+\iota\sin\gamma\sin\left(\theta-\frac{\mathrm{R}}{2}\right)\right\}\operatorname{Exp}\left\{\iota\left(\epsilon+\frac{\delta}{2}\right)\right\} = \cos\left(\alpha+\frac{\mathrm{R}}{2}\right)\left(\cos\frac{\Delta}{2}+\iota\sin\frac{\Delta}{2}\right),$$

$$c\left\{\cos\gamma\sin\left(\theta-\frac{\mathrm{R}}{2}\right)-\iota\sin\gamma\cos\left(\theta-\frac{\mathrm{R}}{2}\right)\right\}\operatorname{Exp}\left\{\iota\left(\epsilon+\frac{\delta}{2}\right)\right\} = \sin\left(\alpha+\frac{\mathrm{R}}{2}\right)\left(\cos\frac{\Delta}{2}-\iota\sin\frac{\Delta}{2}\right);$$

whence

$$\frac{\cos\gamma\cos\left(\theta-\frac{\mathrm{R}}{2}\right)+\iota\sin\gamma\sin\left(\theta-\frac{\mathrm{R}}{2}\right)}{\cos\gamma\sin\left(\theta-\frac{\mathrm{R}}{2}\right)-\iota\sin\gamma\cos\left(\theta-\frac{\mathrm{R}}{2}\right)} = \cot\left(\alpha+\frac{\mathrm{R}}{2}\right)(\cos\Delta+\iota\sin\Delta),$$

which gives

$$\left.\begin{aligned}\frac{\cos 2\gamma\sin(2\theta-\mathrm{R})}{1-\cos 2\gamma\cos(2\theta-\mathrm{R})} &= \cot\left(\alpha+\frac{\mathrm{R}}{2}\right)\cos\Delta\\ \frac{\sin 2\gamma}{1-\cos 2\gamma\cos(2\theta-\mathrm{R})} &= \cot\left(\alpha+\frac{\mathrm{R}}{2}\right)\sin\Delta\end{aligned}\right\} \quad . \quad . \quad (2)$$

From these we obtain the equations

$$\left.\begin{aligned}\tan(2\theta-\mathrm{R}) &= \tan(2\alpha+\mathrm{R})\cos\Delta,\\ \sin 2\gamma &= \sin(2\alpha+\mathrm{R})\sin\Delta,\\ \tan^2\gamma &= \tan(\alpha+\theta)\tan(\alpha+\mathrm{R}-\theta)\end{aligned}\right\} \quad . \quad . \quad . \quad (3)$$

that determine the character of the stream emerging from the quartz, β and δ and consequently R and Δ being supposed to be known.

2. For the discussion of the elliptic polarization it is more

convenient to express $\tan 2(\theta-\alpha)$ and $\sin 2\gamma$ in terms of $\tan \mathrm{R}$. Solving for $\tan 2(\theta-\alpha)$ we obtain

$$\tan 2(\theta-\alpha)=\frac{\tan(2\alpha+\mathrm{R})\cos\Delta-\tan(2\alpha-\mathrm{R})}{1+\tan(2\alpha+\mathrm{R})\tan(2\alpha-\mathrm{R})\cos\Delta}$$

$$=\frac{\sin 2\mathrm{R}\cos^2\Delta/2-\sin 4\alpha\sin^2\Delta/2}{\cos 2\mathrm{R}\cos^2\Delta/2+\cos 4\alpha\sin^2\Delta\,2}$$

$$=\frac{\sin 2\mathrm{R}-\sin 4\alpha\cot^2 2\beta\sin^2\mathrm{R}}{\cos 2\mathrm{R}+\cos 4\alpha\cot^2 2\beta\sin^2\mathrm{R}}$$

$$=\frac{2\tan\mathrm{R}-2\sin 2\alpha\cos 2\alpha\cot^2 2\beta\tan^2\mathrm{R}}{1+(\cos 4\alpha\cot^2 2\beta-1)\tan^2\mathrm{R}}.$$

Also

$$\sin 2\gamma=\sin 2\alpha\cos\mathrm{R}\sin\Delta+\cos 2\alpha\sin\mathrm{R}\sin\Delta$$

$$=\frac{2\sin 2\alpha\cot 2\beta\tan\mathrm{R}+2\cos 2\alpha\cot 2\beta\tan^2\mathrm{R}}{1+(\cot^2 2\beta+1)\tan^2\mathrm{R}},$$

since

$$\sin\Delta=\frac{2\tan\Delta/2}{1+\tan^2\Delta/2}=\frac{2\cot 2\beta\sin\mathrm{R}}{1+\cot^2 2\beta\sin^2\mathrm{R}}.$$

These expressions have the form

$$y=2(ax+bx^2)/(1+cx^2),$$

and the maximum and minimum values of y are $y=ax_m$ occurring when

$$x=x_m=(b\pm\sqrt{b^2+a^2c})/(ac).$$

3. First considering $\sin 2\gamma$, we see that $\sin 2\gamma=0$, or the light is plane polarized, when $\tan\mathrm{R}=0$ and when $\tan\mathrm{R}=-\tan 2\alpha$, the corresponding values of $\tan 2(\theta-\alpha)$ being 0 and $-\tan 4\alpha$. $\sin 2\gamma$ attains its maximum and minimum values $\sin 2\alpha\cot 2\beta\tan r_m$, when

$$\tan\mathrm{R}=\tan r_m=\frac{1\pm\sqrt{1+\tan^2 2\alpha/\sin^2 2\beta}}{\tan 2\alpha/\sin^2 2\beta};$$

writing $\tan 2\alpha/\sin 2\beta=\tan\psi$, ψ being a positive angle less than π,

$$\tan r_m=\sin 2\beta\cot\psi/2,\ \text{or}\ -\sin 2\beta\tan\psi/2,$$

and since $\tan\mathrm{R}=\sin 2\beta\tan\delta/2$, the maximum and minimum values of $\sin 2\gamma$ occur when $\delta=n\pi-\psi$.

The values of tan R corresponding to the maximum and minimum values of sin 2γ may be written

$$\tan R = \sin 2\beta \,.\, x, \text{ where } x = \cot \psi/2 \text{ or } -\tan \psi/2,$$

and
$$\begin{aligned}\tan 2\alpha &= -2 \sin 2\beta \cot \psi/2/(1-\cot^2 \psi/2)\\ &= 2 \sin 2\beta \tan \psi/2/(1-\tan^2 \psi/2)\\ &= -2 \sin 2\beta\, x/(1-x^2);\end{aligned}$$

substituting these values in the expression for $\tan 2(\theta-\alpha)$, we find

$$\tan 2(\theta-\alpha) = 2 \sin 2\beta \frac{x(1+x^2)(1-\cos 4\beta \,.\, x^2)}{(1-x^2)(1+x^2)(1-\cos 4\beta \,.\, x^2)};$$

and unless $x^2 = \sec 4\beta$,

$$\tan 2(\theta-\alpha) = 2 \sin 2\beta \frac{x}{1-x^2} = -\tan 2\alpha.$$

But when $x^2 = \sec 4\beta$,

$$\tan 2\alpha = \sqrt{\cot^2 2\beta - 1}, \text{ or } \tan^2 2\beta = \cos^2 2\alpha;$$

hence corresponding to the maximum and minimum values of sin 2γ, we have $\tan 2(\theta-\alpha) = -\tan 2\alpha$, unless $\cos 2\alpha = \pm \tan 2\beta$, a case that will be considered later.

Further, sin $2\gamma = \pm 1$, only if

$$\tan \psi/2 = \sin 2\alpha \cos 2\beta, \text{ or } \cot \psi/2 = \sin 2\alpha \cos 2\beta,$$

that is if $\cos 2\alpha = \pm \tan 2\beta$, and consequently, except in these cases, θ determines the plane of maximum polarization.

Hence, reserving for future consideration the cases in which $\cos 2\alpha = \pm \tan 2\beta$, we see that as R increases from 0 to π, corresponding to a change of δ from $2n\pi$ to $2(n\pm1)\pi$ according as β is positive or negative, the light is initially plane polarized in the primitive plane of polarization; it then becomes elliptically polarized of a sign, the same as or opposite to that of the plate according as sin 2α is positive or negative: the ratio of the axes of the elliptic vibration attains its maximum value, the plane of maximum polarization being then either parallel or perpendicular to the principal section of the quartz, when $\tan R = \sin 2\beta \cot \psi/2$ or $= -\sin 2\beta \tan \psi/2$ according as β is positive or negative; when $\tan R = -\tan 2\alpha$ the light is again plane polarized in

an azimuth symmetrical to the primitive plane of polarization with respect to the principal section of the plate; the sense of rotation then changes, and the ratio of the axes is again a maximum when $\tan R = -\sin 2\beta \tan \psi/2$ or $= \sin 2\beta \cot \psi/2$, the plane of maximum polarization being again either parallel or perpendicular to the principal section; and finally, when $R = \pi$, the light is plane polarized in the original plane *.

4. Turning now to $\tan 2(\theta - \alpha)$, we see that this attains its maximum and minimum values $\tan R_m$, when

$$\tan R = \tan R_m = \frac{-\sin 2\alpha \cos 2\alpha \pm \sqrt{(\cos^2 2\alpha - \tan^2 2\beta)(\sin^2 2\alpha + \tan^2 2\beta)}}{\cos^2 2\alpha - \sin^2 2\alpha - \tan^2 2\beta}.$$

This, however, only gives a real value for $\tan R$, if $\cos^2 2\alpha > \tan^2 2\beta$, and when this is the case, writing

$$\cos^2 2\alpha - \tan^2 2\beta = \cos^2 \chi,$$

where χ is a positive angle less than $\pi/2$, we have

$$\tan R_m = \tan (\chi - 2\alpha) \text{ or } -\tan (\chi + 2\alpha),$$

the value of $\sin 2\gamma$ in these two cases being $\tan 2\beta / \cos 2\alpha$.

Thus if $\cos^2 2\alpha < \tan^2 2\beta$, the plane of maximum polarization rotates continuously as R increases; while if $\cos^2 2\alpha > \tan^2 2\beta$, it oscillates between two extreme positions, making an angle $\chi/2$ on either side of the principal section of the plate or the perpendicular plane according as $\cos 2\alpha$ is positive or negative †.

The following tables give some of the principal cases of the changes of the plane of maximum polarization and of $\sin 2\gamma$ as R increases from 0 to π, β being taken as positive.

* The changes in the value of $\sin 2\gamma$ may also be conveniently traced from the formula

$$\sin 2\gamma = 2 \sin 2\alpha \cos 2\beta \sin \delta/2 \sin (\psi + \delta/2)/\sin \psi.$$

† This oscillation of the plane of maximum polarization was deduced by Croullebois (*Ann. de Ch. et de Phys.* [4] xxviii. p. 382 (1873)) for the special case of light initially polarized in the principal section of the quartz. He appears to have overlooked the fact that it requires in this case that $\tan 2\beta$ should be less than unity or $\beta < \pi/8$, a condition found by Monnory (*J. de Phys.* [2] ix. p. 277 (1890)). The condition for the general case does not appear to have been previously given.

TABLE I.—$\cos^2 2\alpha > \tan^2 2\beta$.

$0 < \alpha < \pi/4$.

	tan R.	tan $2(\theta-\alpha)$.	θ.	sin 2γ.
1.	0	0	α	0
2.	$\tan(\chi-\alpha)$	$\tan(\chi-2\alpha)$	$\frac{\chi}{2}$	$\frac{\tan 2\beta}{\cos 2\alpha}$
3.	$\frac{\tan^2 2\beta}{\sin 2\alpha \cos 2\alpha}$	0	α	$\sin 2\left(\tan^{-1}\frac{\tan 2\beta}{\cos 2\alpha}\right)$
4.	$\sin 2\beta \cot\frac{\psi}{2}$	$-\tan 2\alpha$	0	$\sin 2\alpha \cos 2\beta \cot\frac{\psi}{2}$
5.	$\cot 2\alpha$	$-\tan 4\alpha$	$-\alpha$	$\sin 2\left(\tan^{-1}\frac{\tan 2\beta}{\cos 2\alpha}\right)$
6.	$-\tan(\chi+2\alpha)$	$-\tan(\chi+2\alpha)$	$-\frac{\chi}{2}$	$\frac{\tan 2\beta}{\cos 2\alpha}$
7.	$-\tan 2\alpha$	$-\tan 4\alpha$	$-\alpha$	0
8.	$-\sin 2\beta \tan\frac{\psi}{2}$	$-\tan 2\alpha$	0	$-\sin 2\alpha \cos 2\beta \tan\frac{\psi}{2}$
9.	0	0	α	0

$\pi/4 < \alpha < \pi/2$.

	tan R.	tan $2(\theta-\alpha)$.	θ.	sin 2γ.
1.	0	0	α	0
2.	$\sin 2\beta \cot\frac{\psi}{2}$	$-\tan 2\alpha$	$\frac{\pi}{2}$	$\sin 2\alpha \cos 2\beta \cot\frac{\psi}{2}$
3.	$-\tan 2\alpha$	$-\tan 4\alpha$	$\pi-\alpha$	0
4.	$\tan(\chi-2\alpha)$	$\tan(\chi-2\alpha)$	$\frac{\pi}{2}+\frac{\chi}{2}$	$\frac{\tan 2\beta}{\cos 2\alpha}$
5.	$\cot 2\alpha$	$-\tan 4\alpha$	$\pi-\alpha$	$\sin 2\left(\tan^{-1}\frac{\tan 2\beta}{\cos 2\alpha}\right)$
6.	$-\sin 2\beta \tan\frac{\psi}{2}$	$-\tan 2\alpha$	$\frac{\pi}{2}$	$-\sin 2\alpha \cos 2\beta \tan\frac{\psi}{2}$
7.	$\frac{\tan^2 2\beta}{\sin 2\alpha \cos 2\alpha}$	0	α	$\sin 2\left(\tan^{-1}\frac{\tan 2\beta}{\cos 2\alpha}\right)$
8.	$-\tan(\chi+2\alpha)$	$-\tan(\chi+2\alpha)$	$\frac{\pi}{2}-\frac{\chi}{2}$	$\frac{\tan 2\beta}{\cos 2\alpha}$
9.	0	0	α	0

TABLE II.—$\cos^2 2\alpha < \tan^2 2\beta$.

$0 < \alpha < \pi/4$.

	tan R.	tan $2(\theta-\alpha)$.	θ.	sin 2γ.
1.	0	0	α	0
2.	$\cot 2\alpha$	$-\tan 4\alpha$	$\frac{\pi}{2}-\alpha$	$\sin 2\left(\tan^{-1}\frac{\tan 2\beta}{\cos 2\alpha}\right)$
3.	$\sin 2\beta \cot\frac{\psi}{2}$	$-\tan 2\alpha$	$\frac{\pi}{2}$	$\sin 2\alpha \cos 2\beta \cot\frac{\psi}{2}$
4.	$\frac{\tan^2 2\beta}{\sin 2\alpha \cos 2\alpha}$	0	$\frac{\pi}{2}+\alpha$	$\sin 2\left(\tan^{-1}\frac{\tan 2\beta}{\cos 2\alpha}\right)$
5.	$-\tan 2\alpha$	$-\tan 4\alpha$	$\pi-\alpha$	0
6.	$-\sin 2\beta \tan\frac{\psi}{2}$	$-\tan 2\alpha$	π	$-\sin 2\alpha \cos 2\beta \tan\frac{\psi}{2}$
7.	0	0	$\pi+\alpha$	0

$\pi/4 < \alpha < \pi/2$.

	tan R.	tan $2(\theta-\alpha)$.	θ.	sin 2γ.
1.	0	0	α	0
2.	$\sin 2\beta \cot\frac{\psi}{2}$	$-\tan 2\alpha$	$\frac{\pi}{2}$	$\sin 2\alpha \cos 2\beta \cot\frac{\psi}{2}$
3.	$-\tan 2\alpha$	$-\tan 4\alpha$	$\pi-\alpha$	0
4.	$\frac{\tan^2 2\beta}{\sin 2\alpha \cos 2\alpha}$	0	$\frac{\pi}{2}+\alpha$	$\sin 2\left(\tan^{-1}\frac{\tan 2\beta}{\cos 2\alpha}\right)$
5.	$-\sin 2\beta \tan\frac{\psi}{2}$	$-\tan 2\alpha$	π	$-\sin 2\alpha \cos 2\beta \tan\frac{\psi}{2}$
6.	$\cot 2\alpha$	$-\tan 4\alpha$	$\frac{3\pi}{2}-\alpha$	$\sin 2\left(\tan^{-1}\frac{\tan 2\beta}{\cos 2\alpha}\right)$
7.	0	0	$\pi+\alpha$	0

5. The transition from one of these cases to the other occurs when $\cos^2 2\alpha = \tan^2 2\beta$, and then

$$\tan 2(\theta-\alpha) = \frac{2\tan \mathrm{R}(1-\tan 2\alpha \tan \mathrm{R})}{1-\tan^2 2\alpha \tan^2 \mathrm{R}} = \frac{2\tan \mathrm{R}}{1+\tan 2\alpha \tan \mathrm{R}}.$$

In this case θ increases from α to $\alpha+\pi/2$, as R increases from 0 to π: the light is circularly polarized when $\tan \mathrm{R} = \cot 2\alpha$, and while for values of R less than that given by this equation θ gives the plane of maximum polarization, for greater values it determines the plane of minimum polarization*.

* When $\tan \mathrm{R} = \cot 2\alpha$, $\tan 2(\theta-\alpha)$ of course becomes indeterminate, but it is easy to see that the change from determining the plane of maximum polarization to giving that of minimum polarization, or *vice versâ*, occurs as θ passes through the values $(2n+1)\pi/4$.

The ratio of the axes of the elliptic vibrations may be easily determined from the formula

$$\tan^2\gamma = \tan(\alpha+\theta)\tan(\alpha+\mathrm{R}-\theta)$$

which in this case reduces to the simple form

$$\tan\gamma = \pm\tan(\theta+\alpha)\tan(\theta-\alpha),$$

the sign to be attributed to $\tan\gamma$ being determined from that of $\sin 2\gamma$.

The following table gives the principal changes relating to this case.

TABLE III.—$\cos^2 2\alpha = \tan^2 2\beta$.

$0 < \alpha < \pi/4$.

	tan R.	tan 2(θ − α).	θ.	tan γ.
1.	0	0	α	0
2.	$\cot 2\alpha$	$\cot 2\alpha$	$\frac{\pi}{4}$	1
3.	$-\cot 2\alpha$	∞	$\frac{\pi}{4}+\alpha$	$\tan\left(\frac{\pi}{4}+2\alpha\right)$
4.	$-\tan 2\alpha$	$-\tan 4\alpha$	$\frac{\pi}{2}-\alpha$	∞
5.	$-\frac{\sin 2\alpha\cos 2\alpha}{1+\cos^2 2\alpha}$	$-\tan 2\alpha$	$\frac{\pi}{2}$	$-\cot^2\alpha$
6.	0	0	$\frac{\pi}{2}+\alpha$	∞

$\pi/4 < \alpha < \pi/2$.

	tan R.	tan 2(θ − α)	θ.	tan γ.
1.	0	0	α	4
2.	$-\frac{\sin 2\alpha\cos 2\alpha}{1+\cos^2 2\alpha}$	$-\tan 2\alpha$	$\frac{\pi}{2}$	$\cot^2\alpha$
3.	$-\tan 2\alpha$	$-\tan 4\alpha$	$\pi-\alpha$	0
4.	$-\cot 2\alpha$	∞	$\frac{\pi}{4}+\alpha$	$-\tan\left(\frac{\pi}{4}+2\alpha\right)$
5.	$\cot 2\alpha$	$\cot 2\alpha$	$\frac{3\pi}{4}$	-1
6.	0	0	$\frac{\pi}{2}+\alpha$	$-\infty$

If α be between $\pi/8$ and $3\pi/8$, cases (3) and (4) in the above table are interchanged.

6. When the light is analysed in a plane making an angle ϕ with the principal section of the plate, the intensity of the radiation characterized by (θ, γ) is

$$\begin{aligned} \mathrm{I} &= c^2\{\cos^2\gamma\cos^2(\phi-\theta)+\sin^2\gamma\sin^2(\phi-\theta)\} \\ &= \tfrac{1}{2}c^2\{1+\cos 2\gamma\cos 2(\phi-\theta)\}. \end{aligned}$$

Now from (2) and (3)

$$\begin{aligned} &\cos 2\gamma\sin 2\theta\cos \mathrm{R}-\cos 2\gamma\cos 2\theta\sin \mathrm{R} \\ &\qquad = \sin 2\gamma\cot\Delta = \sin(2\alpha+\mathrm{R})\cos\Delta \\ &\cos 2\gamma\sin 2\theta\sin \mathrm{R}+\cos 2\gamma\cos 2\theta\cos \mathrm{R} \\ &\qquad = 1-\tan\left(\alpha+\frac{\mathrm{R}}{2}\right)\sin 2\gamma/\sin\Delta \\ &\qquad = 1-\tan\left(\alpha+\frac{\mathrm{R}}{2}\right)\sin(2\alpha+\mathrm{R}) = \cos(2\alpha+\mathrm{R}); \end{aligned}$$

whence

$$\begin{aligned} \cos 2\gamma\sin 2\theta &= \cos 2\alpha\sin \mathrm{R}\cos \mathrm{R}(1+\cos\Delta) \\ &\quad +\sin 2\alpha(\cos^2 \mathrm{R}\cos\Delta-\sin^2 \mathrm{R}) \\ \cos 2\gamma\cos 2\theta &= \cos 2\alpha(\cos^2 \mathrm{R}-\sin^2 \mathrm{R}\cos\Delta) \\ &\quad -\sin 2\alpha\sin \mathrm{R}\cos \mathrm{R}(1+\cos\Delta). \end{aligned}$$

Now

$$\begin{aligned} 1+\cos\Delta &= \frac{2}{1+\tan^2\Delta/2} = \frac{2}{1+\cot^2 2\beta\sin^2 \mathrm{R}} \\ &= \frac{2}{\cos^2 \mathrm{R}+\sin^2 \mathrm{R}/\sin^2 2\beta}, \end{aligned}$$

$\therefore$

$$\sin \mathrm{R}\cos \mathrm{R}(1+\cos\Delta) = \frac{2\tan \mathrm{R}}{1+\tan^2 \mathrm{R}/\sin^2 2\beta} = \sin 2\beta\sin\delta$$

$$\begin{aligned} \cos^2 \mathrm{R}\cos\Delta-\sin^2 \mathrm{R} &= \frac{2}{1+\tan^2 \mathrm{R}/\sin^2 2\beta}-1 \\ &= 2\cos^2\frac{\delta}{2}-1 = \cos\delta \end{aligned}$$

$$\begin{aligned} \cos^2 \mathrm{R}-\sin^2 \mathrm{R}\cos\Delta &= 1-\frac{2\tan^2 \mathrm{R}}{1+\tan^2 \mathrm{R}/\sin^2 2\beta} \\ &= 1-2\sin^2 2\beta\sin^2\frac{\delta}{2} \\ &= \cos^2 2\beta+\sin^2 2\beta\cos\delta. \end{aligned}$$

Hence

$$\cos 2\gamma \cos 2\theta = \cos 2\alpha(\cos^2 2\beta + \sin^2 2\beta \cos\delta) - \sin 2\alpha \sin 2\beta \sin\delta$$
$$= \cos 2\alpha \cos^2 2\beta + \sin 2\beta \sqrt{\sin^2 2\alpha + \cos^2 2\alpha \sin^2 2\beta}\cos(\delta+\psi)$$
$$\cos 2\gamma \sin 2\theta = \cos 2\alpha \sin 2\beta \sin\delta + \sin 2\alpha \cos\delta$$
$$= \sqrt{\sin^2 2\alpha + \cos^2 2\alpha \sin^2 2\beta}\sin(\delta+\psi),$$

where $\tan\psi = \tan 2\alpha/\sin 2\beta$,

and

$$I = \tfrac{1}{2}c^2[1 + \cos 2\alpha \cos^2 2\beta \cos 2\phi + \sqrt{\sin^2 2\alpha + \cos^2 2\alpha \sin^2 2\beta}\{\sin 2\beta \cos 2\phi \cos(\delta+\psi) + \sin 2\phi \sin(\delta+\psi)\}]$$
$$= \tfrac{1}{2}c^2[1 + \cos 2\alpha \cos^2 2\beta \cos 2\phi + \sqrt{\sin^2 2\alpha + \cos^2 2\alpha \sin^2 2\beta}\sqrt{\sin^2 2\phi + \cos^2 2\phi \sin^2 2\beta}\cos(\delta+\psi-\psi')],$$

where $\tan\psi' = \tan 2\phi/\sin 2\beta$.

This formula contains the theory of the interference phenomena that are obtained when a conical pencil of polarized light traverses the plate and is subsequently analysed.

It may also be employed for the case in which the stream that passes through the polarizer, quartz plate and analyser is a parallel pencil of white light that is investigated with a spectroscope. With a sufficiently thick plate, β may be regarded as constant over a region of the spectrum corresponding to a variation of δ by several multiples of π, and we see that the minima of intensity occur for the radiations for which $\delta = (2n+1)\pi + \Psi$, where

$$\tan\Psi = \frac{\sin 2\beta(\tan 2\phi - \tan 2\alpha)}{\sin^2 2\beta + \tan 2\phi \tan 2\alpha},$$

the intensity then being

$$I = \tfrac{1}{2}c^2\{1 + \cos 2\alpha \cos^2 2\beta \cos 2\phi - \sqrt{1-\cos^2 2\alpha \cos^2 2\beta}\sqrt{1-\cos^2 2\beta \cos^2 2\phi}\}.$$

Thus as the analyser is turned, the dark bands travel along the spectrum, changing their intensity as they progress: they are perfectly black when $\phi = \pi/2 \pm \alpha$, and are least marked when $\phi = 0$ and $\pi/2$.

With a plate perpendicular to the optic axis, $\beta=\pi/4$, and

$$I = c^2\cos^2\frac{\delta-\Psi}{2} = c^2\cos^2\frac{\delta-2(\phi-\alpha)}{2};$$

hence as the analyser is turned, the bands traverse the spectrum, retaining the same intensity.

When the plate is parallel to the optic axis, $\beta=0$ or is at any rate very small, and

$$I = \tfrac{1}{2}c^2\{1+\cos 2\alpha\cos 2\phi+\sin 2\alpha\sin 2\phi\cos\delta\};$$

when $\phi=0$ or $\pi/2$, there are no bands, and in other cases the bands occur where $\delta=(2n+1)\pi$ or $2n\pi$, according to the sign of $\sin 2\alpha\sin 2\phi$. Thus when $0<\alpha<\pi/2$, the bands occur at the points for which $\delta=(2n+1)\pi$ so long as $0<\phi<\pi/2$, their intensity being zero, when $\phi=\pi/2-\alpha$; and as ϕ increases through $\pi/2$, they change their position to that of the former maxima of intensity, becoming perfectly black when $\phi=\pi/2+\alpha$ *.

7. Let us now suppose that the light emergent from the plate is examined with a Savart's analyser.

On leaving the quartz, the stream may be represented by the components

$$(u, v)=c(\cos\gamma,\ -\iota\sin\gamma)\ \mathrm{Exp}\{\iota(nt+\epsilon)\}$$

polarized in planes making angles θ and $\theta+\pi/2$ with the principal section of the plate. Hence if μ be the angle that the principal section of the quartz makes with that of the first plate of the analyser, the stream emerging from the Savart's plate will have components

$$\xi = \{\cos\gamma\cos(\theta+\mu)+\iota\sin\gamma\sin(\theta+\mu)\}\ \mathrm{Exp}\{\iota(nt+\epsilon'+\mathrm{D}/2)\}$$

$$\eta = \{\cos\gamma\sin(\theta+\mu)-\iota\sin\gamma\cos(\theta+\mu)\}\ \mathrm{Exp}\{\iota(nt+\epsilon'-\mathrm{D}/2)\}$$

polarized in the principal sections of the first and second plate, where $\mathrm{D}=\mathrm{D}_2-\mathrm{D}_1$, D_1, D_2 being the relative retardations of phase introduced by the two plates.

* The above results agree with Beaulard's description of the phenomena (*J. de Phys.* (3) ii. p. 399 (1893)). Croullebois, however, gives a different account of the phenomena in the first and last cases (*loc. cit.* p. 391).

On analysation in an azimuth ϕ, these give a stream

$$\begin{aligned}\Xi &= \xi\cos\phi+\eta\sin\phi\\ &=[\{\cos\gamma\cos(\theta+\mu)+\iota\sin\gamma\sin(\theta+\mu)\}\cos\phi\,\mathrm{Exp}\,(\iota\mathrm{D}/2)\\ &\quad+\{\cos\gamma\sin(\theta+\mu)-\iota\sin\gamma\cos(\theta+\mu)\}\sin\phi\,\mathrm{Exp}\,(-\iota\mathrm{D}/2)]\\ &\qquad\qquad\times\mathrm{Exp}\,\{\iota(nt+\epsilon')\},\end{aligned}$$

and the intensity, obtained by multiplying this by the conjugate expression, is

$$\begin{aligned}\mathrm{I} &= \{\cos^2\gamma\cos^2(\theta+\mu)+\sin^2\gamma\sin^2(\theta+\mu)\}\cos^2\phi\\ &\quad+\{\cos^2\gamma\sin^2(\theta+\mu)+\sin^2\gamma\cos^2(\theta+\mu)\}\sin^2\phi\\ &\quad+\{\cos 2\gamma\sin 2(\theta+\mu)\cos\mathrm{D}-\sin 2\gamma\sin\mathrm{D}\}\sin\phi\cos\phi\\ &=\tfrac{1}{2}[1+\cos 2\gamma\cos 2(\theta+\mu)\cos 2\phi\\ &\qquad+\sin 2\phi\{\cos 2\gamma\sin 2(\theta+\mu)\cos\mathrm{D}-\sin 2\gamma\sin\mathrm{D}\}]\end{aligned}$$

and when, as is ordinarily the case, $\phi=\pi/4$

$$\mathrm{I}=\tfrac{1}{2}\{1+\cos 2\gamma\sin 2(\theta+\mu)\cos\mathrm{D}-\sin 2\gamma\sin\mathrm{D}\}.$$

In order, then, that the bands may disappear, we must have

$$\sin 2\gamma=0 \quad\text{and}\quad \sin 2(\theta+\mu)=0\,;$$

that is, the light emergent from the quartz must be plane polarized in the principal section of one of the plates of the Savart's analyser—a result that is obvious from elementary considerations.

Now the light on leaving the quartz is plane polarized (1) when $\delta=2n\pi$, the plane of polarization being then the same as it was initially ; (2) when $\tan\delta/2=-\tan 2(\alpha-\mu)/\sin 2\beta$, α being now the azimuth of the initial plane of polarization with respect to the principal section of the first plate of the analyser, and in this case $\tan 2(\theta+\mu-\alpha)=-\tan 4(\alpha-\mu)$. Hence in the first case, in order that the bands may disappear, the light must be initially polarized in one of the principal sections of the Savart's plate ; and in the second case the principal section of the quartz must be in the azimuth $\mu=k\pi/4+\alpha/2$.

It follows, then, that if the initial polarization be such that there are no bands before the introduction of the quartz plate, there will be a disappearance of the bands whenever the

relative retardation of phase due to the quartz plate is $\delta = 2n\pi$, and if the principal section of the quartz bisect the angle between the principal sections of the Savart's plate *, there will be a further disappearance whenever $\delta = (2n+1)\pi$.

These results afford a method of setting the principal section of a quartz plate perpendicular to the axis of a spectrometer and of determining the error in the cutting of a plate that is supposed to be normal to the optic axis.

The collimator is furnished with a web, placed by the ordinary method perpendicular to the axis of the instrument. The telescope is turned to one side, and a telescope with a Savart's analyser is inserted in its place with its axis in the direction of that of the collimator, and the analyser is adjusted so that, when plane polarized light passes, the fringes are parallel to the web of the collimator. This sets the plane bisecting the angle between the principal sections of the plates of the analyser perpendicular to the turning axis of the spectrometer. This adjustment being made, the polarizer is turned until the fringes disappear exactly in the centre of the field.

The quartz plate is then placed at the centre of the table of the instrument, and by means of the telescope of the spectrometer its faces are made parallel to the turning axis, and the reading determined for which its normal is along the axis of the collimator.

With the polarizer adjusted as above, the fringes will in general be restored when the light traverses the quartz, but a series of positions can be found for which they vanish over a narrow streak extending across the field, and if the quartz plate be turned about its normal so as to bring its optic axis perpendicular to the turning axis of the spectrometer, there will be, intermediate to these, further positions at which the fringes are absent over a small area at the centre of the field.

In this way the principal section of the quartz is set perpendicular to the turning axis of the instrument, and a

* The second of the above cases becomes identical with first when $a=0$, if k be even, and when $a=\pi/2$, if k be odd, since the value of δ then becomes $2n\pi$.

determination of the angles of incidence, for which the centres of the zones of disappearance of the fringes are at the centre of the field, will give by calculation or by the graphic method employed by Beaulard * the direction of the optic axis with respect to the faces of the quartz.

XXXVI. *The Use of the Potentiometer on Alternate Current Circuits.* *By* CHARLES V. DRYSDALE, *D.Sc.*†

ONE of the greatest inconveniences in connexion with alternate current measurements has always been the question of range. Owing to the fact that all instruments employed for direct indication of alternate currents or P.D. have a square law, their range is small and they becomes impracticably delicate when low voltages are concerned. There is therefore a very great need for some instrument which, like the direct-current potentiometer, should be capable of measuring P.Ds. and currents of any range with accuracy. It would also be a great convenience if this instrument were capable of indicating the phase as well as the magnitude of the P.D. or current tested.

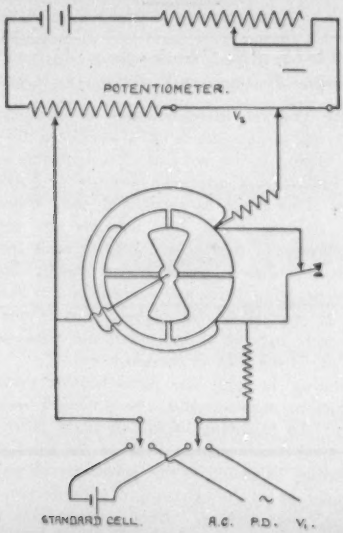
In attempting to apply the potentiometer principle to alternate current measurements, two processes seem to be possible—(*a*) the balancing of the alternate current P.D. against an equal direct current P.D. by the employment of some differential balancing device preferably of an electrostatic character, or (*b*) the balancing of two alternate current P.Ds. against one another. Such propositions as have hitherto been made for an alternate-current potentiometer have, the writer believes, always been based on the former idea, owing to the fact that the latter principle would involve equality of phase as well as of P.D. in the two voltages compared. He understands that some time ago Mr. Swinburne suggested the use of a differential electrometer in connexion with a potentiometer, but has not seen any

* *Thèse de Doctorat*, Marseille, 1893.

† Read January 22, 1904.

details. An obvious device would be that shown in fig. 1, in which the P.D. to be measured is applied between the needle and one pair of quadrants, while that derived from the potentiometer is connected between the needle and the other pair of quadrants, in each case through a considerable

Fig. 1.



resistance, say one megohm. By connecting a key to the opposite pairs of quadrants, which normally short-circuits them, but disconnects them when depressed, a deflexion should be obtained to right or left proportional to $V_1^2 - V_2^2$ or $2VdV$ when balance is nearly obtained. Any want of symmetry in the electrometer should be eliminated by connecting the standard cell as shown, so as to obtain a double weighing method.

The procedure in this case is very simple, and exactly

resembles ordinary direct-current potentiometry. The sole difficulty lies in the electrometer. It should be possible by multicellular devices, or otherwise, to obtain a fair amount of sensitiveness with 1 or 1½ volts, the P.D. of the standard cell, as has been done by Mr. Addenbrooke, but when we come to a tenth of a volt or less, as is so often necessary in current measurement, electrostatic devices seem hopeless. All other magnetic or thermal devices are incorrect in principle as requiring an appreciable current to operate them, which militates against the accuracy of the potentiometer; but by employing a low-resistance potentiometer, and a differential thermal device, such as could be made from the Duddell thermo-galvanometer employing two junctions and two heaters, or two crossed thermo-junctions in series, a sufficiently sensitive and accurate potentiometer could perhaps be made.

In view of the difficulty of making such devices, however, the writer has turned his attention to the second method, *i. e.* of balancing the alternate current P.D. to be measured against a known alternate current P.D., by the interposition of a sensitive alternate-current detector, such as a telephone or vibration galvanometer. This of course implies that the two voltages under comparison must have the same magnitude, frequency, and phase, and, approximately, the same wave-form*.

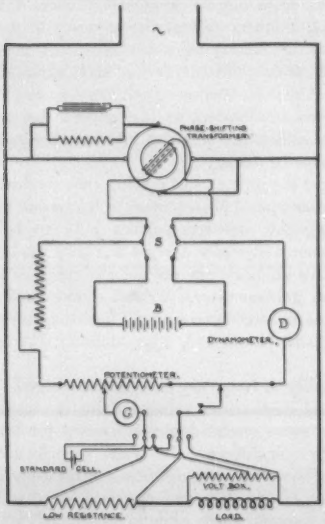
The equality of frequency is of course secured by deriving the test and comparison voltages from the same source of supply, but some special device is needed for bringing the phases into coincidence. In some experiments recently made with the writer's phase-shifting transformer, however †, the accuracy of the phase-variation was found to be so good that it was thought worth while to introduce it into potentiometer work; and fig. 2 shows the diagram of the connexions which have been employed. The phase-shifter is here shown supplied with single phase current through a phase-splitting

* Since this was written the writer has been informed that Dr. Sumpner has somewhere suggested that an alternating-current potentiometer could be made, if any means were available for bringing the phase of the current in the slide-wire into coincidence with that of the P.D. to be measured.

† 'Electrician,' Dec. 11, 1908.

device, which can be built up with it so that it merely requires connecting straight on to the mains. The current derived from the secondary of this transformer traverses the rheostat coils, and slide-wire of an ordinary potentiometer, such as the Crompton form; and also some form of dynamometer-current measuring device. The writer has found

Fig. 2.



a Weston A. C. *voltmeter* excellent for this purpose, as it gives a very good reading with the current (·05 ampere) employed with the Crompton potentiometer. The remainder of the potentiometer is connected up precisely as for ordinary direct current working, except that a vibration galvanometer is substituted for the ordinary form. By having an auxiliary battery B and throw-over switch S, the arrangement is

always ready for either direct-current or alternate-current comparisons.

The procedure is then as follows :—The throw-over switch is first arranged to supply direct current from the battery B, and the rheostat altered until the standard cell is balanced at its nominal value, as in ordinary direct-current working. The dynamometer reading is then carefully noted, or a fiducial mark is made, which always represents the exact ·05 ampere. On now throwing over to alternate current, the rheostat is altered to reproduce the same R.M.S. current as indicated by the dynamometer D ; and the position of the potentiometer contact is shifted and the phase-shifter turned, until the vibration galvanometer shows no vibration. The adjustment is very similar to that of the two resistances in an Anderson's bridge, and presents no difficulties. The voltage and phase are then read off on the potentiometer and phase-shifter respectively. In the illustration a low resistance is connected in series with the load, and a volt-box in parallel with it, and the current and P.D. can be determined directly in the ordinary way. Of course the phase-shifting transformer must always be connected to the same source of supply as the load.

Instead of the vibration galvanometer, a dynamometer or electrometer in which one pair of terminals is connected to the supply so as to be "separately excited," and the other to the ordinary galvanometer terminals, has been employed with good results. The deflexions are then to left or to right as in an ordinary galvanometer. But it must not be forgotten in this case that balance will be obtained not only when the vectors of the two P.Ds. compared are coincident, but also when their vector difference is in quadrature with the P.D. of the supply. For this purpose, if such an instrument is used it must have its "exciting terminals" changed over from one phase to the other, and balance secured in both cases.

Before giving examples of readings obtained in this manner, we must consider what is really measured. It is obvious that this device does not indicate the effective or R.M.S. value of the P.D., except when the supply and tested wave-forms are both sinusoidal or of identical form. When

a vibration galvanometer is employed, its sensitiveness to the fundamental wave is so great in comparison with that to the harmonics, that we shall be practically correct in assuming that it is the fundamental wave only which is measured, and the comparison is really between the mean and not the R.M.S. values. On the other hand, if there is any serious difference of wave-form, the sensitiveness to the upper harmonics should be sufficient to prevent an exact balance being secured, and this warns us when great accuracy is not to be expected.

The effective value of an irregular periodic voltage is

$$\overline{V} = \sqrt{\Sigma \overline{V}_n^2},$$

where $\overline{V}_n$ is the effective value of the harmonic of order n. If the sum of the squares of the higher harmonics is small in comparison with the square of the fundamental, we may take the approximate square root, and

$$\overline{V} = \overline{V}_1 + \frac{1}{2\overline{V}_1} \Sigma \overline{V}_n^2,$$

or

$$\overline{V} = \overline{V}_1 \left\{ 1 + \frac{1}{2\overline{V}_1^2} \Sigma \overline{V}_n^2 \right\},$$

where $\Sigma \overline{V}_n^2$ in the last two expressions implies the sum of the squares of the effective values of all the harmonics, excluding the fundamental.

In the case of a single harmonic this becomes

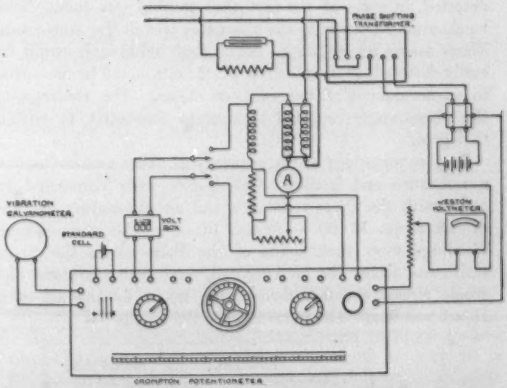
$$\overline{V} = \overline{V}_1 \left\{ 1 + \frac{1}{2} \left(\frac{\overline{V}_n}{\overline{V}_1} \right)^2 \right\},$$

and the expression in brackets only differs 1 per cent. from unity when $\overline{V}_n$ is 14 per cent. of $\overline{V}_1$. Such a difference should be fairly easily recognizable, and would probably be sufficient to prevent the obtaining of a perfect balance on the vibration galvanometer; and, moreover, if the distortion of wave-shape is great, there is little object, as a rule, in great accuracy of measurement. If accuracy is required in such cases, it can of course be secured by tracing the wave-shape and making the above correction, but if care is taken to have a supply of fairly sinusoidal wave-form, the measurements

may be considered sufficiently accurate for the majority of purposes. It should be observed that this difficulty, or a similar one, is inseparable from any method which can be devised, in which the detector is employed to indicate the *difference* between the test and potentiometer P.Ds. as is done with direct currents.

Measurements.—Two illustrations will be sufficient to show the possibilities of the method. An ordinary Crompton potentiometer was used with a Weston dynamometer voltmeter in series with it, and the connexions were made as in fig. 3. A Rubens vibration galvanometer was at first used,

Fig. 3.



but was replaced by a Campbell bifilar instrument, which was found much more sensitive and convenient. A check with direct current and a Cadmium cell showed that the normal current was obtained when the dynamometer indicated 82 "volts," and this reading was reproduced in all the alternate-current measurements.

The first test was a comparison of the readings of a Kelvin Deka-ampere-balance, with the current as given by the P.D. across a Crompton ·04 ω manganin resistance.

The following results were obtained :—

Current by Kelvin.	P.D. by Potentiometer.	Current by Potentiometer.
25	1·002	25·05
20	·8018	20·045
15	·6022	15·055

Frequency of supply 72 ∼.

The readings were obtained with great ease and definiteness, half a division on the slide-wire, or ·0005 volt, being easily detected, in spite of the fact that trouble was found from mechanical vibration, as the alternator was in the same room. There seems no reason to doubt that ·0002 volt could be easily detected, or that a drop of ·1 volt could be measured to an accuracy of ·2 per cent. or closer. The readings of the phase angle repeated themselves constantly to within ·1 degree.

To give some idea of the accuracy of phase measurements, a resistance and ironless choking-coil were connected in series with the ·04 ω resistance, and an alternating current of 15 amps. at 60 ∼ passed through the combination. Readings were then taken of the P.D. across the ·04 ω resistance, across the choking-coil, total resistance, and the whole circuit, the last three being taken by the aid of a 10 × 1 volt box. The following are the results :—

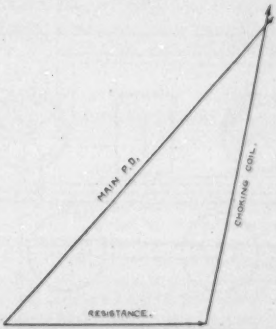
Circuit.	First Test.		Second Test.	
	P.D.	Phase-angle, degrees.	P.D.	Phase-angle.
·04 resistance ...	·5960	46·5	.....	.....
Total resistance .	3·117	47·1	3·108	47·3
Choking-coil	4·880	125·9	4·928	125
Mains	6·185	95·3	6·218	94·9

The difference between the first and second readings is due

to the fact that a carbon resistance was employed which became heated.

Fig. 4 is a graphical representation of the result in the last case, the horizontal line representing the resistance drop, and the other two lines the drop across the choking-coil and whole circuit respectively. Although the triangle does not quite close, the result may be taken as satisfactory in consideration of the fact that single phase supply was used in

Fig. 4.



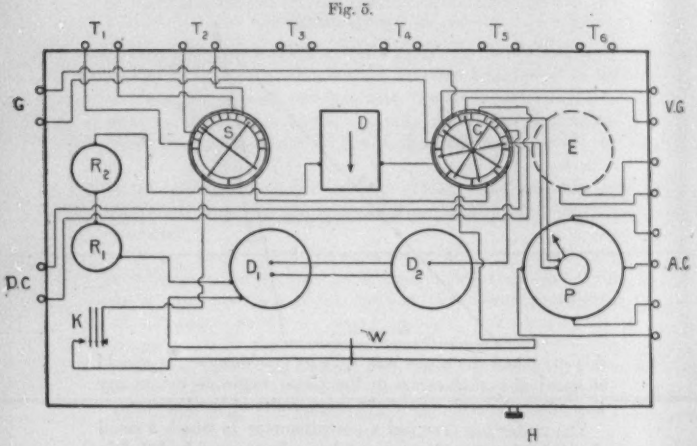
this case, and the phase was split by condensers. It should be noted also that errors in the phase angle do not in any way affect the accuracy of the magnitudes of the P.Ds.

The writer has designed a potentiometer in which a small phase-shifting transformer and dynamometer are included in the case, with a throw-over switch, permitting the intrument to be used as either a direct or alternate current potentiometer.

Fig. 5 shows the connexions, which need little explanation. The main circuit of the potentiometer, consisting of the rheostats R_1 and R_2, the two dials D_1 and D_2, the side-wire W, and the dynamometer D, is connected to two of the blocks of the change-over switch C, which is similar in form to the well-known selector switch used in the Crompton Potentiometer. The derived P.D. from the dial D_2 and slide-wire contact is led by the key K and the selector switch S to two more of the contacts on C. The remaining eight contacts on

C are connected in pairs to the phase-shifting transformer P, the battery terminals D.C., the ordinary galvanometer terminals G, and the vibration galvanometer terminals V.G. The phase-shifter primary is excited from the terminals A.C., which are here shown four in number, for two-phase or split-phase. A single motion of the switch C to right or left connects the main circuit of the potentiometer on to the battery or secondary of the transformer, and simultaneously

Fig. 5.



connects the sliding contacts through the selector switch to the appropriate galvanometer. The key K being provided with a holding-down cam, leaves the hands free to move the potentiometer contacts and rotate the phase-shifter simultaneously. A frequency meter F of the vibrating reed type may be fixed on the base if desired, in which case the instrument is available for inductance and capacity measurements.

The dynamometer D is not provided with a scale, but with a fiducial mark which can be slightly adjusted to suit the standard cell-check. It will also be made astatic to avoid risk of error from stray field.

This apparatus should be capable of giving very good

results. In the meantime the results already obtained appear to justify the belief that this method is capable of being used with considerable accuracy and convenience for all P.D., current, and phase measurements.

The numerous applications of such an instrument are obvious, and include P.D., current, phase, and power measurement over almost any range ; inductance and capacity test, and determination of dielectric losses, &c. One of the most valuable applications, however, is to search-coil work with alternate current, whereby the distribution of main and leakage fluxes in cores, and the propagation of magnetic impulses can be studied, and the variation of current and P.D. along a cable fed with alternate currents.

The writer's thanks are heartily tendered to Mr. A. C. Jolley for setting up the apparatus and great assistance in the experiments.

Discussion.

Mr A. Campbell remarked that the first method suggested by the author had already been described by Mr. J. Swinburne (Phys. Soc. Dec. 1893). It was also used with success at the Reichsanstalt in an improved form by Dr. Drewall (Zeitschr. für Instrumentenkunde, April 1903). Dr. Drysdale's phase-turning method was interesting and would give good results in many cases. The vibration galvanometer, however, was so very much more sensitive for its own frequency than for others, that it would not be likely to give any indication of errors due to other harmonics unless they were very pronounced. Hence the method must be used with great caution.

Mr W. Duddell expressed his interest in the paper, and asked if the limit of accuracy depended on the sensitiveness of the galvanometer and if this was limited by the back E.M.F. of the instrument. He pointed out that the harmonics could be investigated by tuning up the galvanometer to be in unison with them. The device by which the phase was changed without changing the current was very useful.

Mr Rayner pointed out that the accuracy attainable depended on the sensitiveness of the Weston voltmeter employed. A vital point in the instrument was the phase-shifter with the sinusoidal windings. By using two galvanometers the fundamental and a harmonic could both be measured.

Mr. Paterson joined with others in expressing appreciation of the paper. Dr. Drysdale claimed that low resistances dropping 0·1 volt could be used with this potentiometer. He presumed that the object of measuring the phase displacement between the two E.M.F.s was to obtain a value for the watts in the circuit. In this case the employment of low resistances was a great disadvantage on account of the large time

constant which these shunts necessarily have, even though constructed with the utmost care. He thought the failure of the vector diagram exactly to close was more likely to be due to some such error coming in to the measurement of the phase-angle than to any inaccuracy in the Kelvin Balance—as hinted by the Author.

The AUTHOR, in reply, said that he regretted that time had not permitted his making further investigations into previous work in this direction, but he had alluded to Mr Swinburne's suggestions, and had claimed no credit for the electrostatic device. Mr Campbell's criticisms mainly dealt with wave-form errors. Of course where iron cores at considerable saturation were employed, the distortion of wave-shape would be considerable, but with the single exception of power measurement which should then be made with a wattmeter, he knew of no object in such cases for accurate P.D. or current measurements. For ordinary standardisation of P.D. and current, this difficulty did not present itself, as there was no difficulty whatever in obtaining a sufficiently closely sinusoidal wave-form, and the wave-forms of the measured and balancing P.Ds. were usually similar, being derived from the same source. The use of a second vibration galvanometer tuned to the third harmonic would be excellent if it was desired to measure them, but he believed that it would rarely be necessary, and there should be no difficulty in making measurements to an accuracy of 0·1 per cent. with the ordinary alternators and resistances. For inductance and capacity measurements there was of course a positive advantage in an instrument which disregarded the harmonics. In reply to Mr Duddell, there was no doubt that the vibration galvanometer did give a back E.M.F., as was shown by a telephone connected to its terminals when the coil was set in motion, and this would probably increase the damping, but he did not know the amount of the influence on the sensitiveness. He was glad the importance of the sinusoidal winding of the phase-shifting transformer was appreciated, and hoped that it would be found effective in preserving perfect equality of the potentiometer current as the phase was varied. It was true that slots or tunnels were employed in the stator core, but they were of considerable number, and the rotor and stator pitches were incommensurable. As to accuracy, it was true that this was limited by the dynamometer instrument, but this was fairly sensitive as it was always used at its best range, and it was to be doubted whether it was possible under any ordinary circumstances to make alternate current measurements to within an accuracy of a few parts in ten thousand, owing to variations in the alternator speed or E.M.F. Finally, he was certain that no ordinary low resistance standards would give any appreciable inductive errors. To produce an error in magnitude of ·1 per cent. would require a phase displacement of about 2·7 degrees, which was only slightly exceeded by helical iron wire resistances at 50~, and the straight strip resistances of the Reichsanstalt form were almost the perfection of non-inductive resistances. In any case inductance would not affect the closing of the vector triangle.

XXXVIII. *Note on the Luminous Efficiency of a Black Body.* By CHARLES V. DRYSDALE, *D.Sc.**

AT the present moment, when the question of light production is so much to the front, it is of interest to ascertain the possibilities of various substances as incandescent bodies. Until comparatively recently nearly all artificial illumination has been derived from the incandescence of carbon in some form or other, and consequently from a substance which approximates to a perfectly black body. It is therefore of interest to ascertain the possibilities of such a body as a light radiator at various temperatures, and this can be readily obtained from the radiation formulæ.

The best known and most convenient law for calculation is that of Wien †,

$$R_\lambda = C_1 \lambda^{-5} \epsilon^{-\frac{C_2}{\lambda T}}, \quad \ldots \ldots \quad (1)$$

where λ is the wave-length, T the absolute temperature, R_λ the intensity of radiation corresponding to the wave-length λ, and C_1 and C_2 constants. The value of C_2 according to Paschen & Wanner ‡ should be 14,400, and to Lummer & Pringsheim § 14,700 when λ is in microns, and T in degrees centigrade.

Wien's law is not quite as exact as that of Max Planck ‖

$$R_\lambda = C_1 \lambda^{-5} / (\epsilon^{\frac{C_2}{\lambda T}} - 1),$$

but only differs from it by 1 per cent. when λT is greater than $C_2/4{\cdot}6$ or about 3000. It may therefore be used for most calculations.

The energy radiated between any two wave-lengths is therefore

$$\int_{\lambda_1}^{\lambda_2} R_\lambda d\lambda = \frac{C_1}{C_2} T \left[\epsilon^{-\frac{C_2}{\lambda T}} \left\{ \frac{1}{\lambda^2} + \frac{3T}{C_2 \lambda^2} + \frac{6T^2}{C_2^2} + \frac{6T^3}{C_2^3} \right\} \right]_{\lambda_1}^{\lambda_2}, \quad (2)$$

* Read January 22, 1909.

† *Berl. Ber.* 9th February, 1893.

‡ *Berlin Akad. Sitzungsberichte*, ii. pp. 5–11, 1899.

§ *Verh. d. deutsch. Phys. Ges.* 1899, p. 23.

‖ *Ann. der Phys.* i. 1900.

and the total radiation

$$R = \int_0^\infty R_\lambda d\lambda = 6\frac{C_1}{C_2^4}T^4,$$

corresponding to the Stefan law

$$R = \sigma T^4. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence

$$6\frac{C_1}{C_2^4} = \sigma \quad \text{and} \quad C_1 = \frac{\sigma C_2^4}{6}.$$

Several determinations of the constant σ have been made, but that of Kurlbaum * is most generally accepted. His value is $5{\cdot}32 \times 10^{-12}$ watts per sq. cm. Combining this with Lummer & Pringsheim's value of 14700 for C_2 we have $C_1 = 41{,}400$.

At the same time by logarithmic differentiation

$$\frac{1}{R_\lambda} \cdot \frac{\partial R_\lambda}{\partial \lambda} = -\frac{5}{\lambda} + \frac{C_2}{\lambda^2 T}$$

from which we have the well-known relation

$$\lambda_{max.}T = \frac{C_2}{5} \quad \text{or} \quad \lambda_{max.} = \frac{C_2}{5T},$$

where $\lambda_{max.}$ is the dominant wave-length.

Hence for numerical calculation we have the following formulæ:—

Total radiation

$$R = \int_0^\infty R_\lambda d\lambda = 5{\cdot}32 \times 10^{-12}T^4 \text{ watts per sq. cm.} \quad (4)$$

$$R_\lambda = 41400\,\lambda^{-5}\epsilon^{-\frac{14700}{\lambda T}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\lambda_{max.}T = 2940. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\int_{\lambda_1}^{\lambda_2} R_\lambda d\lambda = 2{\cdot}82T\left[\epsilon^{-\frac{14700}{\lambda T}}\left(\frac{1}{\lambda^3} + 2{\cdot}04 \times 10^{-4}\frac{T}{\lambda^2} + 2{\cdot}78 \times 10^{-8}\frac{T^2}{\lambda} + 1{\cdot}89 \times 10^{-12}T^3\right)\right]_{\lambda_1}^{\lambda_2}. \quad (7)$$

To obtain the radiant luminous efficiency we have only to

* *Ann. Phys. Chem.* lxv. 4. pp. 746–748, 1898.

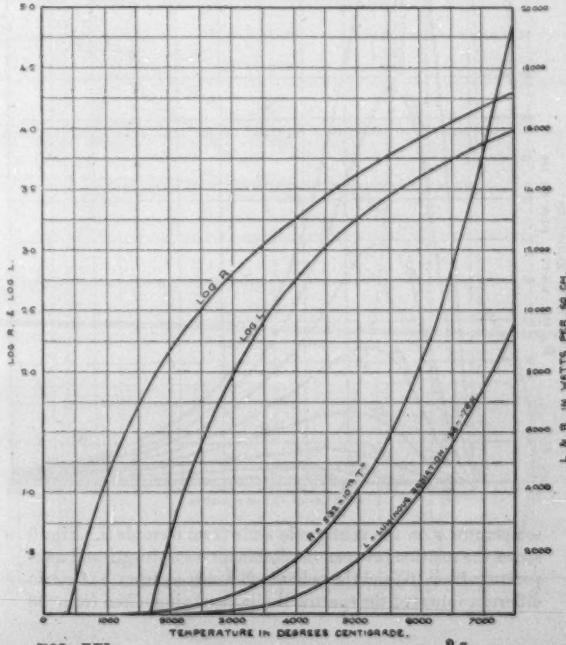
calculate the value of the last integral between the spectral limits and divide by the total radiation R, and we have

Radiant efficiency η_R

$$= 5{\cdot}3 \times 10^{11} \left[\epsilon^{-\frac{14700}{\lambda T}} \left(\frac{1}{\lambda^2 T^3} + \frac{2{\cdot}04 \times 10^{-4}}{\lambda^2 T^2} + \frac{2{\cdot}78 \times 10^{-8}}{\lambda T} + 1{\cdot}89 \times 10^{-12} \right) \right]_{\lambda_1}^{\lambda_2} \quad . \quad (8)$$

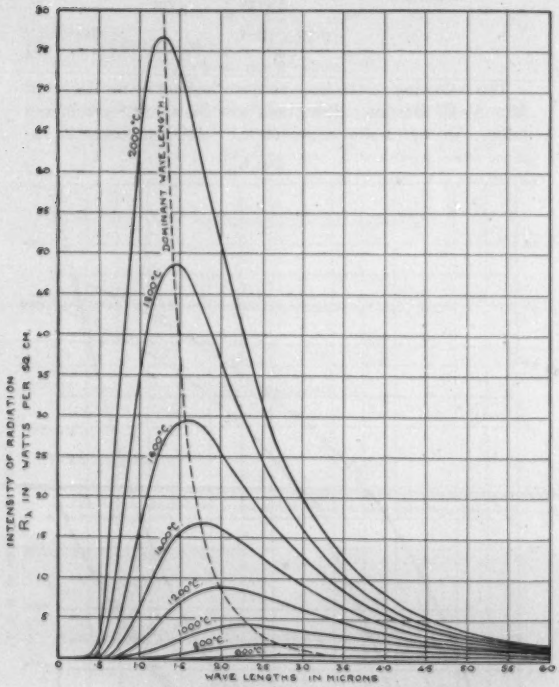
The following results have been worked out by my Assistant Mr. A. F. Burgess, B.Sc., and are shown in the annexed table. In fig. 1 the values of the total and luminous ra-

Fig. 1.



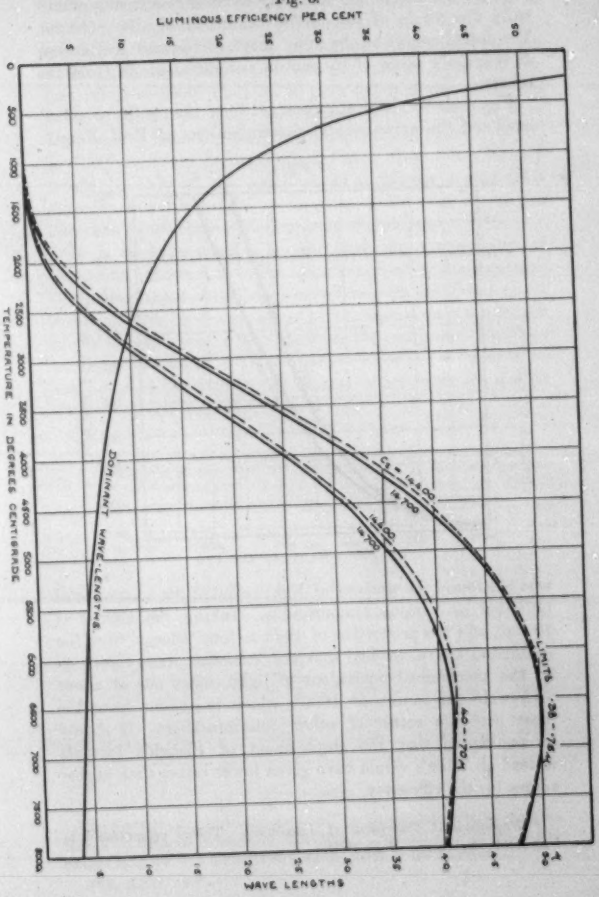
diation and their logarithms are plotted against the ordinary temperature. Fig. 2 shows the relation between the intensity of radiation in watts per sq. cm. and wave-length for various

Fig. 2.



temperatures on the centigrade scale from formula 5. Fig. 3 shows the relation between the dominant wave-length and temperature from (6) and the radiant efficiency η_R from (8) for two different values of the spectral limits, one being taken from ·38

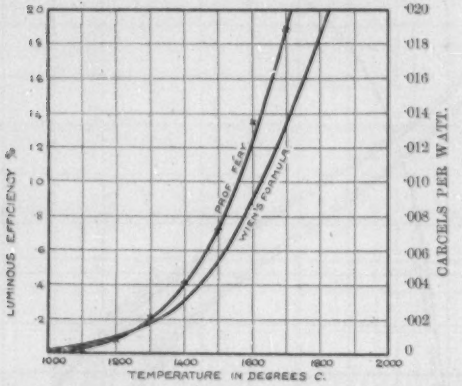
Fig. 3.


LUMINOUS EFFICIENCY PER CENT
TEMPERATURE IN DEGREES CENTIGRADE
WAVE LENGTHS
DOMINANT WAVE-LENGTHS
Cp = 14400 / λT
.38 - .72 μ
.40 - .70 μ

to ·76μ, the other from ·4 to ·7μ, the latter comprising practically the whole of the useful luminosity of the spectrum. The dotted curves on the same sheet correspond to Paschen & Wanner's value of C_2 and do not differ much from the others.

Fig. 4 shows the comparison between the results so calculated and the experimental determinations of Prof. Féry *,

Fig. 4.



who has found the amount of light radiated per watt of total radiation at various temperatures. Taking the Carcel as 10 c.p., and the proportion of light to total energy from the calculated curve, we find that the luminous energy per c.p. or the mechanical equivalent of light comes out at about ·075 watt per candle @ 1600°, which is not far from the most probable result of other determinations. It should be mentioned that the employment of Planck's formula instead of Wien's would have given lower rather than higher values for the efficiency.

* *Rayonnement Calorifique et Lumineux.* Thèses présentées à la Faculté des Sciences. Paris 1902.

These curves show at a glance the enormous extent to which the luminous efficiency is dependent upon temperature, and how extremely low it is at ordinary temperatures. At 1500° C. the efficiency is only of the order of 1 per cent. or less, while at 2000° C. it is still only about 3 per cent. In order to get anything like an efficient source of light by pure temperature radiation therefore, temperatures of from 4000° to 6000° C. would be required, and even then the efficiency would be only of the order of 30 to 50 per cent. This strongly points therefore to the necessity for working in the direction of selective radiation or luminescence.

It is perhaps worthy of note that the temperature of maximum efficiency is in the neighbourhood of the temperature of the sun as found by Prof. Féry, viz. 5500° C., which is due to the close agreement of the dominant wave-length of the sun with that which produces the maximum luminous effect on the retina. This correspondence is doubtless to be attributed to the influence of natural selection on the eye, as Prof. Féry has himself pointed out.

Temp^{r} 0° C.	λ max. μ.	Total Radiation Watts/sq. cm.	Luminous Radiation Watts/sq. cm.		Luminous Efficiency per cent.	
			·4—·7 μ.	·38—·76 μ.	·4—·7 μ.	·38—·76 μ.
1000 ...	2·31	13·97	$8{\cdot}653\times10^{-4}$	$2{\cdot}525\times10^{-3}$	·0062	·0181
1500 ...	1·66	52·57	·1358	·2766	·258	·526
2000 ...	1·29	142·01	2·534	4·264	1·784	3·003
2500 ...	1·06	314·5	17·47	25·96	5·554	8·253
3000 ...	·90	610·5	69·39	98·24	11·367	16·092
3500 ...	·78	1078·1	195·5	257·7	18·137	23·919
4000 ...	·69	1773·4	452·2	562·8	25·498	31·737
4500 ...	·615	2760·8	849·5	1060·3	30·771	38·405
5000 ...	·56	4112·5	1454·0	1789·0	35·355	43·499
5500 ...	·51	5909·0	2280·2	2779·6	38·589	47·041
6000 ...	·47	8237·3	3339·8	4051·0	40·544	49·180
6500 ...	·435	11194	4639·6	5606·3	41·446	50·081
7000 ...	·405	14883	6173·5	7442·5	41·477	50·003
7500 ...	·38	19428	7931·7	9536·2	40·827	49·087
8000 ...	·355	24921	10079	11920	40·449	47·835

Discussion.

The Secretary asked the Author if he would state exactly what he meant by the term "luminous efficiency." The ratio of luminosity to energy radiated was different in different parts of the spectrum.

Dr RUSSELL thought that the physiological effect on the eye ought to be considered. If the paper had been entitled "The radiation efficiency, for rays contained within the limits of the visible spectrum, of a black body" it would perhaps have been described better. The sensibility of the eye to light rays varied not only with the wave-length but also with the intensity of the rays and the time during which they had been acting on the retina. If we considered only the objective stimulus of the luminosity, the Author's implied definition might be, he thought, accepted. But if we consider, as we ought, that the subjective sensation produced in the normal eye has to be taken into account when judging luminous efficiency, another definition is required and the problem becomes far more complicated.

The AUTHOR, in reply, said that the term "luminous efficiency" was not very generally understood. There were three modes of defining it, of which two were due to Prof. Nichols of Cornell University, who distinguished between "total efficiency" and "radiant efficiency." The former of these quantities was the ratio of the luminous energy radiated between the spectral limits to the total energy consumption of the source of light, and the latter the ratio of the luminous energy as above to the total radiation from the source. As had just been pointed out, however, neither of these definitions took any cognizance of the very different luminosities of the various spectral colours, and this had been realized by Dr C. Guilleaume, who had proposed to correct the luminous radiation by multiplying each ordinate of the radiation curve by a factor depending on the luminosity at that ordinate, which was unity at the point of maximum effectiveness ($\cdot 54\,\mu$). The Author had proposed (Proc. R. S. 1908) the term "reduced" total or radiant efficiency, for the efficiency expressed on this basis, which was evidently the most rational one. The term "radiant luminous efficiency" was, however, recognized and had been employed correctly in the paper.

XXXIX. *The Effective Resistance and Inductance of a Concentric Main, and Methods of Computing the Ber and Bei and Allied Functions.* By ALEXANDER RUSSELL, *M.A., D.Sc.**

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I. INTRODUCTION.

IN connexion with the investigation of certain phenomena which occur when alternating currents of high frequency flow in thick wires, a knowledge of how the effective resistances and inductances of these wires vary with the frequency is most important. It is of little use to have "standard" inductances in high frequency circuits when we do not know how their values alter with the frequency, and therefore also with the wave shape of the alternating currents. The "measurement" of inductances by means of alternating currents of unknown wave shape often leads to waste of time. Even

* Read January 22, 1909.

when the wave shape is similar to a sine curve, yet if we do not know how the inductance will vary with the frequency the results have only a very limited application.

It would obviously be extremely useful to have formulæ which would take into account the appreciable variation of the density of the current which occurs over the cross section of the wires with high frequency currents. Even to fix inferior and superior limits to the possible values of the inductance would be a great help in many cases. Unfortunately the mathematical difficulties in the way of arriving at a solution in the case of a helical coil are very great. The author, therefore, has made a study of the simplest problem of all, namely that of a concentric main, as a preliminary to attacking the more difficult problems. A study of this problem is also of importance at the present time * in connexion with the discussion that is taking place amongst electrical engineers as to the magnitude of the skin losses in power transmission cables.

The problem was first discussed by Maxwell †. He obtains a few of the terms of a series by means of which the effective resistance of the inner conductor can be computed at low frequencies. Apparently, however, he did not fully appreciate the importance of the results given by the formula. In May ‡ 1884, Oliver Heaviside discussed the "throttling" effect in a core, that is, the increased resistance, the reduced inductance, and the tendency to surface concentration. He uses two functions M and N in his solution, which Kelvin subsequently called the ber and bei functions. In January § 1885, he described clearly the true nature of the current flow in a wire, laying particular stress on the initial surface effects and subsequent penetration. Lord Rayleigh ‖, adopting

* See the report of the evidence on the London Electric Power Bills given before Sir Luke White's Committee in the House of Lords (Nov. 1908).

† 'Electricity and Magnetism,' vol. ii. § 690.

‡ 'The Electrician,' p. 583, May 3, 1884, or 'Electrical Papers,' vol. i. p. 353.

§ 'The Electrician,' Jan. 3, 1885, or 'Electrical Papers,' vol. i. p. 429.

‖ Phil. Mag. xxii. pp. 381–394 (1886), or 'Scientific Papers,' vol. iii. p. 86.

Maxwell's method, next discussed the problem and gave a formula for the effective resistance of the inner conductor at very high frequencies. Oliver Heaviside * subsequently made most important contributions to our knowledge of the subject. He was the first, for example, to give the approximate formula† for the effective resistance of a hollow cylindrical conductor carrying a low frequency current. This formula is a particular case of the general formula given in this paper. He also gives general descriptions of how the current-density varies in the conductors. He omits so many steps, however, in some places that it is very laborious to follow his reasoning. Three years afterwards Lord Kelvin ‡ gave a practical solution for the effective resistance of a solid inner conductor. It is virtually the same as that given by Heaviside. He gave a table, which we shall examine later, of the numerical values in important practical cases. Sir Joseph Thomson § also gives practical formulæ for the effective resistance and inductance of a concentric main having a solid inner conductor when traversed by very high frequency currents.

The complete solution given in this paper is obtained from elementary electrical considerations, a knowledge of Ohm's law and of Faraday's law of induction being all that is assumed. The author proves the mathematical formulæ at length, as most of them are new and some of them will be helpful in other physical problems. It will also enable any slips he may have made in the algebraical work to be readily detected and easily rectified. He shows, however, that from his solutions all the previous solutions can be readily deduced, and as most of them are complex functions of the electrical and geometrical data of the main, the errors, if any, must be very minor ones.

* 'Electrical Papers,' vol. ii. p. 64 *et seq.*

† *L. c. ante*, p. 192, formula (72).

‡ Journ. of the Inst. of. El. Eng. vol. xviii. p. 4 (1889) or Math. and Phys. Papers, vol. iii. p. 491.

§ 'Recent Researches,' p. 295.

II. Mathematical Formulæ.

1. *The Differential Equation.*

The differential equation * to which our problem leads is

$$\frac{\partial^2 i}{\partial r^2}+\frac{1}{r}\frac{\partial i}{\partial r}=\frac{m^2}{\omega}\frac{\partial i}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad (A)$$

where m and ω are constants and i is a periodic function. If we assume that i varies according to the harmonic law, and that its frequency is $\omega/2\pi$, we may write $i=ue^{\omega t\iota}$, where u is a function of r but not of t, and ι stands for $\sqrt{-1}$.

The equation now becomes

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-m^2\iota u=0. \quad . \quad . \quad . \quad . \quad (B)$$

The solution of this equation is known † to be

$$u=\mathrm{A}\,.\,\mathrm{I}_0(mr\sqrt{\iota})+\mathrm{B}\,.\,\mathrm{K}_0(mr\sqrt{\iota}),$$

where A and B are constants, and $\mathrm{I}_0(x)$ and $\mathrm{K}_0(x)$ can be computed by means of the following series :—

When x is small,

$$\mathrm{I}_0(x)=1+\frac{x^2}{2^2}+\frac{x^4}{2^2\,.\,4^2}+\ldots\ldots \quad . \quad . \quad . \quad (1)$$

and

$$\mathrm{K}_0(x)=\alpha\,.\,\mathrm{I}_0(x)-\log x\,.\,\mathrm{I}_0(x)+\frac{x^2}{2^2}+\left(1+\frac{1}{2}\right)\frac{x^4}{2^2\,.\,4^2}+\ldots \quad (2)$$

where $\alpha=\log 2-\gamma$ and γ is Euler's constant, and so

$$\alpha=0{\cdot}1159315.$$

When x is large,

$$\mathrm{I}_0(x)=\frac{\epsilon^x}{\sqrt{2\pi x}}\left\{1+\frac{1^2}{8x}+\frac{1^2\,.\,3^2}{2(8x)^2}+\ldots\ldots\right\}, \quad . \quad . \quad (3)$$

and

$$\mathrm{K}_0(x)=\sqrt{\frac{\pi}{2x}}\,.\,\epsilon^{-x}\left\{1-\frac{1^2}{8x}+\frac{1^2\,.\,3^2}{2(8x)^2}-\ldots\ldots\right\}. \quad (4)$$

The values of the series (2) and (4), for many values of x,

* This equation was first discussed by Joseph Fourier [*Mémoires de l'Académie*, Tome iv. (for the year 1819)].

† See Gray and Matthews, 'Bessel's Functions.'

have been computed to a high degree of accuracy by W. S. Aldis *, and tables of $I_0(x)$ from 0 to 5·1 are given in the British Assoc. Reports, 1896, the interval of the argument being 0·001.

2. *Kelvin's* ber *and* bei *functions.*

Kelvin † showed that the effective resistance of the inner conductor of a concentric main may be conveniently expressed in terms of two functions which he called the ber and the bei functions ‡. He published tables of the values of these functions for a few values of the argument, and from these tables he computed the numerical value of the effective resistance in various cases.

We may define Kelvin's functions by means of the equation

$$I_0(mr\sqrt{\imath}) = \mathrm{ber}\, mr + \imath\, \mathrm{bei}\, mr.$$

From (1) we deduce at once that

$$\mathrm{ber}\, mr = 1 - \frac{m^4r^4}{2^2 \cdot 4^2} + \frac{m^8r^8}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} - \ldots \quad . \quad . \quad (5)$$

and

$$\mathrm{bei}\, mr = \frac{m^2r^2}{2^2} - \frac{m^6r^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots, \quad . \quad . \quad . \quad . \quad (6)$$

which are the definitions of them given by Kelvin.

In Mascart and Joubert's *l'Electricité et le Magnetisme*, vol. i. p. 718, several of the values of the functions given in Kelvin's paper have been recomputed and certain corrections made. As the author uses these functions in the solutions given below, it was necessary therefore to recheck the calculations. He at first attempted to do this by direct calculation, but the work proved so laborious that he was led to devise shorter methods of calculating the functions. He found that this was easy and that comparatively simple formulæ can be obtained for them in those cases where the direct computation by (5) and (6) would be laborious.

* Proc. Roy. Soc. vol. lxiv. p. 203.

† *L. c. ante.*

‡ Noticing that in Heaviside's notation, ber = M and bei = N, Kelvin's formula follows at once from the formula (36) given in Heaviside's 'Electrical Papers,' vol. ii. p. 183.

The formulæ given below can also be usefully employed in simplifying the formulæ ordinarily given for computing the eddy-current losses * in a metallic cylinder, the inductance and resistance of two parallel cylindrical conductors †, the impedance of a solenoid with a cylindrical metal core ‡, &c.

3. *Approximate formulæ for the* ber *and* bei *functions.*

When the argument is small the functions can be readily computed from the formulæ (5) and (6). These functions generally occur associated together in one or other of the following ways :—

$$X(x)=\mathrm{ber}^2x+\mathrm{bei}^2x, \qquad Y(x)=\mathrm{ber}'^2x+\mathrm{bei}'^2x,$$
$$Z(x)=\mathrm{ber}\,x\,\mathrm{ber}'\,x+\mathrm{bei}\,x\,\mathrm{bei}'\,x,$$

and $$W(x)=\mathrm{ber}\,x\,\mathrm{bei}'\,x-\mathrm{bei}\,x\,\mathrm{ber}'\,x.$$

In these definitions of X, Y, Z and W, ber′ x and bei′ x stand for the differential coefficients of ber x and bei x with respect to x. The combinations Y/X, Z/X, W/X, Z/Y, and W/Y also occur in this and allied problems, and so we shall give formulæ for these functions as well.

By squaring the series for ber x and bei x and adding them together, we get

$$X=1+\frac{1}{2}\left(\frac{x}{2}\right)^4+\frac{1}{\underline{4}\,\underline{4}}\left(\frac{x}{2}\right)^8+\frac{1}{6^2\,\underline{6}}\left(\frac{x}{2}\right)^{12}+\frac{1}{8^2\,\underline{9}}\left(\frac{x}{2}\right)^{16}, \quad (7)$$

approximately.

When x is not greater than 4 this formula may be used. For instance, when x is 4, (7) gives X(4)=11·8275. From the tables given in Gray and Matthews' 'Bessel's Functions' where the values of ber x and bei x are tabulated for values of x up to 6, the interval of the argument being 0·2, we find that ber 4 = −2·56342 and bei 4 = 2·29269, and thus X(4) = ber² 4 + bei² 4 = 11·8275.

* A. Russell, 'Alternating Currents,' vol. i. p. 374.

† J. W. Nicholson, Phil. Mag. [6] xvii. p. 255, 1909.

‡ O. Heaviside, 'Electrical Papers,' vol. i. and ii., or R. T. Wells, Phys. Rev. xxvi. p. 357 (1908).

Similarly we get the approximate formulæ :—

$$Y = \frac{x^2}{4}\left\{1+\frac{3}{(3)^2}\left(\frac{x}{2}\right)^4+\frac{10}{(5)^2}\left(\frac{x}{2}\right)^8+\frac{35}{(7)^2}\left(\frac{x}{2}\right)^{12}\right\}, \quad . \quad . \quad (8)$$

$$Z = \frac{x^3}{16}\left\{1+\frac{1}{\lfloor 4}\left(\frac{x}{2}\right)^4+\frac{1}{\lfloor 6}\left(\frac{x}{2}\right)^8+\frac{1}{4\,.\,12^2\,.\,\lfloor 7}\left(\frac{x}{2}\right)^{12}\right\}, \quad (9)$$

and

$$W = \frac{x}{2}\left\{1+\frac{6}{(3)^2}\left(\frac{x}{2}\right)^4+\frac{30}{(5)^2}\left(\frac{x}{2}\right)^8+\frac{140}{(7)^2}\left(\frac{x}{2}\right)^{12}\right\}. \quad . \quad (10)$$

When x is not greater than 2 the following formulæ can be employed :—

$$\frac{Y}{X} = \frac{x^2}{4}\left\{1-\frac{5}{12}\left(\frac{x}{2}\right)^4+\frac{143}{720}\left(\frac{x}{2}\right)^8-\frac{7661}{4^2\,\lfloor 7}\left(\frac{x}{2}\right)^{12}\right\}, \quad . \quad . \quad (11)$$

$$\frac{Z}{X} = \frac{x^3}{16}\left\{1-\frac{11}{24}\left(\frac{x}{2}\right)^4+\frac{473}{3\,\lfloor 6}\left(\frac{x}{2}\right)^8-\frac{304107}{4\,.\,12^2\,\lfloor 7}\left(\frac{x}{2}\right)^{12}\right\}, \quad . \quad (12)$$

$$\frac{W}{X} = \frac{x}{2}\left\{1-\frac{1}{3}\left(\frac{x}{2}\right)^4+\frac{19}{\lfloor 5}\left(\frac{x}{2}\right)^8-\frac{687}{7\,.\,6^4}\left(\frac{x}{2}\right)^{12}\right\}, \quad . \quad . \quad (13)$$

$$\frac{Z}{Y} = \frac{x}{4}\left\{1-\frac{1}{24}\left(\frac{x}{2}\right)^4+\frac{13}{4320}\left(\frac{x}{2}\right)^8-\frac{647}{12^2\,.\,360\,.\,56}\left(\frac{x}{2}\right)^{12}\right\}, \quad (14)$$

and

$$\frac{W}{Y} = \frac{2}{x}\left\{1+\frac{1}{12}\left(\frac{x}{2}\right)^4-\frac{1}{180}\left(\frac{x}{2}\right)^8+\frac{11}{12\,.\,28\,.\,30}\left(\frac{x}{2}\right)^{12}\right\}. \quad (15)$$

Formula (15) agrees with that found by Lord Rayleigh * by another method. Heaviside † gives $73/(12^2\,.\,28\,.\,80)$, that is $657/(12^2\,.\,360\,.\,56)$ as the coefficient of $(x/2)^{12}$ in (14).

Formulæ (7) to (10) should not be used if x is greater than 4, and formulæ (11) to (15) should not be used if x is greater than 2. In practice, therefore, they have only a limited use.

We shall now find approximate formulæ for $\mathrm{ber}\,x$, $\mathrm{bei}\,x$, X, Y, Z, and W, which can be used with sufficient accuracy for practical purposes when x is not less than 5, and can

* *L. c. ante.*

† 'Electrical Papers,' vol. ii. p. 64. [Dr. Heaviside has written to me that he discovered this slip in 1894.]

always be used for computing with a maximum inaccuracy of less than 1 in 10,000, when x is not less than 10, that is in those cases where the labour involved in the direct computation of the series becomes practically prohibitive.

If y denote $\mathrm{ber}\,x+\iota\,\mathrm{bei}\,x$ we see from the definition we gave of these functions that

$$\frac{\partial^2 y}{\partial x^2}+\frac{1}{x}\frac{\partial y}{\partial x}=\iota y. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Putting $y=\mathrm{A}'\epsilon^{\theta}/\sqrt{x}$, where A' is a constant, we get

$$\frac{\partial^2\theta}{\partial x^2}+\left(\frac{\partial\theta}{\partial x}\right)^2-\iota+\frac{1}{4x^2}=0. \quad . \quad . \quad . \quad (17)$$

When x is large,

$$\theta=x\sqrt{\iota}+a_0+b_0\iota$$

is obviously an approximate solution of (17), where a_0 and b_0 are constants. Let us assume therefore that

$$\theta=x\sqrt{\iota}+a_0+b_0\iota+a_1/x+a_2/x^2+\ldots$$

is a solution of (17). Substituting this value for θ, equating the coefficients of x^{-2}, x^{-3}, ... to zero, and noticing that $\sqrt{\iota}=1/\sqrt{2}+(1/\sqrt{2})\iota$, and $1/\sqrt{\iota}=1/\sqrt{2}-(1/\sqrt{2})\iota$, we find that

$$a_1=\frac{1}{8\sqrt{2}}-\frac{\iota}{8\sqrt{2}}, \qquad a_2=-\frac{\iota}{16},$$

$$a_3=-\frac{25}{384\sqrt{2}}-\frac{25\,\iota}{384\sqrt{2}}, \qquad a_4=-\frac{13}{128},$$

&c.

Hence we may write

$$y\sqrt{x}=\mathrm{A}'\epsilon^{x+a_0+\beta\iota}$$

$$=\mathrm{A}\epsilon^{\alpha}\cos\beta+\iota\mathrm{A}\epsilon^{\alpha}\sin\beta,$$

where

$$\alpha=\frac{x}{\sqrt{2}}+\frac{1}{8\sqrt{2}x}-\frac{25}{384\sqrt{2}x^3}-\frac{13}{128x^4}-\ldots \quad . \quad (18)$$

and

$$\beta=\frac{x}{\sqrt{2}}+b_0-\frac{1}{8\sqrt{2}x}-\frac{1}{16x^2}-\frac{25}{384\sqrt{2}x^3}+\ldots . \quad (19)$$

Hence, since $y = \text{ber}\, x + \imath\, \text{bei}\, x$, we have

$$\text{ber}\, x = (A/\sqrt{x})\, e^{\alpha} \cos\beta, \quad \text{and} \quad \text{bei}\, x = (A/\sqrt{x})\, e^{\alpha} \sin\beta.$$

To determine the values of A and b_0 we notice that

$$\text{ber}\, x + \imath\, \text{bei}\, x = I_0(x\sqrt{\imath}),$$

and thus, from equation (3), we see that A is $1/\sqrt{2\pi}$, and b_0 is $-\pi/8$.

We thus find that when x is large

$$\text{ber}\, x = \frac{e^{\alpha}}{\sqrt{2\pi x}} \cos\beta, \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and

$$\text{bei}\, x = \frac{e^{\alpha}}{\sqrt{2\pi x}} \sin\beta, \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where α and β are given by (18) and (19), and b_0 is $-\pi/8$.

The series for α and β are semi-convergent and a rigorous mathematical justification of (20) and (21) is difficult. It is easy, however, to verify that, if we only include the terms of the series given above, (20) and (21) give the values of the functions with great accuracy when x is greater than 5. The values of α and β are easily computed by the formulæ

$$\alpha = 0{\cdot}707105\, x + 0{\cdot}08839/x - 0{\cdot}046/x^3, \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and

$$\beta = 0{\cdot}707105\, x - 0{\cdot}39270 - 0{\cdot}08839/x - 0{\cdot}0625/x^2 - 0{\cdot}046/x^3. \quad (23)$$

Differentiating (20) and (21) we find that

$$\text{ber}'\, x = \left(\frac{1}{\sqrt{2}} - \frac{1}{2x} - \frac{1}{8\sqrt{2}x^2}\right) \text{ber}\, x - \left(\frac{1}{\sqrt{2}} + \frac{1}{8\sqrt{2}x^2} + \frac{1}{8x^3}\right) \text{bei}\, x, \quad . \quad (24)$$

and

$$\text{bei}'\, x = \left(\frac{1}{\sqrt{2}} + \frac{1}{8\sqrt{2}x^2} + \frac{1}{8x^3}\right) \text{ber}\, x + \left(\frac{1}{\sqrt{2}} - \frac{1}{2x} - \frac{1}{8\sqrt{2}x^2}\right) \text{bei}\, x. \quad (25)$$

Squaring equations (20) and (21), and adding, we get

$$X = \frac{e^{2\alpha}}{2\pi x}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

When the value of x is not less than 7, the inaccuracy of the formula

$$X = \frac{e^{x\sqrt{2} + 1/4\sqrt{2}x}}{2\pi x}$$

is less than 1 in 10,000.

Heaviside * has given the formula $X=\epsilon^{x\sqrt{2}}/2\pi x$, but in order to get a four-figure accuracy with this formula x would have to be greater than 1500.

In a similar manner, we find that

$$Y=X\left(1-\frac{1}{x\sqrt{2}}+\frac{1}{4x^2}+\frac{3}{8\sqrt{2}x^3}\right), \quad . \quad . \quad (27)$$

$$Z=X\left(\frac{1}{\sqrt{2}}-\frac{1}{2x}-\frac{1}{8\sqrt{2}x^2}\right), \quad . \quad . \quad . \quad . \quad (28)$$

$$W=X\left(\frac{1}{\sqrt{2}}+\frac{1}{8\sqrt{2}x^2}+\frac{1}{8x^3}\right). \quad . \quad . \quad . \quad . \quad (29)$$

These formulæ also give the ratios Y/X, Z/X, and W/X. They correspond to (11), (12), and (13), and give a four-figure accuracy when x is not less than 8.

By the binomial theorem, we also readily deduce the following formulæ corresponding to (14) and (15) :—

$$\frac{Z}{Y}=\frac{1}{\sqrt{2}}-\frac{3}{8\sqrt{2}x^3}-\frac{3}{8x^3} \quad . \quad . \quad . \quad (30)$$

and

$$\frac{W}{Y}=\frac{1}{\sqrt{2}}+\frac{1}{2x}+\frac{3}{8\sqrt{2}x^3} \quad . \quad . \quad . \quad . \quad (31)$$

To test these formulæ let us take the low value of 6 for x. We find from (20) and (21) that

$$\text{ber } 6=-8{\cdot}858 \quad \text{and} \quad \text{bei } 6=-7{\cdot}335.$$

These results are in exact agreement with their values found by direct computation from the series given in (5) and (6). The accuracy of the formulæ rapidly increases as x increases.

In the following table the numbers obtained by substituting 10 for x in formulæ (20), (21), (24), and (25) are compared with the numbers given by Kelvin † and by Mascart and Joubert †.

* 'Electrical Papers,' vol. ii. p. 184.

† *L. c. ante.*

	ber 10.	bei 10.	ber′ 10.	bei′ 10.
Kelvin	138·8405	56·3704	51·373	135·23
Mascart & Joubert ...	138·840	56·370	51·207	135·31
Formulæ	138·840	56·368	51·202	135·29

Before the author worked out the approximate formulæ, he had verified Mascart and Joubert's corrections as far as four figures by direct calculation from the series given in (5) and (6) and the series obtained by differentiating them. As the corrections to Kelvin's table when the argument is 15 and when it is 20 are large, it will be interesting to calculate these values by our formulæ.

	ber 15.	bei 15.	ber′ 15.	bei′ 15.
Kelvin	−2969·79	−2952·33	86·648	−4089·2
Mascart & Joubert ...	−2967·26	−2952·72	91·061	−4088·5
Formulæ	−2967·26	−2952·66	91·010	−4087·7

	ber 20.	bei 20.	ber′ 20.	bei′ 20.
Kelvin	47583·7	11500·8	24325·1	41491·5
Mascart & Joubert ...	47489·2	114774·4	−48802·8	111853
Formulæ	47491·6	114770·1	−48797·9	111853

In evaluating the formulæ we have only used 7-figure logarithmic tables. To compute ber′ 15 correctly we ought to have calculated ber 15 and bei 15 to eight significant figures, as ber 15 and bei 15 are nearly equal to one another and from (24) we see that ber′ 15 is nearly equal to their difference. It will be seen that the accuracy of Mascart and Joubert's corrections is satisfactory. The values, however, they give for the functions when the argument is 30 are not correct.

	ber 30.	bei 30.	ber′ 30.	bei′ 30.
Mascart & Joubert ...	-4544×10^{4}	11024×10^{4}	-10933×10^{4}	4383×10^{4}
Formulæ	-4611×10^{4}	10995×10^{4}	-10959×10^{4}	4330×10^{4}

Hence the ordinary tables need revision. A more useful set of tables, however, might be constructed of the functions X, Y, Z, W, Y/X, Z/X, W/X, Z/Y, and W/Y. The ber and bei functions never occur alone in any of the practical formulæ using these functions, with which the author is acquainted.

For instance, Heaviside * and Kelvin * have proved that the ratio of the effective resistance of the inner core of a concentric main with high frequency currents to its resistance with direct currents can be written down almost at once, when the ratio of W to Y is known. It was in fact in order to find the value of this ratio that Kelvin had a table of ber and bei functions computed.

The values of W/Y given in Kelvin's paper and found by (15) and also by direct computation are compared in the following table :—

x.	Values given in Kelvin's paper.	Values computed by (15).	True Values.
0·5	4·0000	4·0013	4·0013
1·0	2·00014	2·0104	2·0104
1·5	1·3678	1·3678	1·3678
2·0	1·0805	1·0782	1·0782

In Kelvin's table bei′1 is given as 0·4999, but its true value is 0·4974. Making this correction and using Kelvin's figures, we get 2·0104 for the value of W/Y when x is 1.

Including the next term in the expansion of W/Y given in (31), we find that

$$\frac{W}{Y} = 0{\cdot}7071 + \frac{1}{2x} + \frac{0{\cdot}265}{x^{3}} - \frac{0{\cdot}35}{x^{4}}. \quad . \quad . \quad (32)$$

* *L. c. ante.*

The values of W/Y given in the first column of the following table have been computed by this formula.

x.	Kelvin.	Mascart & Joubert.	Formula (32).
5	0·8172	.....	0·8172
5·5	0·8069	.....	0·8064
6	0·7979	.....	0·7976
8	0·7739	.....	0·7737
10	0·7588	0·7596	0·7597
15	0·7431	0·7416	0·7416
20	0·7325	0·7328	0·7328
30	.....	0·7251	0·7241
40	.....	0·7196	0·7198
50	.....	0·7172	0·7172
∞	0·7071	.....	0·7071

As the approximate formulæ given above are very simple, it will be seen that tables need only be constructed for values of x lying between 2 and 5 or 6, although to have tables of other values would doubtless be a great convenience to those who have to use the formulæ.

4. *Particular solution.*

We have seen that a solution of the equation

$$\frac{\partial^2 i}{\partial r^2} + \frac{1}{r}\frac{\partial i}{\partial r} = \frac{m^2}{\omega}\frac{\partial i}{\partial t}$$

is
$$i = \mathrm{I}_0\left(mr\sqrt{\iota}\right)\epsilon^{\omega t\iota}$$
$$= (\mathrm{ber}\, mr + \iota\, \mathrm{bei}\, mr)(\cos\omega t + \iota \sin\omega t).$$

Hence, since both the real and imaginary parts of this solution must satisfy the differential equation, we see that a particular solution may be written in either of the following forms :

$$i = (\mathrm{A}\,\mathrm{ber}\, mr + \mathrm{B}\,\mathrm{bei}\, mr)\cos\omega t + (-\mathrm{A}\,\mathrm{bei}\, mr + \mathrm{B}\,\mathrm{ber}\, mr)\sin\omega t, \quad . \quad . \quad (33)$$

or
$$i = (\mathrm{A}^2 + \mathrm{B}^2)^{1/2}(\mathrm{ber}^2 mr + \mathrm{bei}^2 mr)^{1/2}\cos(\omega t - \epsilon), \quad . \quad (34)$$

where A and B are constants, and

$$\tan\epsilon = (-\mathrm{A}\,\mathrm{bei}\, mr + \mathrm{B}\,\mathrm{ber}\, mr)/(\mathrm{A}\,\mathrm{ber}\, mr + \mathrm{B}\,\mathrm{bei}\, mr).$$

For a solid core and an infinitely thin return conductor of infinite conductivity this solution suffices, and is the one given by Kelvin. When, however, the core is hollow * or when we wish to take into account the effects of the return conductor, the complete solution has to be found, as the above solution cannot be made to satisfy all the boundary conditions.

5. *The* ker *and* kei *functions.*

Imitating Kelvin we shall write the second type of solution of (B), namely $K_0(mr\sqrt{\iota})$, in the form $\ker mr + \iota \operatorname{kei} mr$. Substituting $x\sqrt{\iota}$ for x in equation (2) and equating the coefficients of the real and imaginary terms on the sides of the equation, we find that

$$\ker x = (\alpha - \log x)\operatorname{ber} x + (\pi/4)\operatorname{bei} x - (1+\tfrac{1}{2})\frac{x^4}{2^2.4^2} + (1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4})\frac{x^8}{2^2.4^2.6^2.8^2} - \ldots, \quad (35)$$

and

$$\operatorname{kei} x = (\alpha - \log x)\operatorname{bei} x - (\pi/4)\operatorname{ber} x + \frac{x^2}{2^2} - (1+\tfrac{1}{2}+\tfrac{1}{3})\frac{x^6}{2^2.4^2.6^2} + \ldots, \quad \ldots\ldots \quad (36)$$

where $\alpha = 0{\cdot}1159315\ldots$.

6. *Approximate formulæ for the* ker *and* kei *functions.*

When x is small, we may use the following approximate formulæ, which can easily be proved by (5) and (6) :—

$$\ker mx = \alpha - \log mx + \frac{\pi}{4}\cdot\frac{m^2x^2}{4} - (\alpha + \tfrac{3}{2} - \log mx)\frac{m^4x^4}{64} - \frac{\pi}{4}\cdot\frac{m^6x^6}{2^2.4^2.6^2}, \quad . \quad (37)$$

$$\operatorname{kei} mx = -\frac{\pi}{4} + (\alpha + 1 - \log mx)\frac{m^2x^2}{4} + \frac{\pi}{4}\cdot\frac{m^4x^4}{64} - (\alpha + \tfrac{11}{6} - \log mx)\frac{m^6x^6}{2^2.4^2.6^2}, \quad (38)$$

$$\ker' mx = -\frac{1}{mx} + \frac{\pi}{4}\cdot\frac{mx}{2} - (\alpha + \tfrac{5}{4} - \log mx)\frac{m^3x^3}{16} - \frac{\pi}{4}\cdot\frac{m^5x^5}{2^2.4^2.6}, \quad . \quad . \quad . \quad (39)$$

and

$$\operatorname{kei}' mx = (\alpha + \tfrac{1}{2} - \log mx)\frac{mx}{2} + \frac{\pi}{4}\cdot\frac{m^3x^3}{16} - (\alpha + \tfrac{5}{8} - \log mx)\frac{m^5x^5}{2^2.4^2.6} \quad . \quad (40)$$

* The solution given in Mascart & Joubert, vol. i. p. 719 (1897), for the effective resistance of a hollow inner core is incorrect.

When mx is small we see that

$$\ker mx = a - \log mx \text{ and } \operatorname{kei} mx = -\pi/4, \text{ approximately;}$$

and hence, when mx is very small, we may write

$$\ker^2 mx + \operatorname{kei}^2 mx = (\log mx)^2. \quad . \quad . \quad . \quad (41)$$

When x is not small we get suitable formulæ for calculating these functions by noticing that

$$\theta = -x\sqrt{i} + a_0' + b_0'\sqrt{i}$$

is also an approximate solution of (17). Hence, finding a series in descending powers of x for θ, and determining the constants by (4), we find that

$$\ker x = \sqrt{\frac{\pi}{2x}}\, e^{\alpha'} \cos \beta', \quad . \quad . \quad . \quad . \quad (42)$$

and

$$\operatorname{kei} x = \sqrt{\frac{\pi}{2x}}\, e^{\alpha'} \sin \beta', \quad . \quad . \quad . \quad . \quad (43)$$

where

$$\alpha' = -\frac{x}{\sqrt{2}} - \frac{1}{8\sqrt{2}x} + \frac{25}{384\sqrt{2}x^3} - \frac{13}{128x^4} + \ldots \quad . \quad (44)$$

and

$$\beta' = -\frac{x}{\sqrt{2}} - \frac{\pi}{8} + \frac{1}{8\sqrt{2}x} - \frac{1}{16x^2} + \frac{25}{384\sqrt{2}x^3} - \ldots. \quad (45)$$

It will be seen that α' and β' can be deduced from the formulæ for α and β, (18) and (19), by merely changing the sign of x in the latter. In making calculations it is best to use the formulæ obtained by writing $-x$ for x on the right-hand side of the equations (22) and (23).

By differentiating (42) and (43) we find that

$$\ker' x = -\ker x \left\{\frac{1}{\sqrt{2}} + \frac{1}{2x} - \frac{1}{8\sqrt{2}x^2}\right\} + \operatorname{kei} x \left\{\frac{1}{\sqrt{2}} + \frac{1}{8\sqrt{2}x^2} - \frac{1}{8x^3}\right\}, \quad . \quad . \quad . \quad (46)$$

and

$$\operatorname{kei}' x = -\ker x \left\{\frac{1}{\sqrt{2}} + \frac{1}{8\sqrt{2}x^2} - \frac{1}{8x^3}\right\} - \operatorname{kei} x \left\{\frac{1}{\sqrt{2}} + \frac{1}{2x} - \frac{1}{8\sqrt{2}x^2}\right\}. \quad . \quad . \quad . \quad (47)$$

We also have

$$X_1(x) = \ker^2 x + \mathrm{kei}^2 x = \frac{\pi e^{2a'}}{2x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

and

$$Y_1(x) = \ker'^2 x + \mathrm{kei}'^2 x = X_1(x)\left\{1 + \frac{1}{x\sqrt{2}} + \frac{1}{4x^2} - \frac{3}{8\sqrt{2}x^3}\right\}. \quad (49)$$

When mx is very large we shall use the formulæ

$$\ker mx = \sqrt{\frac{\pi}{2mx}}\, e^{-\frac{mx}{\sqrt{2}}} \cos\left(\frac{mx}{\sqrt{2}} + \frac{\pi}{8}\right), \quad . \quad . \quad (50)$$

$$\mathrm{kei}\, mx = -\sqrt{\frac{\pi}{2mx}}\, e^{-\frac{mx}{\sqrt{2}}} \sin\left(\frac{mx}{\sqrt{2}} + \frac{\pi}{8}\right), \quad . \quad . \quad (51)$$

$$\ker' mx = -\sqrt{\frac{\pi}{2mx}}\, e^{-\frac{mx}{\sqrt{2}}} \cos\left(\frac{mx}{\sqrt{2}} - \frac{\pi}{8}\right), \quad . \quad . \quad (52)$$

and

$$\mathrm{kei}' mx = \sqrt{\frac{\pi}{2mx}}\, e^{-\frac{mx}{\sqrt{2}}} \sin\left(\frac{mx}{\sqrt{2}} - \frac{\pi}{8}\right). \quad . \quad . \quad (53)$$

7. *Formulæ containing both functions.*

We require the following formulæ, also, in our solutions. When mc is small,

$$S_c = \mathrm{ber}' mc \ker' mc + \mathrm{bei}' mc \,\mathrm{kei}' mc$$

$$= \frac{m^2c^2}{4}(a + \tfrac{3}{4} - \log mc) + \frac{m^6c^6}{12 \,.\, 64}(a - \log mc) + \frac{37 m^6c^6}{2^2 \,.\, 4^2 \,.\, 6^2 \,.\, 8^2} \quad . \quad (54)$$

and

$$T_c = \mathrm{bei}' mc \ker' mc - \mathrm{ber}' mc \,\mathrm{kei}' mc$$

$$= -\frac{1}{2} + \frac{\pi}{4} \cdot \frac{m^2c^2}{4} - \frac{m^4c^4}{48} \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

When mc is very great, we may write

$$S_c = -\frac{\cos mc\sqrt{2}}{2mc}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

and

$$T_c = -\frac{\sin mc\sqrt{2}}{2mc}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

8. *The complete solution.*

On the assumption that i follows the harmonic law, we see that the complete solution of the equation (A) is

$$i = (\mathrm{A\ ber}\, mr + \mathrm{B\ bei}\, mr + \mathrm{C\ ker}\, mr + \mathrm{D\ kei}\, mr)\cos \omega t + (-\mathrm{A\ bei}\, mr + \mathrm{B\ ber}\, mr - \mathrm{C\ kei}\, mr + \mathrm{D\ ker}\, mr)\sin \omega t, \quad (58)$$

where A, B, C, and D are constants.

III. The Formulæ for the Effective Resistance and Inductance of a Concentric Main with a Solid Inner Conductor.

In order to simplify the problem, we shall first suppose that the inner conductor is a solid metal cylinder of radius a, and that the outer conductor is a coaxial hollow cylinder of inner and outer radii b and c respectively. Let μ be the value of the permeability of the metals forming the conductors, and let μ' be the permeability of the insulating material separating them. Let ρ be the volume resistivity of the conducting metal. We shall suppose that μ, μ', and ρ are constants and, for the present, that both the capacity and leakage currents in the dielectric can be neglected. We can assume, therefore, that the flow of current in the conductors is parallel to their common axis, and hence, that the equipotential surfaces in each conductor are planes perpendicular to this axis.

Let us now consider the current in a cylindrical tube of unit length in the inner conductor, whose inner and outer radii are r and $r+dr$ respectively. If e_1 be the potential difference between the ends of this tube, the equation to determine the current-density i in it is, by Ohm's law and Faraday's law,

$$e_1 = (\rho/2\pi r\partial r)(i \,.\, 2\pi r\partial r) + \partial\phi/\partial t = \rho i + \partial\phi/\partial t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

where ϕ is the number of magnetic lines linked with the current in this cylindrical tube.

By hypothesis, the equipotential surfaces in the inner conductor are planes perpendicular to the axis. Hence e_1 is

independent of the value of r, and thus

$$0 = \rho \frac{\partial i}{\partial r} + \frac{\partial}{\partial t} \frac{\partial \phi}{\partial r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

From the symmetry of a concentric main, we see that the intensity of the current is the same at all points equidistant from the axis. Hence, since the magnetic force outside an infinite cylindrical tube, carrying a current flowing parallel to its axis, is the same as if all the current were concentrated at this axis, we have, at all points of the inner conductor,

$$\phi = \mu \int_r^a \frac{2I_x}{x} \partial x + 2\mu' I \log \frac{b}{a} + \mu \int_b^c \frac{2(I - I'_x)}{x} \partial x, \quad (61)$$

where I_x is the algebraical sum of the currents flowing through the cross section of a coaxial cylinder whose radius is x, I is the total current flowing in the inner conductor, and I'_x is the sum of the currents flowing in the tube of the outer conductor whose inner radius is b and outer radius is x.

By differentiating ϕ with respect to r, we get

$$\frac{\partial \phi}{\partial r} = -\frac{2\mu}{r} \int_0^r 2\pi i r \partial r,$$

and hence, by (60),

$$\rho \frac{\partial i}{\partial r} = \frac{4\pi\mu}{r} \int_0^r r \frac{\partial i}{\partial t} \partial r, \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

and finally, by differentiating,

$$\frac{\partial^2 i}{\partial r^2} + \frac{1}{r} \frac{\partial i}{\partial r} = \frac{4\pi\mu}{\rho} \frac{\partial i}{\partial t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

Writing

$$m^2 = \frac{4\pi\mu\omega}{\rho} = \frac{8\pi^2\mu f}{\rho}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

where f is the frequency of the alternating currents, we see that the value of i is given by (58).

Let us suppose that the current-density along the axis of the cylinder is given by $i = i_0 \cos \omega t$. At the axis r is zero; and since ker 0 is infinite both C and D must be zero, as otherwise the current-density would be infinite. Since also ber 0 = 1 and bei 0 = 0, we have $A = i_0$ and $B = 0$, and thus

$$i = i_0 \operatorname{ber} mr \cos \omega t - i_0 \operatorname{bei} mr \sin \omega t. \quad . \quad . \quad . \quad (65)$$

It is easy to verify by the help of the equations

$$\int r \operatorname{ber} mr\,\partial r = (r/m)\operatorname{bei}' mr \quad . \quad . \quad . \quad . \quad (66)$$

and

$$\int r \operatorname{bei} mr\,\partial r = -(r/m)\operatorname{ber}' mr \quad . \quad . \quad . \quad (67)$$

that this value of i, which is the same as that given by Kelvin, satisfies (62). It therefore gives the solution of the problem of finding the current-density at any point in the substance of the inner conductor.

From (65) we find that

$$\mathrm{I}_r = 2\pi\int_0^r ir\,\partial r$$

$$= (2\pi/m)\, r \operatorname{bei}' mr \,.\, i_0 \cos\omega t + (2\pi/m)\, r \operatorname{ber}' mr \,.\, i_0 \sin\omega t. \quad (68)$$

Putting r equal to a in this equation, we get for the total current,

$$\mathrm{I} = (2\pi/m)\, a \operatorname{bei}' ma \,.\, i_0 \cos\omega t + (2\pi/m)\, a \operatorname{ber}' ma \,.\, i_0 \sin\omega t. \quad (69)$$

By differentiating this equation with regard to t, we find that

$$\frac{\partial \mathrm{I}}{\partial t} = -(2\pi\omega/m)\, a \operatorname{bei}' ma \,.\, i_0 \sin\omega t + (2\pi\omega/m)\, a \operatorname{ber}' ma \,.\, i_0 \sin\omega t, \quad (70)$$

and hence, solving these equations for $i_0 \cos\omega t$ and $i_0 \sin\omega t$, we get

$$i_0 \cos\omega t = \frac{m \operatorname{bei}' ma}{2\pi a \Upsilon(ma)}\mathrm{I} + \frac{m \operatorname{ber}' ma}{2\pi a\omega \Upsilon(ma)}\frac{\partial \mathrm{I}}{\partial t}, \quad . \quad (71)$$

and

$$i_0 \sin\omega t = \frac{m \operatorname{ber}' ma}{2\pi a \Upsilon(ma)}\mathrm{I} - \frac{m \operatorname{bei}' ma}{2\pi a\omega \Upsilon(ma)}\frac{\partial \mathrm{I}}{\partial t}. \quad . \quad (72)$$

Let us now consider the currents in the outer conductor. Let i' be the current density in this cylinder at a distance r from the axis. Let e' be the potential-difference per unit length measured from the distributing station to the alternator, i' being considered positive when flowing in the same direction.

We have, therefore,

$$e' = (\rho/2\pi r\,\partial r)(2\pi r\,\partial r \,.\, i') - \partial\phi'/\partial t$$

$$= \rho i' - \partial\phi'/\partial t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

where

$$\phi' = \mu\int_r^\infty \frac{2(\mathrm{I} - \mathrm{I}'_x)}{x}\,\partial x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

Thus
$$r\frac{\partial\phi'}{\partial r} = -2\mu(\mathrm{I}-\mathrm{I}'_r)$$
$$= -4\pi\mu\int_r^e ri'\partial r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

Also, since we have supposed that e' does not vary with r, we get from (73) and (75)
$$\rho\frac{\partial i'}{\partial t} = -\frac{4\pi\mu}{r}\int_r^e r\frac{\partial i'}{\partial t}\partial r\,; \quad . \quad . \quad . \quad (76)$$

and thus
$$\frac{\partial^2 i'}{\partial r^2}+\frac{1}{r}\frac{\partial i'}{\partial t} = \frac{m^2}{\omega}\frac{\partial i'}{\partial t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

By (58), the solution of (77) may be written as follows:—
$$i' = (\mathrm{A}\,\mathrm{ber}\,mr+\mathrm{B}\,\mathrm{bei}\,mr+\mathrm{C}\,\mathrm{ker}\,mr+\mathrm{D}\,\mathrm{kei}\,mr)\,i_0\cos\omega t$$
$$+(-\mathrm{A}\,\mathrm{bei}\,mr+\mathrm{B}\,\mathrm{ber}\,mr-\mathrm{C}\,\mathrm{kei}\,mr+\mathrm{D}\,\mathrm{ker}\,mr)\,i_0\sin\omega t,$$
$$. \quad . \quad . \quad . \quad (78)$$

where A, B, C, and D are constants which have to be determined from the data of the problem.

Now noticing that
$$\int r\,\mathrm{ker}\,mr\,\partial r = (r/m)\,\mathrm{kei}'\,mr \quad . \quad . \quad . \quad (79)$$
and
$$\int r\,\mathrm{kei}\,mr\,\partial r = -(r/m)\,\mathrm{ker}'\,mr, \quad . \quad . \quad (80)$$

we find, by substituting the value of i' given by (78) in (76), that
$$\mathrm{A}\,\mathrm{bei}'\,mc-\mathrm{B}\,\mathrm{ber}'\,mc+\mathrm{C}\,\mathrm{kei}'\,mc-\mathrm{D}\,\mathrm{ker}'\,mc = 0, \quad (81)$$
and
$$\mathrm{A}\,\mathrm{ber}'\,mc+\mathrm{B}\,\mathrm{bei}'\,mc+\mathrm{C}\,\mathrm{ker}'\,mc+\mathrm{D}\,\mathrm{kei}'\,mc = 0. \quad (82)$$

By equating also the integral value $\int_b^e 2\pi ri'\partial r$ of the current in the return conductor to the value of I given by (69), we get
$$\mathrm{A}\,\mathrm{bei}'\,mb-\mathrm{B}\,\mathrm{ber}'\,mb+\mathrm{C}\,\mathrm{kei}'\,mb-\mathrm{D}\,\mathrm{ker}'\,mb = -\frac{a}{b}\,\mathrm{bei}'\,ma, \quad (83)$$
and
$$\mathrm{A}\,\mathrm{ber}'\,mb+\mathrm{B}\,\mathrm{bei}'\,mb+\mathrm{C}\,\mathrm{ker}'\,mb+\mathrm{D}\,\mathrm{kei}'\,mb = -\frac{a}{b}\,\mathrm{ber}'\,ma. \quad (84)$$

The four equations (81)–(84) completely determine the four constants A, B, C, and D. Hence the current density at all points on the outer conductor is found.

From (73) and (74) we can see at once that

$$e' = \rho i'_b - \mu \frac{\partial}{\partial t} \int_b^e \frac{2(\mathrm{I}-\mathrm{I}'_x)}{x} \partial x, \quad . \quad . \quad . \quad (85)$$

where i'_b is the current density on the inner surface of the outer conductor. We also see from (59) and (61) that

$$e = \rho i_a + 2\mu' \log \frac{b}{a} \frac{\partial \mathrm{I}}{\partial t} + \mu \frac{\partial}{\partial t} \int_a^b \frac{2(\mathrm{I}-\mathrm{I}'_x)}{x} \partial x, \quad . \quad (86)$$

where i_a is the current density on the outer surface of the inner conductor. Thus, by addition,

$$e + e' = \rho i_a + \rho i'_b + 2\mu' \log \frac{b}{a} \frac{\partial \mathrm{I}}{\partial t}. \quad . \quad . \quad . \quad (87)$$

Hence writing for i_a and i'_b their values given by (65) and (68) respectively, and also writing for $i_0 \cos \omega t$ and $i_0 \sin \omega t$ their values in terms of I and $\partial \mathrm{I}/\partial t$ from (71) and (72), we get, after a little reduction, that

$$e + e' = \mathrm{RI} + \mathrm{L} \frac{\partial \mathrm{I}}{\partial t}, \quad . \quad . \quad . \quad . \quad . \quad (88)$$

where

$$\mathrm{R} = (\rho m/2\pi a \mathrm{Y}_a)(\mathrm{ber}\, ma\, \mathrm{bei}'\, ma - \mathrm{bei}\, ma\, \mathrm{ber}'\, ma)$$
$$+ \frac{\rho m\, \mathrm{bei}'\, ma}{2\pi a \mathrm{Y}_a} \{\mathrm{A}\, \mathrm{ber}\, mb + \mathrm{B}\, \mathrm{bei}\, mb + \mathrm{C}\, \mathrm{ker}\, mb + \mathrm{D}\, \mathrm{kei}\, mb\}$$
$$- \frac{\rho m\, \mathrm{ber}'\, ma}{2\pi a \mathrm{Y}_a} \{\mathrm{A}\, \mathrm{bei}\, mb - \mathrm{B}\, \mathrm{ber}\, mb + \mathrm{C}\, \mathrm{kei}\, mb - \mathrm{D}\, \mathrm{ker}\, mb\}, \quad . \quad . \quad . \quad . \quad (89)$$

and

$$\mathrm{L} = 2\mu' \log \frac{b}{a} + \frac{2\mu}{ma \mathrm{Y}_a} (\mathrm{ber}\, ma\, \mathrm{ber}'\, ma + \mathrm{bei}\, ma\, \mathrm{bei}'\, ma)$$
$$+ \frac{2\mu\, \mathrm{ber}'\, ma}{ma \mathrm{Y}_a} \{\mathrm{A}\, \mathrm{ber}\, mb + \mathrm{B}\, \mathrm{bei}\, mb + \mathrm{C}\, \mathrm{ker}\, mb + \mathrm{D}\, \mathrm{kei}\, mb\}$$
$$+ \frac{2\mu\, \mathrm{bei}'\, ma}{ma \mathrm{Y}_a} \{\mathrm{A}\, \mathrm{bei}\, mb - \mathrm{B}\, \mathrm{ber}\, mb + \mathrm{C}\, \mathrm{kei}\, mb - \mathrm{D}\, \mathrm{ker}\, mb\}. \quad . \quad . \quad . \quad . \quad (90)$$

In (89) and (90), Y_a stands for $\mathrm{ber}'^2\, ma + \mathrm{bei}'^2\, ma$, and A, B, C, and D can be found as follows from the equations (81)–(84), using the notation adopted in formulæ (54) and (55):—

$$\mathrm{AY}_c = -\mathrm{CS}_c + \mathrm{DT}_c \quad . \quad . \quad . \quad . \quad . \quad (91)$$

and

$$\mathrm{BY}_c = -\mathrm{CT}_c - \mathrm{DS}_c. \quad . \quad . \quad . \quad . \quad . \quad (92)$$

Hence substituting the values of A and B, given by (91) and (94), in (83) and (84), we get

$$C(Y_cS_b - Y_bS_c) + D(Y_bT_c - Y_cT_b)$$
$$= -\frac{a}{b}Y_c(\mathrm{ber}'\,ma\,\mathrm{ber}'\,mb + \mathrm{bei}'\,ma\,\mathrm{bei}'\,mb), \quad (93)$$

and

$$-C(Y_bT_c - Y_cT_b) + D(Y_cS_b - Y_bS_c)$$
$$= -\frac{a}{b}Y_c(\mathrm{ber}'\,ma\,\mathrm{bei}'\,mb - \mathrm{bei}'\,ma\,\mathrm{ber}'\,mb). \quad (94)$$

From these equations C and D are easily found, and then A and B can be found from (91) and (92).

The above formulæ give the complete solution of the problem when the applied wave is sine-shaped and the capacity and leakage effects are neglected.

If the applied wave be not sine-shaped, but if it be a periodic function of the time, it may be expanded in a series of sines by Fourier's theorem; and hence we could write down the solution without difficulty, but it would be very cumbrous.

IV. Simplified Formulæ for particular cases.

1. *With direct currents.*

From the next solution, by putting m equal to zero, we find that the resistance R_d and the inductance L_d with direct currents are given by

$$R_d = \frac{\rho}{\pi a^2} + \frac{\rho}{\pi(c^2 - b^2)}, \quad . \quad . \quad . \quad . \quad (95)$$

and

$$L_d = 2\mu' \log\frac{b}{a} + \frac{\mu}{2}$$
$$+ \frac{2\mu c^4}{(c^2 - b^2)^2}\log\frac{c}{b} - \mu\frac{3c^2 - b^2}{2(c^2 - b^2)}. \quad (96)$$

This value for the inductance agrees with that found by Lord Rayleigh*.

* Phil. Mag. [5] vol. xxi. p. 381 (1886), or Russell's 'Alternating Currents,' vol. i. p. 53.

2. *With low frequency currents.*

Substituting the appropriate approximate formulæ (54) and (55) for S_e and T_e in the equations (91)–(94), and noticing that $Y_e=(m^2c^2/4)(1+m^4c^4/192)$ approx., we find that

$$C = C_1 m^4 + C_2 m^8, \quad \ldots \quad (97)$$

where

$$C_1 = \frac{a^2c^2(b^2-a^2)}{16(c^2-b^2)} - \frac{a^2b^2c^4}{4(c^2-b^2)^2}\log\frac{c}{b},$$

and

$$\begin{aligned} C_2 = {} & \frac{c^2a^2}{2^2.4^2.6^2.8(c^2-b^2)}\left\{\begin{array}{l} b^2c^2(61\,b^2+19\,c^2-42\,a^2) \\ \quad +(b^2-a^2)(5\,a^2b^2-a^4-7\,b^4)\end{array}\right\} \\ & - \frac{a^2b^2c^4\log(c/b)}{2^2\,4^2.12.(c^2-b^2)^2}\left\{14\,c^2b^2-a^4-2b^4+3\,a^2b^2\right\} \\ & - \frac{a^2b^4c^6(b^2-a^2)}{4^2(c^2-b^2)^3}\left\{\log\frac{c}{b}\right\}^2 \\ & + \frac{a^2b^6c^8}{4^2(c^2-b^2)^4}\left\{\log\frac{c}{b}\right\}^3. \end{aligned}$$

$$-D = D_1 m^2 + D_2 m^6, \quad \ldots \quad (98)$$

where

$$\begin{aligned} D_1 = {} & \frac{a^2c^2}{2(c^2-b^2)} \\ D_2 = {} & \frac{a^2c^2}{2^2.4^2.6(c^2-b^2)}(7\,c^2b^2-a^4-2b^4+3\,a^2b^2) \\ & + \frac{a^2b^2c^4(b^2-a^2)}{2.4^2.(c^2-b^2)^2}\log\frac{c}{b} \\ & - \frac{a^2b^4c^6}{8(c^2-b^2)^2}\left(\log\frac{c}{b}\right)^2 \end{aligned}$$

$$\begin{aligned} A = {} & \frac{a^2}{c^2-b^2} - \frac{\pi a^2c^2}{8(c^2-b^2)}m^2 + \frac{7\,a^2c^4m^4}{2^2.4^2.3(c^2-b^2)} + \frac{2}{c^2}D_2m^4 \\ & - C_1(\alpha+\tfrac{3}{4}-\log mc)m^4, \quad \ldots \quad (99) \end{aligned}$$

and

$$\frac{B}{m^2} = \frac{2C_1}{c^2} + D_1(\alpha + \tfrac{3}{4} - \log mc) - \frac{\pi}{4} C_1 m^2$$
$$+ \left\{ \frac{7c^2}{4^2.6} C_1 + \frac{2}{c^2} C_2 + D_2(\alpha + \tfrac{3}{4} - \log mc) + \frac{19c^4}{2^2.4^2.6^2.2} D_1 \right\} m^4 \quad . \quad . \quad . \quad (100)$$

Substituting these values in (89) and (90), and using the approximate formulæ for the functions, we get, after lengthy algebraical operations,

$$R = \frac{\rho}{\pi a^2}\left(1 + \frac{m^4 a^4}{192}\right)$$
$$+ \frac{\rho}{\pi(c^2 - b^2)} \left\{ 1 + m^4(\rho_1 + \rho_2 \xi + \rho_3 \xi^2) \right\}, \quad . \quad . \quad (101)$$

where

$$\rho_1 = \frac{(7c^2 - b^2)(c^2 - b^2)}{192}, \quad \rho_2 = \frac{b^2 c^4}{8(c^2 - b^2)},$$

$$\rho_3 = -\frac{b^2 c^6}{4(c^2 - b^2)^2}, \text{ and } \xi = \log \frac{c}{b}.$$

It is interesting to notice that the coefficient of m^2 vanishes and that the numerical constants α and $\log m$ have cancelled out. This formula is in exact agreement with that* given by Oliver Heaviside. It is not quite clear how he obtains the formula, but he states that the work was "very heavy."

In power transmission cables $a^2 = c^2 - b^2$. In this case the greater the value of b the smaller the value of $\rho_1 + \rho_2 \xi + \rho_3 \xi^2$. In power transmission cables, when the frequency is not greater than 50, the increase in the effective resistance of the outer conductor due to the skin effect is negligibly small.

We find in a similar manner that

$$L = L_0 - \frac{\mu}{2} \cdot \frac{a^4}{192} m^4$$
$$- \mu(\lambda_1 + \lambda_2 \xi + \lambda_3 \xi^2 + \lambda_4 \xi^3) m^4, \quad . \quad . \quad (102)$$

* 'Electrical Papers,' vol. ii. p. 192 (72).

where
$$\lambda_1 = \frac{19c^6 + 103c^4b^2 - 41b^4c^2 + 3b^6}{2^2.4^2.6^2(c^2-b^2)},$$

$$\lambda_2 = -\frac{14b^2c^4(2c^2-b^2)}{2^2.4^2.3(c^2-b^2)^2},$$

$$\lambda_3 = -\frac{b^4c^6}{4(c^2-b^2)^3},$$

$$\lambda_4 = \frac{b^6c^8}{2(c^2-b^2)^4}, \quad \text{and} \quad \xi = \log\frac{c}{b}.$$

Heaviside[6] gives the formula as if the coefficient of m^4 were zero. The term $-(\mu a^4/384)m^4$ can be deduced from the formula given in Gray and Matthews' 'Bessel's Functions,' p. 160, as the "self-inductance" of a cylindrical conductor.

Formula (102) shows that at low frequencies the inductance diminishes as the frequency increases, the effect being more pronounced the thicker the shell of the outer conductor.

3. *With high frequency currents.*

When ma is greater than 5 we can use the formulæ (50) to (53) for the ker functions and the corresponding formulæ for the ber functions. Substituting these values in (89) and (90) we get, after a little reduction,

$$R = \frac{\rho m}{2\pi a}\left\{\frac{1}{\sqrt{2}} + \frac{1}{2ma} + \frac{3}{8\sqrt{2}m^2a^2}\right\} + \frac{\rho m}{2\pi b\sqrt{2}}\cdot\frac{\sinh m(c-b)\sqrt{2} + \sin m(c-b)\sqrt{2}}{\cosh m(c-b)\sqrt{2} - \cos m(c-b)\sqrt{2}}, \quad (103)$$

and

$$L = 2\mu'\log\frac{b}{a} + \frac{2\mu}{ma}\left(\frac{1}{\sqrt{2}} - \frac{3}{8\sqrt{2}m^2a^2} - \frac{3}{8m^3a^3}\right) + \frac{2\mu}{mb\sqrt{2}}\cdot\frac{\sinh m(c-b)\sqrt{2} - \sin m(c-b)\sqrt{2}}{\cosh m(c-b)\sqrt{2} + \cos m(c-b)\sqrt{2}}. \quad (104)$$

[6] 'Electrical Papers,' vol. ii. p. 192. [Dr. Heaviside has pointed out to me that he was only giving the first term correction to R + Lω, and thus it was unnecessary to give the coefficient of m^4 in (102).]

It is to be noticed that R becomes infinite when $b=c$, but L becomes

$$2\mu' \log (b/a) + (2\mu/ma)(1/\sqrt{2} - 3/8\sqrt{2}m^2a^2 - 3/8m^3a^3).$$

The first term in (103) gives the resistance of the inner conductor and the second that of the outer conductor. In (104) the first term gives the linkages of the magnetic flux in the dielectric with the current in the inner conductor. The second term gives the linkages of the magnetic flux in the substance of the inner conductor, and the third the linkages of the flux in the outer conductor.

4. *With very high frequency currents.*

When the frequency is very high, provided that c be not nearly equal to b, we may write

$$\mathrm{R} = \frac{\rho m}{2\pi a\sqrt{2}} + \frac{\rho m}{2\pi b\sqrt{2}}$$

$$= \sqrt{\mu f \rho}\left(\frac{1}{a} + \frac{1}{b}\right). \quad . \quad . \quad . \quad . \quad . \quad (105)$$

Similarly when ma is very great we may write

$$\mathrm{L} = 2\mu' \log \frac{b}{a} + \frac{\mu}{2\pi}\sqrt{\frac{\rho}{\mu f}}\left(\frac{1}{a} + \frac{1}{b}\right). \quad . \quad (106)$$

We see, therefore, that as f increases R continually increases, but L approaches the value $2\mu' \log (b/a)$ asymptotically. Whatever the frequency, we see that the value of L lies between the value given by (96) and $2\mu' \log (b/a)$. It is easy to see that it has the latter value when the currents are confined to an infinitely thin skin on the outer and inner surfaces of the inner and outer conductors respectively.

Formulæ (105) and (106) agree with those given by Sir Joseph Thomson (see 'Recent Researches,' p. 295). They are also given in Heaviside's 'Electrical Papers,' vol. ii. p. 193.

V. The Density of the Current in the Inner and Outer Conductors.

1. *With low frequency currents.*

For the inner conductor we find from equation (65), that

$$i = i_0 \{X(mr)\}^{1/2} \cos(\omega t + \epsilon), \quad . \quad . \quad . \quad (107)$$

where $\tan \epsilon = \text{bei } mr / \text{ber } mr$.

It is known * that $X(mr)$ always increases as mr increases. Hence the amplitude of the current density is always greatest at the surface of the inner conductor and least along the axis. When the sixth and higher powers of mr can be neglected (107) becomes

$$i = i_0(1 + m^4 r^4/64) \cos(\omega t + \epsilon), \quad . \quad . \quad . \quad (108)$$

where $\tan \epsilon = m^2 r^2/4$.

If $mr = 1$, (107) gives

$$i = 1{\cdot}015\, i_0 \cos(\omega t + 14^\circ\ 15'),$$

and (108) gives $i = 1{\cdot}016\, i_0 \cos(\omega t + 14^\circ\ 2')$.

Hence when mr is not greater than 1, (108) may be used. Even when $mr = 2$ the inaccuracy in the value of the amplitude of the current density given by (108) is less than 2 per cent. In this case the amplitude of the current-density at the surface of the conductor is about 23 per cent. greater than at its axis, and the current along the axis lags by about 52° behind the surface current.

Similarly when mc is small, we find by (78) that

$$\frac{i'}{i_0} = \frac{a^2}{c^2 - b^2} \left\{ 1 - \frac{m^2 c^2}{2} \log mr \right\} \cos \omega t. \quad . \quad (109)$$

Hence the amplitude of the current-density in the outer conductor diminishes as r increases.

3. *With high frequency currents.*

In this case, by (20) and (21), we get for the inner conductor

$$\frac{i}{i_0} = \frac{\epsilon^{mr/\sqrt{2}}}{\sqrt{2\pi mr}} \cos\left(\omega t + \frac{mr}{\sqrt{2}} - \frac{\pi}{8}\right). \quad . \quad . \quad (110)$$

* Russell's 'Alternating Currents,' vol. i. p. 373.

Similarly in the outer conductor we find that

$$i_0 = \frac{\epsilon^{ma/\sqrt{2}}}{\sqrt{2}\pi mr}\left(\frac{a}{b}\right)^{1/2}\left[\frac{\cosh m(c-r)\sqrt{2}+\cos m(c-r)\sqrt{2}}{\cosh m(c-b)\sqrt{2}-\cos m(c-b)\sqrt{2}}\right]^{1/2}\cos(\omega t+\theta), \quad . \quad . \quad . \quad (111)$$

where
$$\tan\theta = \frac{\epsilon^{-m(c-r)/\sqrt{2}}\sin\epsilon_1+\epsilon^{m(c-r)/\sqrt{2}}\sin\epsilon_2}{\epsilon^{-m(c-r)/\sqrt{2}}\cos\epsilon_1+\epsilon^{m(c-r)/\sqrt{2}}\cos\epsilon_2},$$

and
$$\epsilon_1=\gamma-\frac{mr}{\sqrt{2}}+\frac{\pi}{8};\quad \epsilon_2=\delta+\frac{mr}{\sqrt{2}}+\frac{\pi}{8};$$

$$\tan\gamma=\frac{B}{A};\quad \tan\delta=\frac{D}{C},\quad \text{and } \gamma-\delta=mc\sqrt{2}.$$

Since $\{\cosh m(c-r)\sqrt{2}+\cos m(c-r)\sqrt{2}\}/r$

continually diminishes as r increases, we see that the amplitude of i always diminishes as r increases.

VI. Concentric Main with Hollow Inner Conductor.

Let us suppose that a_1 is the inner radius of the inner conductor. The solution (58) for the current-density i still applies. Assuming that $i=i_0\cos\omega t$, when r is a_1, we get two equations connecting the four constants A, B, C, and D. The equation corresponding to (62) is now

$$\rho\frac{\partial i}{\partial r}=\frac{4\pi\mu}{r}\int_{a_1}^{r} r\frac{\partial i}{\partial t}\partial r. \quad . \quad . \quad . \quad (112)$$

Substituting the value of i in this equation and equating the coefficients of $\cos\omega t$ and $\sin\omega t$ in the resulting equation to zero, we get two other equations connecting the four constants, and hence they can be found.

Adopting the method suggested by O. Heaviside*, however, the values of the effective resistance of each tube may be written down at once from the formulæ given above. From formula (101), for instance, the resistance R_0 of the outer

* 'Electrical Papers,' vol. ii. p. 192.

conductor is given by

$$R_0 = \frac{\rho}{\pi(c^2-b^2)}\left[1+m^4\left\{\frac{(7c^2-b^2)(c^2-b^2)}{192}+\frac{b^6c^4}{8(c^2-b^2)}\log\frac{c}{b}-\frac{b^2c^6}{4(c^2-b^2)^2}\left(\log\frac{c}{b}\right)^2\right\}\right]. \quad . \quad . \quad (113)$$

To deduce the resistance R_i of the inner tube we write a for b, and a_1 for c in the coefficient of m^4, and noticing that the area of the cross section is now $\pi(a^2-a_1^2)$, we get

$$R_i = \frac{\rho}{\pi(a^2-a_1^2)}\left[1+m^4\left\{\frac{(7a_1^2-a^2)(a_1^2-a^2)}{192}+\frac{a^2a_1^4}{8(a^2-a_1^2)}\log\frac{a}{a_1}-\frac{a^2a_1^6}{4(a^2-a_1^2)^2}\left(\log\frac{a}{a_1}\right)^2\right\}\right]. \quad . \quad (114)$$

For a given cross sectional area $\pi(a^2-a_1^2)$ we see from (114) that the coefficient of m^4 diminishes as a_1 increases. Hence, at low frequencies, making the inner conductor hollow diminishes the skin effect. Similarly for a given cross sectional area, the larger the outer tube the smaller the effective resistance of the tube.

From (103) we see that at high frequencies the resistance R_0 of the outer tube is given by

$$R_0 = \frac{\rho m}{2\pi b\sqrt{2}}\cdot\frac{\sinh m(c-b)\sqrt{2}+\sin m(c-b)\sqrt{2}}{\cosh m(c-b)\sqrt{2}-\cos m(c-b)\sqrt{2}}.$$

Hence the resistance R_i of the inner tube is given by

$$R_i = \frac{\rho m}{2\pi a\sqrt{2}}\cdot\frac{\sinh m(a-a_1)\sqrt{2}+\sin m(a-a_1)\sqrt{2}}{\cosh m(a-a_1)\sqrt{2}-\cos m(a-a_1)\sqrt{2}}. \quad (115)$$

The difference, therefore, between the effective resistance of a thin and a thick inner tube having equal outer radii is very small at high frequencies.

VII. The Impedance of a Concentric Main.

If the length of the main be very long compared with its diameter and the insulation resistance of the dielectric be very high, the assumption that the current flow is linear is permissible. In this case, if e be the potential-difference

between the mains at a point at a distance x from the alternator, we have*

$$-\frac{\partial e}{\partial x} = \mathrm{R}i + \mathrm{L}\frac{\partial i}{\partial t},$$

and

$$-\frac{\partial i}{\partial x} = \frac{e}{\mathrm{S}} + \mathrm{K}\frac{\partial e}{\partial t},$$

where K and S are the capacity and the insulation resistance between the mains per unit length. In these equations R and L have the values found earlier in the paper. Hence we have to substitute these values in the well-known expressions for the impedance of the main.

VIII. Numerical example.

In connexion with the transmission of electric power at low frequencies, the question of the magnitude of the skin effect in three-core cables has been recently discussed by engineers. As the question is one of considerable difficulty owing to the very complex nature of the magnetic field inside a three-core cable (see Russell, 'Alternating Currents,' vol. i. p. 321), it is important to know what are the corresponding losses in a concentric main.

Let us suppose that the inner core is solid and that mc is not greater than 2, so that we may use formula (101). Let the radius of the inner main be one centimetre. For a very high pressure cable a suitable value† of b would be 2·4 cms. Hence, since the section of the outer conductor is made equal to that of the inner, c is 2·6 cms. Substituting these numbers in the formula, we find that

$$\frac{\mathrm{R}}{l} = \frac{\rho}{\pi}\left(1 + \frac{m^4}{192}\right) + \frac{\rho}{\pi}\left(1 + \frac{0{\cdot}0072\, m^4}{192}\right),$$

where l is the length of the main in centimetres.

With the frequencies used in practice m is not very different from 1. We see that the skin effect increases the resistance of the inner conductor by about the half of one per cent., and the increase in the effective resistance of the

* Russell, 'Alternating Currents,' vol. ii. p. 458 *et seq.*

† Russell, 'Electric Cables and Networks,' p. 203.

outer conductor is less than the hundredth part of this. For a low voltage cable we might have $c=5/3$ and $b=4/3$. In this case

$$\frac{R}{l}=\frac{\rho}{\pi}\left(1+\frac{m^4}{192}\right)+\frac{\rho}{\pi}\left(1+\frac{0{\cdot}059\,m^4}{192}\right),$$

and hence the increase in the loss of the outer conductor is only about the twentieth part of the corresponding quantity for the inner. If the return conductor were a tight-fitting tube so that $b=1$ and $c=\sqrt{2}$, the formula becomes

$$\frac{R}{l}=\frac{\rho}{\pi}\left(1+\frac{m^4}{192}\right)+\frac{\rho}{\pi}\left(1+\frac{0{\cdot}148\,m^4}{192}\right).$$

Even in this case the increase in the loss of the outer is less than the fifth part of the increase in the loss of the inner, although the losses for very high frequencies would be practically the same in each conductor.

If the main consisted of a hollow inner conductor whose radii were 1 and $\sqrt{2}$ respectively and an outer conductor of radii 2·6 and 2·4, then we would have

$$\frac{R}{l}=\frac{\rho}{\pi}\left(1+\frac{0{\cdot}105\,m^4}{192}\right)+\frac{\rho}{\pi}\left(1+\frac{0{\cdot}0072\,m^4}{192}\right).$$

Hence, making the inner conductor hollow has appreciably reduced the losses due to skin resistance. It will also be noticed that the losses due to this cause in the $(1, \sqrt{2})$ cylindrical conductor are about 20 per cent. less when the return current is outside it than when it is inside it.

It is easy to see that if the current returns by a fine conductor in a minute hole along the axis of a cylindrical conductor, the increase in the losses for low-frequency currents will be nearly seven times greater than if it returned by a concentric tube outside the conductor.

IX. Values of m for Copper Conductors.

The volume resistivity of high conductivity annealed copper at 60° F. (15°·6 C.) is 1696·5. Hence

$$m^2=\frac{4\pi\mu\omega}{\rho}=\frac{8\pi^2\mu}{1696{\cdot}5}f.$$

Assuming that $\mu = 1$, we get the following table:—

f.	m.	f.	m.
5	0·4824	100	2·1573
10	0·6822	200	3·0509
15	0·8355	300	3·7366
20	0·9648	400	4·3147
25	1·0787	500	4·8240
30	1·1816	1000	6·8221
35	1·2763	5000	15·255
40	1·3644	10000	21·573
45	1·4472	100000	68·221
50	1·5255	1000000	215·734

Discussion.

Mr W. Duddell congratulated the Author, and remarked that the results obtained in the paper would be most useful. It was, he said, customary in dealing with high-frequency currents to use several parallel insulated wires instead of a single wire in order to increase the surface and obtain greater conductivity. He asked the Author to what extent it was advisable to do this. He also asked how far it was valuable to silver-plate copper wires for high-frequency currents.

Mr A. Campbell observed with regard to Dr. Russell's warning that inductance standards might be greatly dependent on frequency, that well designed single-layer wavemeter coils of highly stranded wire (*e. g.* 7/36[3]) did not show according to his experiments a variation of more than 1 or 2 in 1000 in their self-inductance for frequencies from 0 up to 1,000,000 ∼ per second, the actual inductances being from 10 to 200 microhenries.

Dr W. H. Eccles remarked that in high-frequency work the resistance of a conductor depended on the nature of the surface. If a copper wire tarnished its resistance altered. Uniform results could be obtained by lacquering the wires. He congratulated the Author upon the mathematical results obtained.

Mr B. S. Cohen said that a knowledge of the variations with frequency of effective resistance and inductance of copper conductors was of considerable importance in connexion with telephonic transmission, and it was to be hoped that Dr Russell would extend his valuable investigations to conductors lying side by side. The heaviest gauge of conductors met with in general telephonic practice is 2·85 mm. in diameter in the case of cables, and these conductors are separated by about 2·5 mm. of combined paper and air dielectric and twisted together with a lay of about 15 cms.

In the case of overhead open wires, the largest conductor in general use is 4 mm. in diameter, although in a number of cases a 5·69 mm. conductor has been used. The mean axial distance apart of these conductors is about 35 cms. as they are rotated in sets of 4 in a foot square.

Applying Lord Rayleigh's formula for effective resistance in these cases we get, for a frequency of 1500 ∼, an increase of 1 per cent. in effective resistance over steady current resistance in the case of the 2·85 mm. cable wire. A frequency of 1500 ∼ can quite safely be taken as the maximum frequency which is of any importance in speech articulation. The increase with the 5·69 mm. open wire at this same frequency is 12 per cent., and with the 4 mm. wire it is 3·55 per cent. In a table given in a well-known Pocket Book, a 5·69 mm. conductor is stated to increase 15 per cent. in effective resistance at 1500 ∼, and he noticed that this figure had been adopted by one writer on telephonic matters. Mr G. M. B. Shepherd some time ago worked out the decrease in inductance with frequency for wires lying side by side by both Heaviside's and Rayleigh's formulæ. The following percentage variations between 0 and 1500 ∼ were found for 5·69 mm. wire:—L ·37 per cent. decrease; r ·22 per cent. increase. This is very small, and is due to the fact that most of the inductance is due to the field between wire and wire. On the other hand, increase of resistance in this wire at 1500 ∼ means an increase in attenuation of 12 per cent.

In the case of 5·69 mm. conductors 33 cms. apart, the inductance as calculated by Maxwell's formula amounts to 3·2 millihenries, which would reduce the attenuation constant to one-fifth of its value at 1500 ∼. It is very desirable therefore to know the order of decrease of inductance with frequency.

Lastly, I should like to ask Dr Russell whether he thinks that the presence of the adjacent conductors in a telephone cable or open wire route is likely to modify the skin effect to any decided extent.

Mr Paterson expressed his interest in the paper, and asked if the formula given for the effective resistance of the core of a concentric main would be of the same form if the outer layer were removed. He also asked if the skin effect in the outer shell would be altered if the shell were cut parallel to its length and laid flat.

Prof. C. H. Lees pointed out that a change of sign in one of the early equations would lead to the use of the Bessel J function instead of the I function.

Dr Drysdale congratulated the Author, and wished to make a suggestion from the point of view of facilitating the use of these and other calculations. Electrical measurements frequently necessitated the employment of higher functions, and this involved either troublesome calculations or the compilation of bulky tables. He had therefore been led to try whether graphical methods could not be employed. The plotting of a function $f(x)$ graphically could only give values to a low degree of accuracy, but if some simple empirical formula $y=\phi(x)$ were found which approximated to the required function, it was possible to express the true value $f(x)$ in the form $r\phi(x)$ or $\phi(x)+a$,

where r or a might be termed a correcting factor or term, which only varied between comparatively narrow limits. By employing a curve in which $\log r$ or a was plotted against x, it was frequently possible to obtain values of a function from a single curve correct to 1 in 10,000 or closer, and which greatly assisted interpolation. He had found this of great service in dealing with elliptic integrals, and solid angles, and thought that it might perhaps also be applied to the ber and bei functions.

The Author, in reply to Mr Duddell, stated that in his opinion stranding and insulating the strands of the wires used in wireless telegraphy would not diminish the skin effect, if the strands were parallel to the axis of the wire. It would probably increase it. Silver-plating the wires would be beneficial. The distribution of the high-frequency current on the surface of the wires can sometimes be found by remembering that the distribution is such that no magnetic induction is produced in the metal. The corresponding electrostatic problems are consequently of great help. Mr Campbell's experimental results were interesting as they show that it is possible to make coils of low time constant so that frequencies even as high as a million have little effect on their inductance. These coils, however, are of little use for many experimental purposes. With ordinary laboratory coils a frequency of 10,000 will alter their inductance 2 or 3 per cent. If we have two parallel cylindrical wires practically touching, the inductance will alter from 3·7726 cms. per unit length to practically zero as the frequency is increased.

Dr Eccles rightly lays stress on the importance of keeping the surface of the wires used in wireless telegraphy in good condition and careful lacquering would be very beneficial. Mr Cohen's data about the telephone cables used in long distance transmission are most interesting. A piece of metal adjacent to the wire would in many cases modify the skin effect, but it would be difficult to deduce any general rule as the main current is affected in different ways by the induced eddy-currents in the neighbouring metal according to the relative positions and magnitudes of the metal and wire. Mr Paterson's question was rather hard to answer. In the hollow tube the current-density was a maximum at the outer surface. When split open and laid out flat, the current-density would be a maximum at the edges. In both cases, the uneven distribution of the current leads to an increase in the apparent resistance. But which would be the greater he could only determine by calculation. In reply to Professor Lees, he stated that he had first given the Kelvin functions in terms of Bessel's J function, but he had changed it to the I function as this made a + instead of a − in the fundamental definition. Dr Drysdale's methods of calculation would be convenient in the case of the elliptic functions, but it would not be so easy to apply them to periodic functions of continuously increasing amplitude.

XI. *A Laboratory Machine for Applying Bending and Twisting Moments simultaneously.* *By* Professor E. G. Coker, *M.A., D.Sc., of the City and Guilds of London Technical College, Finsbury* *.

[Plates XXV. & XXVI.]

Apparatus for applying simple tension or compression stresses, and also for applying bending or twisting moments to materials, are in common use in laboratories, but such machines are, as a rule, not very well adapted for experiments on combined stresses, although the frequency of such cases in practice make it very desirable that experiments on the effects of combinations of stresses should be carried out by engineering students. This is particularly the case with shafts which are generally subjected to stresses due to combined bending and twisting moments. The present paper describes a machine built by students of the City and Guilds Technical College, Finsbury, in which uniform bending and twisting moments can be applied simultaneously over the whole length of the specimen, and in any desired proportion to each other.

The machine is similar to one designed by the author for the testing laboratory at McGill University, Montreal, but with some modifications suggested by experience with the earlier machine. The principle on which the design is based is illustrated by fig. 1 (Pl. XXV.), in which a rod R is suspended at intermediate points A, B, by wires C, D, depending from a fixed support E. The equal overhanging ends of the rod are loaded by weights W, so that the applied couple between the points of support is uniform and of amount Wa, where a is the length of the lever-arm. The rod is also twisted by weights W_1 attached to equal arms of length b, so that there is a uniform twisting moment of amount W_1 between the points of suspension. The two systems of loading are independent and their ratio can be adjusted to any value desired.

In carrying out this arrangement in practice it is con-

* Read February 26, 1909.

venient to arrange that one of the levers for applying the twisting moment shall always remain in a horizontal position, and that the other shall be capable of turning through an arc to bring the first lever back to zero after each application of the load. The most convenient way of carrying this out is to replace the adjustable lever by a worm and worm-wheel gear secured in a casing and turned by a hand-wheel. To allow freedom for bending the worm-wheel casing must be pivoted to rotate around a line intersecting the axis of the specimen and perpendicular thereto, and this method of pivoting must also be adopted at the horizontal lever. This arrangement only differs from that of the perfectly freely suspended arrangement shown by fig. 1 (Pl. XXV.) in fixing one point of the rod, and this has the indirect advantage of stilling vibration which is troublesome in the freely suspended bar.

The arrangement described above is carried into effect in the manner indicated by figs. 2, 3, 4, and 5, showing the apparatus in side elevation, end elevations, and plan respectively.

The various parts are supported in a built up frame consisting of two planished steel shafts, A secured in cast-iron cross frames, B mounted on four standards, one of which latter is adjustable in height to secure steadiness on an uneven floor. Upon the steel shafts are two castings C, D, each of which has a cylindrical bearing E encircling one of the shafts and resting with a flat face F in line contact with the other shaft, and secured in position by a cross-bar G threaded on studs. This connexion is perfectly rigid, since it removes all degrees of freedom and it is readily released by simply turning back one of the cross-bar nuts, leaving the casting free to slide into a new position. It also has the advantage that no accurate fitting is required for the supporting frame. The casting C carrying the worm-wheel gear W has trunnion bearings H at right angles to and intersecting the axis of the specimen. The bearings are fitted with friction rollers, and when the machine is used simply for torsion the worm-wheel is kept in a vertical position by an arm I keyed to the bearing H and locked in position by a thumb-screw. A weight J attached by an arm to the second bearing balances the pivoted casing in all positions.

The weigh levers are supported from a vertical standard K of the frame D by a wire L, terminating in a thin plate M with a keyhole slot encircling the spindle N. Formerly a roller bearing was used for this spindle, but this is an unnessary refinement as the friction is extremely small, and can be easily taken into account. The casting supported in this way has three levers, P, Q, and R, the first two of which are for the application of twisting moments S, and the third R, in the line of the specimen, is for applying a bending moment.

All the loading levers are provided with knife-edges S, fig. 6, of circular form, made by turning an ordinary Whitworth nut down to form a disk with a V-shaped edge. These disks carry rings T with wide angled V-shaped recesses on the inner sides, and light rods V screwed into these rings carry the weights. This arrangement of knife edge is very easy to adjust accurately, and when bending and twisting stresses are applied simultaneously the rolling line contact adjusts itself to the bending and twisting of the specimen. The bending of the specimen causes a change in the effective arm of the bending levers, which is generally negligible, but a correction may be necessary with a very long specimen. For if a is the length of the lever-arm and b is the radius of the circular knife-edge, an angular deviation of amount θ will cause a change of $a-(a\cos\theta+b\sin\theta)$ in the lever-arm, and this is zero when $\theta=0$ and also when $a=a\cos\theta+b\sin\theta$.

In the machine described a is 10 inches and b is 0·5 inches, and the angles $\theta=0$ and $\theta=5°{\cdot}75$ both correspond to an effective length of 10 inches. The maximum correction between these values is easily shown to be at an angle θ given by the equation $b\cos\theta=a\sin\theta$, in the present case $2°{\cdot}9$ approximately, for which value the correction is 0·12 per cent. For values of θ greater than 6° the correction increases more rapidly, and its amount may be obtained from the diagram, fig. 7, which shows the percentage error for all angles up to 10°. In the majority of tests the angular change at the ends rarely exceeds 5°, and the correction is therefore so very small as to be practically negligible.

The worm-wheel W and the casting V for the weigh-

levers are bored out to receive the ends of the specimen, and are provided with fixed keys which slide in corresponding key-ways cut in the specimen. When tubes are subjected to stress they are provided with solid ends secured by transverse pins, thereby avoiding brazed joints since these latter are troublesome owing to the state of the metal being altered by the brazing. The end of the specimen projecting through the worm-wheel is fitted with a lever X for applying bending moment, and both levers for bending may be loaded independently or by a cross-bar suspended from stirrups as shown in fig. 2.

A photograph of the machine is shown by fig. 8 (Pl. XXVI.) with a specimen inserted which has failed under the combined effect of bending moment and twisting moments.

Measurement of the Strains.

The worm-wheel is graduated in degrees and a vernier circle enables 0·1 of a degree to be read with ease, while in order to start with a zero reading this vernier is carried on a ring sliding in a groove in the casing so that it can be adjusted to any angular position. This arrangement measures the twist on the whole specimen, and includes any motion due to back-lash in the keys and keyways during a test, it is therefore only suitable for measurements with long wires having substantial ends, and for plastic strains in which the end effects are negligible in comparison.

For observations within the elastic limit the author prefers to use an instrument* which is secured to the specimen and is self-contained.

This instrument was originally designed to measure the angle of twist within the elastic limit, and with some recent alterations it can be adjusted in a few seconds for measuring the angular change due to bending. The calibration of the readings is effected on the specimen and serves for both bending and twisting. Fig. 9 (Pl. XXV.) shows the apparatus in part longitudinal section.

It consists of a graduated circle A mounted on the specimen

* "On Instruments for Measuring small Torsional Strains," Phil. Mag. December 1899.

B by three screws C in the chuck-plate D. A sleeve E provided with three screws grips the specimen at a fixed distance away from the first set.

The spacing of these two main pieces on the specimen is effected by a clamp, not shown in the figure, which grips the double cones F, G, and maintains them at the correct distance apart, until the set screws are adjusted.

The clamp is afterwards removed, leaving the plane of the graduated circle perpendicular to the axis of the specimen and the sleeve correctly set and ready to receive the reading-microscope H.

The vernier plate carries a sliding tube I, on which a wire J is mounted, and the movement of this latter due to bending or twist is measured by a scale in the eye-piece K, the divisions of which are calibrated by reference to the graduated circle. It is found convenient to have the microscope-tube pivoted about an axis perpendicular to its central line at L, so that any slight difference due to imperfect centering can be adjusted by the screw M to make the calibration value agree for a series of specimens.

The observation wire may be set at any convenient position for calibration, but for observations of the angle of twist when the specimen is also subjected to a uniform bending moment the wire should be in the central plane perpendicular to the specimen. For if the bending is in the plane containing the axis of the specimen and the observation wire, it has the effect of causing new parts of the wire to come into view on the scale, but no error is caused thereby. If the specimen is bent in a plane at right angles to the former, then the change in the reading is $(\theta-\phi)l$, where θ and ϕ are the alterations of angle at the ends and $2l$ is the length of the specimen under observation. Since the bending is uniform $\theta=\phi$ and no correction is necessary. Bending in any other plane can be resolved into components in the vertical and horizontal planes, and therefore falls under the preceding cases. In order to effect the adjustment required, both the wire and the microscope slide in adjustable tubes provided with graduated scales, and the movement to bring the wire into focus is divided between them.

To check the setting of the wire in the central position it is convenient to apply a uniform bending moment, and then to observe if any change takes place in the reading. The position for no change in the reading can be found in a few seconds.

In experiments where the bending moment is constant and the twisting moment is varied, no adjustment is practically required during the elastic life of the specimen; and even when the bending moment is variable the adjustment is practically negligible, as the length of the specimen under test is only a few inches.

The instrument is used for observations of the angular change due to bending by adjusting the wire in the horizontal plane passing through the axis of the specimen, and at a fixed distance away from the central plane, as shown in fig. 7. Thus if the wire is at a distance x from the central plane, and the specimen is subjected to a uniform bending moment, the reading will be $(l+x)\theta-(l-x)\theta=2x\theta$, and this is a measure of the angular change θ between the ends, since any correction involves higher powers of θ which are negligible for elastic strains.

The instrument may therefore be used for measuring strains due to bending or twisting, and the single calibration required for both sets of readings is effected when the instrument is in position on the specimen.

With the usual notation the value of Young's modulus can be calculated from the equation $EI_1 dy^2/dx^2=M$, from which we obtain $EI_1\theta_1=Ml$ for determining E. Similarly the rigidity modulus for specimens of circular sections may be calculated from the equation $C=Tl/I_2\theta_2$. The ratio of C to E can also be readily obtained from the value $a\theta_1/\theta_2$, where a is an instrument constant: values of Poisson's ratio may be determined in this way without calculating C and E separately. As an example, we may take the case of a test on a steel tube 6 inches long, having an external diameter of 0·748 inch and a wall-thickness of 0·024 inch.

The calibration value of the eyepiece scale where referred

to the graduated circle was found to be 10·35 divisions for one minute of arc. The following readings were obtained:—

Twisting Moment. Inch-Pounds.	Reading	Δ.
0	0	
100	237	−237
200	474	−237
300	712	−238
400	920	−238
500	1188	−238

and the value of C = 12,400,000 in inches and pounds was determined from these observations.

The observation wire was then rotated into a horizontal position and set at a distance of one inch from the central plane. Bending moments were applied and readings were observed as follows :—

Bending Moment. Inch-Pounds.	Reading	Δ.
0	0	
100	59	−59
200	118	−59
300	178	−60
400	239	−61
500	297	−58

from which the value of E was calculated. The angular change θ_2 of one end of the specimen was determined from the readings R by the formula $R = 2x\theta_2$, where $x = 1$ inch, and account was taken of the fact that the effective radius of arc was changed from r_1 to r_2 in the new adjustment of the instrustment so that the angle θ_2 must be multiplied by r_1/r_2. In this instance r_1/r_2 was 1·61 and $E/C = r_2/r_1 \cdot \theta_1/\theta_2 = 2{\cdot}485$ or E = 30,800,000 and Poisson's ratio = ·243.

A tension-test on a similar piece of tube gave E = 30,700,000.

In addition to its applications for bending and twisting the apparatus may be used for testing a variety of cases of combined stress if a pump is added to give a fluid pressure in the interior of tubes.

The accompanying diagram (Pl. XXV. fig. 10) shows the results of tests to failure of bicycle-tubing when subjected to (I.) bending, (II.) twisting, and (III.) twisting combined with a uniform bending moment.

Discussion.

Prof. F. T. Trouton said he was interested in the experiments from the point of view of the viscous flow which occurred towards the end of the test. He asked if the flow would be different if the end of the bar was unloaded.

The Chairman expressed his interest in the paper, and asked if the experiments favoured any particular theory of fracture.

Prof. Coker said he had not yet made many experiments with the machine, but those made seemed to show that failure was due to shear stress. He hoped to carry out further experiments on both ductile and brittle materials.

XLI. *On the Self-Demagnetizing Factor of Bar Magnets.* *By* Silvanus P. Thompson, *D.Sc., F.R.S., and* E. W. Moss *.

[Plate XXVII.]

This paper consists of three parts :—(i.) A discussion of the significance and definition of the self-demagnetizing factor of magnets in general, and of bar-magnets in particular; (ii.) a redetermination of the values of the self-demagnetizing factor for bar-magnets of circular section; (iii.) determination of the values of the self-demagnetizing factor for bar-magnets of rectangular cross-sections of various proportions.

Part I.—Preliminary. On the Significance and Definition of the Self-Demagnetizing Factor.

Between any two magnet-poles, whether they are regarded as points, or as regions over which there is a surface-distribution of magnetism, there are magnetic forces. In

* Read February 26, 1909.

the space between any two point-poles the intensity of the magnetic field that is due to these poles, at any point in the line joining them, is expressed by the equation :

$$\mathscr{H}_x = \frac{m_1}{(a+x)^2} - \frac{m_2}{(a-x)^2};$$

where the respective strengths of poles are m_1 and $-m_2$; a the half of the distance between them, and x the distance of the point in question from the mid-point between them. The value of this expression in no way depends on the material in the space between the poles, whether non-magnetic or magnetic, or actually magnetized in any manner.

If m_1 and $-m_2$ are numerically equal, the expression becomes :

$$\mathscr{H}_x = 2m\frac{a^2+x^2}{(a^2-x^2)^2}.$$

At the mid-point, under the same condition, the intensity has the minimum value of

$$\mathscr{H}_{\text{min.}} = 2m \div a^2.$$

If the space between the two point-poles be regarded as occupied by a thin, cylindrical, uniformly-magnetized steel magnet the ends of which constitute the point-poles in question, then these equations will be the expresions for a self-produced magnetic field acting in a direction which opposes the actual magnetism of the magnet, and tending to demagnetize it. Each portion of the filiform magnet will be acted upon by a demagnetizing field, strongest towards the poles, weakest at the middle. The supposed uniform magnetization of the magnet will of course be unstable. If it were produced, even for a moment, there would at once be a retrocession of a portion of the magnetization from the ends, with a new distribution of the polarity. On the supposition that the middle part of the rod retains still its full flux, the retrocession of the pole would shorten the effective length of the magnet, diminishing the magnetic moment, but increasing any self-demagnetizing internal action. This tendency to produce a retrocession of the pole may operate to different degrees according to whether the bar consist of soft iron, or hard tungsten steel. In either case the retreat of the pole can be only incomplete ; because if we suppose the pole to have

actually retreated by any given amount—for example 1 centimetre—the end piece of that length will now be subjected to the magnetizing action of the rest of the bar, and will be remagnetized up to a certain point, namely, such that the reaction of the magnetism of this piece is equal to the magnetizing action of the whole of the rest of the bar, less the demagnetizing reaction of the bar as a whole. The inevitable result is a distributed pole. It cannot remain concentrated at one point, on the end; it must redistribute itself along the bar with a distribution determined by the conditions of equilibrium at every point.

Also the middle piece of the bar will not be exempt from influence, it, too, must diminish its inherent magnetism, because even in weak fields the magnetism of the hardest steel is subject to cyclical changes; and because any retrocession of the poles is, *pro tanto*, productive of an increase in the self-demagnetizing force at the middle. Only in cases where this self-demagnetizing force at the middle is less than that which suffices to produce an irreversible change in the magnetism of the steel, that is only in cases where the bar is very long in proportion to its cross-section, can the action at the middle be regarded as negligible.

It is clear then, in general, that for every bar-magnet there will be a self-demagnetizing action the value of which, at the middle of the bar, depends, for a given intensity of magnetization, on the length of the bar relatively to its cross-section, on the permeability of its parts, and on the distribution of its surface-magnetism. Owing to the circumstance that with every kind of steel the permeability is neither constant, nor stands in any simple or even single-valued relation to the flux-density, any calculation of the actual polar distribution for rods or bars is exceedingly complicated and indeed impracticable.

As is well-known, the one and only form of magnet that is practicable for calculation is that of the ellipsoid, the properties of which are that for any and every value of the permeability, and when placed in any uniform field, the surface magnetism is so distributed that the magnetic force which this distribution of polarity exerts in the interior is uniform at every point within. Hence the internal demagnetizing force everywhere within is constant; the

resultant field at every point of the interior (if the structure is homogeneous and isotropic) is also constant, and the internal flux-density cannot but be uniform.

Du Bois and others have determined by experiment the demagnetizing actions of cylindrical rods of various dimensions, and have compared them with ellipsoids of revolution of similar dimensional proportions.

In the case of ellipsoids, it is natural to compare the value of the intensity of the self-demagnetizing force with the value of the internal magnetization $\mathscr{I}$, because both of these are uniform throughout the interior. For an ellipsoid of revolution of given axial proportions, whether highly or only slightly magnetized, both $\mathscr{H}_d$ the self-demagnetizing force, and $\mathscr{I}$, are proportional to one another. By definition $\mathscr{I}$ is the quotient of the magnetic moment by the volume. For a given size of equatoreal cross-section of the prolate ellipsoid, the magnetic moment and the volume are both proportional to the axial length. But for ellipsoids of given equatoreal section and of different lengths, the self-demagnetizing force $\mathscr{H}_d$ (for a given $\mathscr{I}$, or a given m) does not follow any simple function of the axial length. For small changes of length it is nearly proportional to the inverse square of the axial length, but is accurately expressible only in terms deducible from a rather troublesome elliptic integral. Maxwell and Du Bois (following F. Neumann) have given the general formulæ. But because both $\mathscr{H}_d$ and $\mathscr{I}$ are for an ellipsoid of given ellipticity proportional to one another, it was quite natural to regard the quotient of the former by the latter—that is to say the amount of self-demagnetizing force per unit of intrinsic magnetization—as a sort of natural coefficient, and to recognize it as a *self-demagnetizing factor*. Du Bois (following Maxwell) assigns to it the symbol N. It has a definite value for ellipsoids of revolution of any assigned ellipticity. Thus for an ellipsoid of equatoreal diameter 1 and axial length 10, the value of N is 0·2549 whatever the degree of magnetization. Thus if an ellipsoid of this form be magnetized so that $\mathscr{I}$ has the value 100 C.G.S. units, the self-demagnetizing force within the ellipsoid will everywhere have the value of 25·49 gauss. Denoting the dimension-ratio of axial length l to equatoreal diameter d by the symbol $\mathfrak{m} = l \div d$ (in Du Bois' notation), then $\mathfrak{m}^2 N$

varies from 25·49, when $\mathfrak{m} = 10$, to 80 when $\mathfrak{m} = 1000$. (See Du Bois, *The Magnetic Circuit*, p. 41.)

But, if we now compare the case of the ellipsoid with that of the cylindrical bar, we find that the matter is not so simple. For with the bar, as stated above, $\mathscr{H}_d$ is by no means uniform throughout the interior, neither is $\mathscr{I}$. The former has its minimum at the middle point of the axis, while the latter has its maximum at the equatoreal section of the bar. To compute the value of $\mathscr{H}_d$ at the middle point (or at any other) is impossible without knowing the law of surface distribution, and this depends on too many conditions to be of service. But the nett value of $\mathscr{H}_d$ for the entire bar can be easily determined by comparing the $\mathscr{B}$-$\mathscr{H}$ curve of the bar (found by experiment) with the $\mathscr{B}$-$\mathscr{H}$ curve of a ring (or infinitely long rod) of the same iron, and taking the difference of the values of $\mathscr{H}$ for some assigned value of $\mathscr{B}$. On the other hand, values of $\mathscr{I}$ can be found by experiment, either magnetometrically, giving the mean value, or ballistically, giving either maximum or mean according as whether the exploring coil on the bar is wound over its whole length or over its equatoreal zone only. The ratio $\mathscr{H}_d \div \mathscr{I}$ so deduced may still be called the *self-demagnetizing factor*, and values found for rods of different dimension-ratios.

Magnets of other forms, for example the slit toroid, or anchor-ring with a gap in it, and the horse-shoe magnet with parallel limbs of given proportions, will likewise have self-demagnetizing factors of their own, dependent on their geometry and on the distribution of their polarities. With them also, neither $\mathscr{H}_d$ nor $\mathscr{I}$ will have constant values at all points within the substance of the magnet; and for each form therefore the term "self-demagnetizing factor" bears a significance different from that which it possesses for the ellipsoid of revolution or for the cylindrical bar.

All previous writers have defined the term *dimension-ratio* as applied to a bar as the ratio between its length l and the diameter d of its circular section. But when we come to deal with forms of cross-section other than circular, it is inconvenient to use this mode of expression. For if we were dealing with a flat bar of breadth b, the curve for self-demagnetizing factors in terms of the ratio $l \div b$ would not be comparable with those for cylindrical bars in terms of $l \div d$.

The preferable way, when such comparison has to be made, is to state a dimension-ratio, for bars of all and every form of section, in terms of the ratio which is borne by the length to the square-root of the area of section. The ratio $l \div \sqrt{A}$, we accordingly propose to denote by the symbol λ. For any given bar we have the relation $\lambda = m \times 1{\cdot}128$.

Part II.—Experimental. On the Values of the Self-demagnetizing Factor for Bar-magnets of circular section.

Several investigators, including Ewing, Fromme, Holz, and Ascoli, have written on the factor of self-demagnetization of cylindrical bar-magnets, and have given experimental values for bars having different ratios of length to diameter. The best-known results are those published by Du Bois, who has compared the values obtained with those for ellipsoids having similar axial ratios. More recently, Riborg Mann obtained a series of values slightly higher than those obtained by Du Bois, who has accepted them as more correct than his own figures. Ewing's observations ranged over rods the lengths of which varied from 300 diameters down to 50 diameters. Du Bois' results go from a dimension-ratio of 1000 down to one of 10; those of Riborg Mann from one of 300 down to 5. The magnitude of the outstanding discrepancies may be indicated by stating the values found by different observers for the self-demagnetizing factor N for cylinders having a dimension-ratio of 50. For rods of this proportion Du Bois found $N = 0{\cdot}0162$; Riborg Mann $N = 0{\cdot}01825$. For the ellipsoid of revolution having the same axial ratio of 50, Du Bois and Riborg Mann agree in assigning the value 0·0181, and presumably the true value for the cylinder is less than that figure. Greater discrepancy is found for shorter cylinders. For a dimension-ratio of 10 Du Bois gives $N = 0{\cdot}2160$, while Riborg Mann gives $N = 0{\cdot}25500$.

To clear up, if possible, such discrepancies a research was undertaken in the laboratory at the Technical College, Finsbury.

The bars used were cut from two long rods of best Swedish iron carefully annealed, and for comparison a ring was forged from the same material. To each and all of the rods the

same diameter was given, namely, 1·128 cm., in order that each might have a cross-section of precisely 1 sq. cm. After being turned down to approximate size they were annealed, and then finally turned to the precise size required.

The magnetizing coil used to magnetize the rods was a long coil wound on a brass tube 91·4 cm. in length and 4·75 cm. in external diameter. It was carefully overwound with 5800 turns of wire of No. 20 s.w.g., in seven layers. With this coil a very uniform field could be produced of any desired intensity up to $\mathcal{H}=255$. The uniformity of the field between the ends of this coil was tested by means of a short coil of somewhat smaller diameter, wound on a turned bobbin of hard fibre, of a size fitted to slide inside the brass tube. The wires of this smaller coil were connected with a ballistic galvanometer, the throw of which was observed when the current in the long magnetizing coil was reversed. The field was found to be sensibly uniform for a length of 60 cm.; while the longest specimen of iron was only 40 cm. There was therefore no need to apply any corrections for non-uniformity of field.

The ballistic method was also used for determining the magnetization of the bars. On the middle of each bar was wound an exploring coil of 10 turns of very fine wire, the breadth of each such coil not exceeding 0·25 cm. The galvanometer was calibrated by the short coil previously mentioned, its dimensions being accurately known. The magnetizing current was measured by a standard commercial amperemeter, the readings of which were calibrated at regular intervals of time by a Crompton potentiometer.

Each specimen was mounted on a carrier by means of which it could be inserted centrally in the middle of the long magnetizing coil. The galvanometer calibration having been effected, a test was made of each bar by subjecting it to a series of reversals in fields varying from $\mathcal{H}=20$ to $\mathcal{H}=255$, the throws of the galvanometer being noted; and for each bar a $\mathcal{B}$-$\mathcal{H}$ curve was then plotted.

A similar curve having been plotted from the tests made on the ring, the values of the demagnetizing intensity of field $\mathcal{H}_d$, due to the self-demagnetizing action of the poles of each bar, could then be calculated, for any value of $\mathcal{B}$, by taking the abscissa, corresponding to that ordinate, in the

curve for that bar, and subtracting the corresponding abscissa in the curve for the ring.

Let the field due to self-demagnetization at the mid-point of any bar, for any given flux-density $\mathcal{B}$, be called $\mathcal{H}_d$. Let the total impressed field due to the magnetizing coil be called $\mathcal{H}$; and let the impressed field required in the ring to produce the same given value of $\mathcal{B}$ be called $\mathcal{H}_r$. Then

$$\mathcal{H}_d = \mathcal{H} - \mathcal{H}_r.$$

Then since, by definition, the self-demagnetizing factor N has the value

$$N = \frac{\mathcal{H}_d}{\mathcal{I}},$$

and

$$\mathcal{I} = \frac{\mathcal{B} - \mathcal{H}}{4\pi},$$

we get

$$N = \frac{4\pi \mathcal{H}_d}{\mathcal{B} - \mathcal{H}}.$$

Fig. 1 (Pl. XXVII.) gives the $\mathcal{B} - \mathcal{H}$ curves for our rods, the dimension-ratios of which varied from 35·6 to 2·66. These curves were sensibly straight lines up to $\mathcal{B} = 12{,}000$, or as high as the curves could be carried. The value $\mathcal{B} = 10{,}000$ was chosen for the calculation of the self-demagnetizing force and deduction of the self-demagnetizing factor, except for the very short rods in which lesser values of $\mathcal{B}$ were alone available.

Fig. 2 gives as the final result the curve exhibiting the values of the self-demagnetizing factors found, for rods of different lengths, the corresponding values found by Du Bois and by Riborg Mann being added for comparison.

It will be seen (1) that our values are throughout lower than those found by either of these experimenters; (2) that we have carried the determinations down to shorter rods than those examined by either of them; (3) that the discrepancies between their results and ours are smaller as the dimension-ratios are larger.

The fact that our values are throughout lower than those of Du Bois and Riborg Mann is doubtless due to the circumstance that they used a magnetometric method, whilst we have returned to the ballistic method of Ewing. The values of $\mathcal{I}$ which they employ are the mean values deduced from the magnetic moment, and are presumably mean values throughout the length of the bar, whilst our values of $\mathcal{I}$ are

TABLE I.—Demagnetizing Factors for Cylindrical Bars.

$\frac{l}{d}$.	$\frac{l}{\sqrt{\text{Area}}}$.	Demagnetizing Factors.			
		Cylinder.			Ellipsoid of Revolution.
		Du Bois.	Riborg Mann.	Thompson & Moss.	
2·66	3·0	...	...	1·2	
3·55	4·0	...	...	0·83	
4·44	5·0	...	...	0·618	
5·0	5·64	...	0·6800	*0·53*	
5·34	6·0	...	...	0·483	
6·66	7·5	...	...	0·3518	
8·86	10·0	...	...	0·233	
10·0	11·28	0·216	0·2550	*0·198*	0·2549
10·67	12·0	...	...	0·18	
13·3	15·0	...	...	0·1287	
15·0	16·92	0·1206	0·1400	*0·108*	0·135
17·72	20·0	...	...	0·0826	
20·0	22·56	0·0775	0·08975	*0·069*	0·0848
25·0	28·2	0·0533	0·06278	*0·049*	0·0579
26·6	30·0	...	...	0·0438	
30·0	33·84	0·0393	0·04604	*0·036*	0·0432
35·6	40·0	...	...	0·0255	
40·0	45·12	0·0238	0·02744	*0·0223*	0·0266

N.B.—Figures in italics are values got by interpolation.

the values deduced from the action of an exploring coil wound round the equator of each bar, and presumably measure the maximum value of $\mathscr{I}$. As the self-demagnetizing action of a bar depends on neither the mean value, nor the maximum value of $\mathscr{I}$, as we have seen, but on a mean that is impossible to calculate unless the actual surface distribution of the magnetism is known, it appeared to us preferable to take the value of $\mathscr{I}$ that can be ascertained with precision at the place where the self-demagnetizing force has its minimum, namely the centre of the bar.

One point of criticism on Riborg Mann's results may be permitted us. To give us confidence in our results, we have throughout used substantial bars of 1·128 cm. in diameter, and have raised the lengths. Riborg Mann used a single cylinder of iron 11·850 cm. in length, originally of a diameter 1·526 cm., therefore of a dimension-ratio of 7·76. This he turned down successively to smaller and smaller diameters until he reached a diameter of 0·237 cm., giving a dimension-ratio of 50. How he contrived to turn so thin a wire is

remarkable. It would have a sectional area of only 0·0561 sq. cm. Further, while his cylinder was 11·850 cm. in length, his magnetizing coil was only 30 centimetres long and 4 cm. in diameter. The ends of his rod were therefore at points only $2\frac{1}{4}$ diameters distant from open ends of the coil, where therefore the value of the field would differ by some $2\frac{1}{2}$ per cent. from the value of the uniform field at the middle of the coil.

Part III.—Experimental. On the Values of the Self-Demagnetizing Factor for Bar-magnets of rectangular cross-sections of various proportions.

We are not aware that any previous investigator has determined the self-demagnetizing factor for square bars or flat bars of rectangular section such as are often used in magnetic work.

A priori we should expect the self-demagnetizing factors to be less than for bars of equal section of circular form and equal length; since the greater perimeter of the rectangular forms is magnetically equivalent to giving to the end parts a polar expansion, reducing the reluctance of the air-paths of the external magnetic flux, and so bettering the magnetic circuit. And such has proved to be the case.

The experiments were made in exactly the same manner as those for the bars of circular section. Rectangular rods of the softest Swedish iron of various proportions were procured, and reduced by milling-cutter to the required form, so as in every case to have a sectional area of 1 square centimetre; the ratios of breadth to thickness being respectively 1 : 1 ; 2 : 1 ; 4 : 1 ; 6 : 1, and 10 : 1. From each of these rectangular rods pieces were cut of lengths of 10, 8, 6, 5, 4, and 3 centimetres respectively. In all 35 different ones were examined. For each of these a $\mathscr{B}$-$\mathscr{H}$ curve was plotted; and the self-demagnetizing-factors were deduced as before.

In figs. 3, 4, 5, 6, 7, and 8 these various curves are plotted; and in fig. 9 the final results are summed up by plotting the several demagnetizing-factors as functions of λ the ratio of the length to the square-root of the area of section.

Table II. gives numerically the values of the self-demagnetizing factors obtained for various ratios of breadth b to

thickness t of the cross-section, and also for various values of λ. The individual bars were carefully gauged for breadth and thickness, and the slight discrepancies (never exceeding 1 per cent. of the intended ratio) were allowed for; but being small they occasioned no difference in the plotting.

TABLE II.—Demagnetizing Factors for Bars of different lengths and equal Sectional Area, having Rectangular Sections from 10 : 1 to 1 : 1.

$\frac{l}{\sqrt{A}}=3.$		$\frac{l}{\sqrt{A}}=4.$		$\frac{l}{\sqrt{A}}=5.$		$\frac{l}{\sqrt{A}}=6.$		$\frac{l}{\sqrt{A}}=8.$		$\frac{l}{\sqrt{A}}=10.$	
$b/t.$	N.	$b/t.$	N.	$b/t.$	N.	$b/t.$	N.	$b/t.$	N.	$b/t.$	N.
10·03/1	0·828	10/1	0·586	10·05/1	0·44	10/1	0·354	10·2/1	0·2358	10/1	0·178
5·95/1	0·925	5·96/1	0·66	5·99/1	0·488	6/1	0·3885	5·96/1	0·264	5·99/1	0·19
3·98/1	1·02	3·99/1	0·726	3·98/1	0·528	3·96/1	0·415	4/1	0·28	3·98/1	0·205
2·0 /1	1·098	2/1	0·775	2/1	0·575	2/1	0·448	2/1	0·3	2/1	0·22
1·492/1	1·13	1·5/1	0·7980	1·49/1	0·59	1·5/1	0·4645	1·5/1	0·3075	1·495/1	0·224
0·9914/1	1·13	0·99/1	0·8	0·993/1	0·59	0·996/1	0·465	...	...	0·993/1	0·224

It will be noticed that the Table records values also for bars having the ratio of 1·5 : 1; but no curves are given for this ratio, as they were practically the same as those for square bars. In plotting the $\mathscr{B}$-$\mathscr{H}$ curves for these particular bars, it was possible in two cases only to distinguish the curves from those for the square bars, and in these two cases the difference was extremely small.

For equal values of the ratio of l to $\sqrt{A}$, it was found in general that the self-demagnetizing factor, for bars having a sectional ratio of 2 to 1, was about 93 per cent. of that for bars of square section; while for flat bars, having a sectional ratio of 10 to 1, the value of the self-demagnetizing factor went down to about 75 per cent. of that for bars of square section.

DISCUSSION.

Mr. S. SKINNER said the Authors had referred to the ageing of their magnets, but in the actual experiments the field was applied and the measurement taken at once before any ageing could take place.

Prof. F. T. Trouton pointed out that the self-demagnetizing factor became less as the bar became thinner, and asked if this was likely to continue until the bar became a thin tape.

Mr. A. Campbell asked if experiments had been made on different materials. The curves obtained for soft iron might be very different from those obtained for hard steel. With regard to the standard magnet referred to by Prof. Thompson, he had used one at the National Physical Laboratory and found it a satisfactory way of calibrating ballistic galvanometers.

Mr. Rayner asked if any experiments had been made on hollow magnets, such as those used in the Kew pattern unifilar magnetometer.

Prof. C. H. Lees asked how the iron ring used in their experiments had been prepared, and suggested that it might be advisable to make use of two or three rings.

Mr. R. S. Whipple said the question of the preparation of a standard magnet was an important one. Had the Authors made any experiments with ball-ended magnets, especially those with thin wires and large balls?

The Secretary read a letter from Dr. A. Russell, stating that the paper was a notable contribution to the practical theory of magnetism. The Authors deserved thanks for their lucid comments on the self-demagnetizing factor and for the important experimental results they had obtained. He asked how they had calculated the magnetizing force. Had they assumed that the magnetizing solenoid was infinitely long and that the radius of the circular axis of the ring was infinite? The errors introduced by these assumptions were small, and Dr. Russell indicated how they might be calculated. It could be shown that if the length of the axis of a helical current was greater than six times its diameter, the magnetic flux entering or leaving the end planes was within about one per cent. of half the total flux through the central plane. Seeing that the external field produced by a cylindrical bar-magnet was almost equivalent, except near the ends, to that produced by a helical current, he would not be surprised if the flux leaving the end of a cylindrical bar-magnet was appreciably less than half the total flux through the central plane. He should be grateful if Dr. Thompson could give any data on this point. Further researches on the demagnetizing effect of the free magnetism in the air-gap of a split toroid would be of great interest, and would throw some needed light on the magnitude of the errors involved in the ordinary magnetic formulæ used in engineering.

Prof. S. P. Thompson, in reply to Mr. Skinner, said that in their experiments there was an immediate ageing of their magnets on the application of the field. With regard to Mr. Campbell's remarks, different materials would of course give different curves. A forged ring was used in the research, but the forging could not have had much effect on the curves obtained. They had not tested hollow magnets, but experiments had been made on magnets with holes in the ends and the results were similar to those obtained for solid magnets. He had not performed experiments with Robison ball-ended magnets, but the effect of magnetic material at the ends of a magnet was to reduce its self-demagnetizing factor. Referring to Dr. Russell's remarks, he said the corrections he mentioned could be neglected.

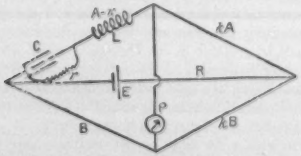
XLII. *On Pirani's Method of Measuring the Self-Inductance of a Coil.* *By* E. C. Snow, *B.A.**

In Pirani's method of measuring the self-inductance of a coil, it is assumed that the self-inductance of the galvanometer used may be neglected. That this assumption is legitimate in the case in which the discharge of the condenser employed (which, it will be shown, is of the same nature as the discharge through the galvanometer) is continuous has been proved †.

The object of the present analysis is to investigate the case in which the discharge of the condenser is oscillatory.

A condenser of capacity C, the "internal" resistance of which can be neglected, is placed in series with the coil whose self-inductance is to be measured. The condenser is shunted by a non-inductive resistance r. This combination forms one arm of a Wheatstone's bridge, the resistances of the other arms of which are adjusted so that a steady balance holds between the four arms.

One or other of two methods can now be used: (i.) keeping r constant, C can be varied until there is no throw on closing the galvanometer-key before the battery-key; or (ii.) keeping C constant r can be varied, the total resistance of the arm being kept constant, until the same result is arrived at.



The following symbols will be used:—

I, the current in the battery.

x the current in the galvanometer or (in the case of an alternating current) the telephone.

a, the current in the coil whose self-inductance is required.

* Read March 12, 1909.

† O. de A. Silva, *L'Eclairage Electrique*, vol. l. pp. 113–116.

a, the current in the shunt resistance r.
c, the current flowing into the condenser.
b, the current in the resistance B.
a', the current in the resistance kA.
b', the current in the resistance kB.
R, the resistance of the battery.
ρ, the resistance of the galvanometer or telephone.
L, the self-inductance of the coil.
L′, the self-inductance of the galvanometer or telephone.
E, the applied E.M.F., not necessarily constant.

The resistances of the arms of the bridge are adjusted as indicated.

The following seven equations are derived immediately from Kirchhoff's laws applied to the various circuits :—

$$a=\mathrm{C}+a \quad . \quad . \quad . \quad (1) \qquad b=b'-x \quad . \quad . \quad . \quad (2)$$

$$a=a'+x \quad . \quad . \quad . \quad (3) \qquad \mathrm{I}=a+b \quad . \quad . \quad . \quad (4)$$

$$\mathrm{RI}+\mathrm{B}b+k\mathrm{B}b'=\mathrm{E}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\mathrm{L}\frac{da}{dt}+ra+(\mathrm{A}-r)a+\rho x+\mathrm{L}'\frac{dx}{dt}-\mathrm{B}b=0. \quad . \quad . \quad (6)$$

$$\rho x+k\mathrm{B}b'-k\mathrm{A}a'+\mathrm{L}'\frac{dx}{dt}=0. \quad . \quad . \quad . \quad (7)$$

Also
$$c=\mathrm{C}r\frac{da}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

From (1), (2), (3), (4), and (7) we find that

$$a=\frac{fx+k\mathrm{BE}}{h}+p\frac{dx}{dt},$$

and

$$b=\frac{k\mathrm{AE}-gx}{h}-q\frac{dx}{dt},$$

where

$$f=\{\rho+k\mathrm{B}+k\mathrm{A}\}\{\mathrm{R}+\mathrm{B}+k\mathrm{B}\}-k^2\mathrm{B}^2.$$

$$g=\mathrm{R}\{\rho+k\mathrm{A}+k\mathrm{B}\}+k^2\mathrm{AB}.$$

$$h=k\mathrm{A}\{\mathrm{R}+\mathrm{B}+k\mathrm{B}\}+k\mathrm{RB}.$$

$$ph=\mathrm{L}'\{\mathrm{R}+\mathrm{B}+k\mathrm{B}\}.$$

$$qh=\mathrm{L}'\mathrm{R}.$$

Substituting these values of a and b in (6) we have

$$pL\frac{d^2x}{dt^2}+\left(L\frac{f}{h}+M\right)\frac{dx}{dt}+\left(m-\frac{f}{h}r\right)x$$
$$=-r\alpha+\frac{kBr}{h}E-\frac{kBL}{h}\frac{dE}{dt}, \quad . \quad . \quad (9)$$

where

$$M=L'+Ap+Bq-pr,$$

and

$$hm=Af+Bg+\rho h.$$

Differentiating (9), and eliminating α and $\frac{d\alpha}{dt}$ between the resulting equation together with (8) and (9), we finally obtain

$$LCp\frac{d^3x}{dt^3}+\left\{MC+LC\frac{f}{h}+\frac{Lp}{r}\right\}\frac{d^2x}{dt^2}$$
$$+\left\{p+mC+\frac{M}{r}+\frac{f}{hr}(L-Cr^2)\right\}\frac{dx}{dt}+\frac{m}{r}x$$
$$=\frac{kB}{hr}(Cr^2-L)\frac{dE}{dt}-\frac{kBCL}{h}\frac{d^2E}{dt^2}. \quad . \quad . \quad (10)$$

The solution of this differential equation will consist of two parts, a complementary function and a particular integral. The former is the solution obtained by putting the right-hand side of (10) equal to zero. This part will, consequently, be the complete solution in the case for which E is constant.

In this case the value of x is, therefore,

$$x=A_1e^{-\alpha_1 t}+A_2e^{-\alpha_2 t}+A_3e^{-\alpha_3 t},$$

where α_1, α_2, and α_3 are the roots of the cubic equation

$$LCpy^3-\left\{MC+LC\frac{f}{h}+L\frac{p}{r}\right\}y^2$$
$$+\left\{p+mC+\frac{M}{r}+\frac{f}{hr}(L-Cr^2)\right\}y-\frac{m}{r}=0,$$

and A_1, A_2, and A_3 are constants determined by the initial conditions.

From the above value of x—the current through the galvanometer—the value of α, the current through the shunted resistance, can be obtained.

For, from (1) and (8) above, we have

$$Cr\frac{d\alpha}{dt}+\alpha=a=\frac{f}{h}x+\frac{kBE}{h}+p\frac{dx}{dt}.$$

Putting in the values of x and $\frac{dx}{dt}$, and solving for α, we find

$$\alpha=A_0e^{-\frac{t}{Cr}}+\frac{kBE}{h}+A_1\left(\frac{f}{h}-p\alpha_1\right)\frac{e^{-\alpha_1 t}}{1-Cr\alpha_1}$$

$$+A_2\left(\frac{f}{h}-p\alpha_2\right)\frac{e^{-\alpha_2 t}}{1-Cr\alpha_2}+A_3\left(\frac{f}{h}-p\alpha_3\right)\frac{e^{-\alpha_3 t}}{1-Cr\alpha_3},$$

A_3 being another constant determined by the initial conditions.

From this we see that the discharge of the condenser will be continuous or oscillatory according as α_1, α_2, and α_3 are all real or one real and two imaginary (these are the only cases which have to be considered for a cubic equation). But from the above value of x, if α_1, α_2, and α_3 are all real it is seen that the discharge of the galvanometer is continuous, while if two roots are imaginary it is oscillatory. Hence the discharge of the condenser is of the same nature as that through the galvanometer.

To consider the case of the oscillatory discharge of the condenser, therefore, we must put

$$\alpha_2=k_2+ik_3,$$

and consequently

$$\alpha_3=k_2-ik_3, \qquad i \text{ being } \sqrt{-1}.$$

The terms $A_2e^{-\alpha_2 t}+A_3e^{-\alpha_3 t}$ now become $A_4e^{-k_2 t}\cos(k_3t-\epsilon)$. A_4 and ϵ being other constants.

The complete value of x now is

$$x=A_1e^{-\alpha_1 t}+A_4e^{-k_2 t}\cos(k_3t-\epsilon).$$

The initial conditions will be of the form

$$(x)_0=0, \quad \left(\frac{dx}{dt}\right)_0=u, \quad \text{and} \quad \left(\frac{d^2x}{dt^2}\right)_0=v,$$

u and v being constants.

Differentiating the expression for x and putting $t=0$, we

shall have the following equations to determine the constants A_1, A_4, and ϵ.

$$A_1 + A_4 \cos\epsilon = 0,$$
$$A_1\alpha_1 + A_4 k_2 \cos\epsilon - A_4 k_2 \sin\epsilon = -u,$$
$$A_1\alpha_1^2 + A_4(k_2^2 - k_2^2)\cos\epsilon - 2A_4 k_2 k_2 \sin\epsilon = v.$$

These give

$$\Delta A_1 = v + 2uk_2,$$
$$\Delta A_4 \cos\epsilon = -v - 2uk_2,$$
$$\Delta A_4 \sin\epsilon = \frac{1}{k_2}\left\{(\alpha_1 - k_2)(v + 2uk_2) + \Delta u\right\}.$$

where $\Delta \equiv (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)$, α_2 and α_3 having the values used above.

In the case of a constant E.M.F. a ballistic galvanometer is used. This measures the total quantity of electricity flowing from the time $t=0$. The conditions of the experiment require that the galvanometer should give no deflexion after the current has flowed for an appreciable time. This condition* can be expressed by $\int_0^\infty x\,dt = 0$, the upper limit of integration being taken as infinity, since the current will always be zero after a small interval has elapsed. This condition will give

$$0 = A_1\int_0^\infty e^{-\alpha_1 t}dt + A_4\cos\epsilon\int_0^\infty e^{-k_2 t}\cos(k_2 t)\,.\,dt + A_4\sin\epsilon\int_0^\infty e^{-k_2 t}\sin(k_2 t)\,.\,dt;$$

$$i.e. \quad \frac{A_1}{\alpha_1} + A_4\cos\epsilon\frac{k_2}{k_2^2 + k_2^2} + A_4\sin\epsilon\frac{k_2}{k_2^2 + k_2^2} = 0.$$

Also $2k_2 = \alpha_2 + \alpha_3$, and $k_2^2 + k_2^2 = \alpha_2\alpha_3$.

* Lord Rayleigh, B. A. Report, 1883, p. 444, has drawn attention to the imperfections of a galvanometer as an instrument for indicating whether the integral sum of the transient currents through it is zero or not. See also a paper by Alex. Russell, M.A., M.I.E.E., Phil. Mag. 6th series, vol. xii. No. 69, 1906, where a full investigation and explanation of the necessary corrections is given.

Putting in the values of the constants, therefore, we have

$$\tfrac{1}{2}\{v+u(\alpha_2+\alpha_3)\}\{\alpha_2\alpha_3+\alpha_1(\alpha_1-\alpha_2-\alpha_3)\}+\Delta\alpha_1 u=0;$$

$$i.e.\ \{v+u(\alpha_2+\alpha_3)\}\{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\}+\Delta\alpha_1 u=0,$$

and, therefore, $$v+u(\alpha_1+\alpha_2+\alpha_3)=0. \quad . \quad . \quad . \quad . \quad (11)$$

Now from equation (9) above we have, when $t=0$,

$$pL\left(\frac{d^2x}{dt^2}\right)_0+\left(L\frac{f}{k}+M\right)\left(\frac{dx}{dt}\right)_0=\frac{kBr}{k}E,$$

the term $\frac{dE}{dt}$ being always zero in the case of a constant E.M.F.

Also, since $(a)_0=0$, we shall have, on putting $t=0$ in the expression for a, $\frac{kBE}{k}+p\left(\frac{dx}{dt}\right)_0=0$.

Hence

$$pL\left(\frac{d^2x}{dt^2}\right)_0+\left\{L\frac{f}{k}+M+pr\right\}\left(\frac{dx}{dt}\right)_0=0.$$

$$i.e.\quad pLv+\left\{L\frac{f}{k}+M+pr\right\}u=0.$$

Also

$$\alpha_1+\alpha_2+\alpha_3=\left\{MC+LC\frac{f}{k}+L\frac{p}{r}\right\}\div LCp.$$

Using these last expressions, (11) becomes

$$C\left\{L\frac{f}{k}+M+pr\right\}-\left\{MC+LC\frac{f}{k}+L\frac{p}{r}\right\}=0.$$

$$i.e.\quad \frac{p}{r}(L-Cr^2)=0, \quad or \quad L=Cr^2.$$

Therefore, in the case of the oscillatory discharge of the condenser, with a constant E.M.F., the relation $L=Cr^2$ holds. Hence the expression for the inductance of the coil is independent of the inductance of the galvanometer.

Cass Institute, E.C.,
Feb. 23, 1909.

Discussion.

Mr. A. Campbell expressed interest in the Author's thorough investigation in the first part of the paper. In the case of alternating currents, however, Mr. Snow's final result involved the resistance and inductance of the galvanometer circuit. Since the condition for a balance was that the current through the galvanometer should be zero at every instant,

the final result could not involve either of these quantities, both could be altered to any extent without affecting the balance. The correct solution for alternating currents could, however, be obtained without difficulty. Thus, let P be the effective resistance of the coil L, and let $\sigma = A/B$, the rest of the notation remaining unchanged.

Then $$P + L\omega\sqrt{-1} + \frac{r}{1 + rC\omega\sqrt{-1}} = \sigma B.$$

Separating the real and imaginary parts,

$$L = rC(\sigma B - P) \quad \text{and} \quad rLC\omega^2 = P - \sigma B + r.$$

Hence $$L = \frac{r^2C}{1 + r^2C^2\omega^2} \quad \text{and} \quad P = \sigma B - \frac{r}{1 + r^2C^2\omega^2}.$$

Thus it is seen that the balance depends on the frequency, which must therefore be accurately known in order to determine L and P. Mr. T. Smith had worked out the problem by the Author's method, and had found the source of the error. With alternating currents, the initial assumption that the resistances of the four arms are A, B, kA, kB was impossible. When this was put right, B_1 and B_2 both could be equal to zero, which conditions easily reduced to the equations given above.

Dr. Russell pointed out that M. Silva began the investigation of the accuracy of Pirani's method two years ago. He had not, however, discussed the case when two of the roots of the cubic were imaginary. He congratulated the Author on having completed Silva's investigation, and said that his solution of the cubic would be helpful in other problems.

The Author, in reply, thanked Mr. Campbell for his remarks and said he would make the necessary alterations in the part of the paper dealing with alternating currents.

XLIII. *A High-Potential Primary Battery.* *By* Wm. S. Tucker, *B.Sc., A.R.C.Sc.**

[Plate XXVIII.]

This battery may be used for maintaining at known high potentials such conductors as the needle of the quadrant electrometer, for charging condensers in capacity and insulation tests, for calibrating electrometers and electrostatic voltmeters, and for producing constant electrostatic fields.

It is composed of a large number of elements in series. These elements consist of carbon and pure zinc with a saturated or nearly saturated solution of calcium chloride

* Read March 12, 1909.

as electrolyte. The carbon is in the form of graphite used for blacklead pencils, and is about 5 cm. long. A strip of zinc foil about 5 mm. wide is bound to one extremity of the carbon by thin copper wire as in the diagram, and the strip is bent so that it nearly makes contact with the carbon of the next element. This contact, however, is just prevented by paraffin wax. The connected ends are melted into a table of paraffin wax contained in a shallow wooden tray. Elements are mounted in parallel rows, the ends of alternate rows being connected to brass terminals. In the battery illustrated (Pl. XXVIII.) 900 of these elements are arranged in series. The table of wax with the elements is then inverted, and the exposed portion of the elements dipped into a shallow tray containing the solution. On being withdrawn, each carbon-zinc pair holds solution between owing to capillary effect. With ordinary care there is no danger of fracturing the elements, but to protect them they are enclosed in a frame of perforated zinc.

The solution, which is highly hygroscopic, is of such strength that it neither increases nor diminishes in bulk on exposure to the air if the temperature and humidity of the latter do not appreciably change.

The battery is arranged so that a row or any portion of a row of elements can be employed, hence any desired voltage from that of one element to that of all the elements can be obtained.

To effect this, sets of 50 elements corresponding to nearly 50 volts can be introduced by changing the connecting-wire from one brass terminal to the next in the row of terminals fixed to the front of the battery, as seen in the figure.

By means of the plug-key shown at the top of the battery on the right-hand side, a set of 25 more elements may be introduced or cut out—the key occupying for this purpose a gap in the front or at the back of the apparatus. Still another row of 25 elements can be introduced, one element at a time, by means of a slide-contact which passes over a number of brass studs insulated by ebonite, and each connected to an element below. This is also shown in the figure.

It has been found possible to obtain 1·02 volts per element

and to keep this constant to within 0·1 per cent. for about two hours and to with 1 per cent. for half a day. With a suitable solution the battery will act for over a week without repetition of the dipping, and another dipping after that time immediately puts it in working order again.

If the battery is short-circuited for a short time no apparent damage is done, owing to the very high resistance of the element. For the same reason no shock can be felt and no appreciable spark can be obtained when the terminals are momentarily touched. If, however, condensers are placed across the terminals, they are quickly charged, and a strong spark can be obtained.

When the terminals are again insulated the elements gradually recover, owing to the apparent depolarizing action of the porous graphite. It is this effect which distinguishes the battery from others of the same type; thus, with the copper-zinc type there is always a tendency for the voltage to fall, and there is no recovery after the battery is depolarized, without drying the elements.

After use the battery may be cleansed of the exciting fluid by immersing it in the tray, through which a stream of cold water is caused to pass. The battery can then be withdrawn and allowed to dry.

The zinc shows remarkably little corrosion after three or four months of use, and separate experiments have shown that the local action with calcium chloride is very much less than with ammonium chloride. Moreover, there is no sign of creeping, and hence the insulation is maintained.

This insulation is a special feature; all terminals are mounted on ebonite, and all wire connexions inside the battery are covered with paraffin wax.

Since the battery can be dried, it can be stored for an indefinite period, but is always ready for immediate use when required. Moreover, should any element be accidentally broken, the defect is easily located, and a new one can be quickly inserted without disturbing the others.

It may be mentioned that the battery is very compact, the tray being only 18 inches long, 10 inches wide, and 2 inches deep. The materials employed cost comparatively little, and the life of the battery when carefully treated should compare

very favourably with that of others used for the same purpose.

DISCUSSION.

Mr. F. E. SMITH congratulated the Author upon his battery, which he said was small, cheap, and effective. He asked if Mr. Tucker had made experiments with other elements, and whether he had constructed a single cell of low resistance and investigated the change of potential with time.

Mr. A. CAMPBELL asked what happened if the battery was short-circuited, and what determined whether "creeping" would occur when using any particular solution.

Mr. S. G. STARLING said one of the important points about the battery was that it could be washed and put away, and was then ready for use on any future occasion.

The AUTHOR, in reply, said that the only other elements he had used were copper and zinc. He had used different liquids, but always got a lower voltage. Short-circuiting of course caused a fall of potential, but there was not much chemical action on account of the very high resistance of the battery.

XLIV. *The Effect of Radiations on the Brush-Discharge.* *By* A. E. GARRETT, *B.Sc.**

THE effect of the radiations from radium on the electric discharge produced by means of a Wimshurst machine carrying three pairs of plates each 2 feet diameter, between two brass spheres of 27 mm. and 48 mm. diameter respectively, has been studied by Willows and Peck †. They found that the sensitivity of the discharge to the action of radium radiations rapidly increased with increase of spark-length; the extinction of the visible discharge was accompanied by a decrease in the current; the radium was most effective when held opposite the positive end of the spark-gap; the effect of the radiations from radium could be stopped by the action of a magnet; Röntgen rays did not produce any apparent effect upon the discharge; and when the anode was made sufficiently small (no. 22 wire was used) the discharge passed almost entirely in the brush form, and the radium had no apparent effect.

* Read March 12, 1909.

† Phil. Mag. March 1905, pp. 378 *et seq.*

It was with the idea of ascertaining how far the above results apply when the discharge is produced by an induction-coil, that the present experiments were undertaken. The experiments are not completed, and there is no opportunity to do so at present; it was, however, thought that the results so far obtained would be of sufficient interest to warrant their publication. The induction-coil used was a 6-inch coil, and the primary current was so regulated, and the coil so arranged, that the discharge took the form of a positive brush, and a luminous glow on the cathode. The radium used in all the experiments about to be described was 5 milligrams of strong radium bromide contained in a capsule and covered with mica.

When using the coil it was always found that the radium had to be placed much nearer, in order to produce any visible effect upon the discharge, than when using the Wimshurst.

When the anode of the coil was a copper wire 3 mms. diameter and the spark-gap was 6·3 cms., the radium had no apparent effect on the discharge whether the end of the wire was flat or carefully rounded.

Some experiments were tried with a second spark-gap in parallel with the first, the knobs of the second being just so far apart that no visible discharge took place between them when a steady discharge occurred across the first gap. When the radium was so placed that the visible discharge in the first gap was extinguished, sparking at once took place across the second gap, so that in this case radium produced an effect equivalent to an increase of the resistance of the first gap. When the gap was small the radium was unable to extinguish the discharge; the smallest gap on which any effect was observed was one of 2·5 cms. Willows and Peck (*loc. cit.*) found that radium produced little effect when the knobs of the Wimshurst were less than 3–4 cms. apart.

As the sensitive nature of the brush was found to depend upon the shape and size of the anode, it was thought probable it might also depend upon the material of which the anode was made. Experiments to test this were next tried.

Effect of Material of which the Knobs are made.

Equal-sized anodes were made of copper, brass, zinc, iron, and carbon. In each case a rod 9 mms. diameter with hemispherically turned end was used.

In the case of the first three no difference in the distance at which radium was effective could be observed. When iron was used the radium had to be placed much nearer; and when carbon was used it was practically impossible to stop the brush. This was tried with spark-gaps of 5 cms., 10 cms., and 11·5 cms., respectively. The cathode was first a zinc ball 3·75 cms. diameter, and then a brass plate, and similar results to the above were obtained in all the experiments.

The brass anode was next carefully covered with soot, and it was found that this made it much more difficult to extinguish the brush. When, however, the soot was scratched off a small spot on the rounded end, the brush became just as sensitive as with clean brass. It was also found, as a general rule, that the radium was more effective if the anode was kept carefully polished; in fact with short spark-gaps it was often impossible to extinguish the brush unless this was done. Again, when a small capacity was added to the machine the brush became more sensitive. This would lead one to imagine that the sensitivity of the brush depends upon the oscillatory nature of the discharge, and that the latter depends upon the nature of the anode.

The nature of the material of which the cathode was made was not found to have any effect upon the kind of brush produced. The above experiments were repeated with the Wimshurst machine, with similar results; and since it was found that a much more sensitive brush could be obtained by this means, the machine was used in the subsequent experiments.

With reference to the peculiar action of the iron, and more particularly the carbon positive electrodes, we know that the cathode and anode falls of potential * depend, among other things, upon the metal of which the electrodes are

* Thomson, 'Conduction through Gases,' pp. 442 & 458.

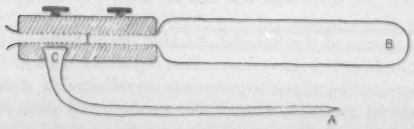
made; unfortunately, however, owing to the difficulties connected with absorbed gases, no such data have been obtained for carbon electrodes. On the other hand, the pressure in the air-gap * is known to be about twice as great for carbon as for the metals used; and this of course would make it more difficult for side influences to upset the discharge after it has once been established.

The oscillatory nature of the sensitive brush was next investigated.

Discontinuous Nature of the Discharge.

The brush was examined by means of a rotating mirror which was driven at a uniform rate by a small motor, and the relative periods of the discharges were ascertained by counting the number of brushes visible in the mirror. By this means it was at once discovered that the degree to which the brush is sensitive to the action of radium depends upon the discontinuity of the discharge. In order that the plates of the Wimshurst might be rotated at a uniform rate, they were driven by a motor; but still it was frequently found impossible to produce a sensitive brush at will. On this account the following arrangement, which was also useful for comparing a steady brush with one which was very sensitive to the action of radium, was used.

Diagram I.



B is the anode, A a piece of thick copper wire pointed at A and soldered to the connexion at C. When A was at the correct distance from end B, a sensitive brush appeared at B, and a brush on which radium had no effect at A.

In the rotating mirror the reflexion of A appeared as an almost continuous band of light, while only a few images of B (3 to 7 according to the anode) were seen; and when the

* Thomson, 'Conduction through Gases,' p. 394.

radium was held sufficiently near, the images of the brush B disappeared, while that of A continued unaltered. Positive knobs of different substances were tested, and it was found that the period of the brush-discharge with carbon as anode was about one half that when copper was used. Again, the images of the carbon brush were evenly spaced, but those formed when zinc or copper were used appeared in little groups.

If the gap was gradually increased when using carbon as anode, the brush ultimately became slightly sensitive, and at the same time the period appeared to change, the number of images seen in the mirror decreased in the proportion 7 : 5. The spacing of these images was also much less regular, and they showed a tendency to arrange themselves in groups.

Iron anodes behaved in the same way as carbon, but the brush was more sensitive than with carbon, although less so than with copper and zinc. [Ordinary iron rod was used, so the effects observed may be due to the carbon contained.]

Since the brush was more sensitive when the images appeared in groups, it was thought possible to see by means of a low-power microscope some difference in the appearances of sensitive and non-sensitive brushes.

Microscopic Appearance of the Brush.

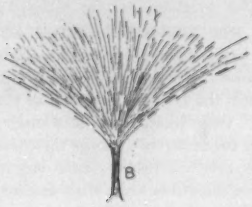
For this purpose a microscope fitted with a 3-inch objective was used. A sensitive brush when viewed with the naked eye appears as in diagram II (a); but with the microscope the stem B is found to be built up of several small stems (6–10). When radium is gradually brought near the brush, these stems are seen to become fewer in number, until only one is left; then if the radium is pushed still nearer, the brush is entirely extinguished. With the non-sensitive brush as produced at the pointed anode, the stems B as seen through the microscope appear thicker, and seem to arise from small mounds as in diagram II (b), which to the naked eye appear as a glow on the anode. This was also visible when the rounded carbon anode was used, but the mounds were then not so conspicuous. When a fairly sensitive brush was examined, and the electrodes were gradually separated so as to produce

a more sensitive brush, the stems B were found to diminish in brightness and at the same time become fewer in number.

Hence it would seem that the sensitive brush is due to the intermittent nature of the discharge, and that the period of the discharge as ascertained by the spacing of the images is

Diagram II.

II (*a*).



II (*b*).

an irregular one; while the non-sensitive brush at the carbon anode is produced by a discharge which not only has a shorter period, but also one of a more regular nature. The non-sensitive brush from a pointed anode formed a practically continuous band of light in the rotating mirror.

Before concluding this part of the experiments the effect produced by a magnetic field was tried.

Effect of Magnetic Field.

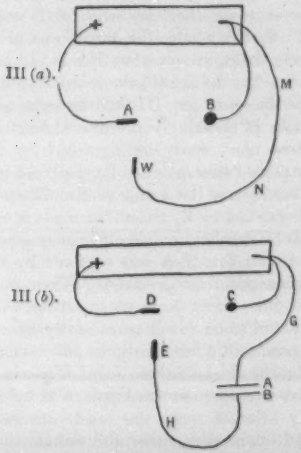
In this connexion I was able to confirm the observation of Willows and Peck (*loc. cit.*); for as soon as the field was applied, the brush, which had been previously stopped by radium, started again vigorously, thus showing that it is the β radiations of radium to which the effect is due. The magnetic field produced no visible effect on the brush, when tried under exactly similar conditions without the radium. This experiment was tried with the induction-coil.

Cook * and Bowlker † have experimented with side attachments, and have found that these have a marked effect upon the nature of the discharge; so it was thought that an investigation in that direction might throw some light upon the manner in which the β radiations effect the brush-discharge.

Experiments with Side Attachments.

A. *With Induction-Coil.*

Diagram III.



M N W and G A B H E are bare copper wire.
A and B in III (*b*) are brass plates 9 cms. square.

Using III (*a*).—When M N W was insulated and attached to the cathode the brush obtained at A was much brighter, more clearly defined, and more constant. The brush so obtained was generally much less sensitive than when W was absent. It was somewhat easier to obtain a good brush when

* Phil. Mag. 1899, vol. xlvii. pp. 40 *et seq.*
† Phil. Mag. 1904, vol. viii. pp. 897 *et seq.*

a plane surface was used for the cathode than when a ball was used. The best brush was obtained when W was held facing the anode at a distance a little too great for sparks to pass between W and the anode.

Using III (*b*).—G A B H E was insulated and attached to the cathode. When E pointed to the anode in a direction at right angles to the gap, and was gradually moved towards it, the brush at D suddenly disappeared. If the distance A to B was altered, the distance of E from D had also to be altered. When A was nearer to B, E had to be placed farther from D, and *vice versâ*, in order that the brush at D might just be extinguished. On the whole, the brush was not so steady with this arrangement, except when E was just on the point of extinguishing it. In III (*b*) the relative position of A B with respect to the spark-gap D C had no influence upon the results obtained. If brushing just occurred from A to B, the brush at D was most easily extinguished by E. When sparking took place between A and B, the brush at D was by no means so sensitive to the action of E. When the brush at D was extinguished by E, it could be made to reappear by placing a sheet of ebonite, cardboard, or even paper, between A and B or D and E. If A was replaced by a point the same phenomena could be observed. When A, B, and E were in almost the correct positions to extinguish the brush at D, it was found to be much more easily affected by the radium than usual; in other words, the action of radium and the side attachment E is additive; for if E was in almost the correct position to extinguish the brush at D and the radium was gradually brought near, the brush was extinguished, but reappeared on moving either the radium or E further from D.

If G A B was attached to the anode it was difficult to obtain a steady brush, and the brush when obtained was not so sensitive to the action of radium as when G A B was attached to the cathode.

B. *With the Wimshurst Machine* the same kind of results were obtained; the following in addition were also noted. When W (III *a*) was attached to the anode and was of the same size and material as the latter, but the wire M N was coiled, W could be placed much nearer B than A could, and

yet a brush occurred at A and no visible discharge at W. When W was pointed and attached to the cathode, if placed facing the anode the brush was extinguished, and no visible discharge occurred at W; but the brush reappeared when any solid body was held between W and A. When a brass ball was used instead of a pointed rod, the brush at A was not extinguished. If the anode was earthed through a high resistance, a steady brush was obtained, and at the same time the machine was not able to reverse. The degree to which the brush was sensitive could be altered by changing the resistance through which the anode was earthed. When the earth was too good it was not possible to work the machine. With the anode so earthed the radium had to be placed on an insulated stand, since it was found that under these conditions radium was more effective if held in the hand; and it was even found possible to stop the brush by placing the hand alone, within 10 cms. of the anode. As a rule, the brush produced when the anode was earthed was far less sensitive to insulated radium than usual. It was generally found that Röntgen rays had no effect on the brush unless the tube was placed very near the anode. Is it possible that the effect produced by the radium &c. may be due to a non-luminous side discharge taking place between the radium &c. and the anode? To test this point an attempt was made to ascertain whether a current between the radium and the anode could be detected, but the results obtained were very irregular.

Willows and Peck (*loc. cit.*) found, however, that a decrease took place in the current across the gap when the brush was extinguished.

A sensitive brush produced by the Wimshurst could not be stopped by a side discharge from an induction-coil, and, in fact, it was found that the side discharge must be produced by the same source as the brush itself for the former to be effective, unless the side arrangement is near enough for sparking to take place.

The natural conclusions from these experiment seems to be that the brush owes its sensitive nature to the oscillatory nature of the discharge; and that it is possible the action of the radium may be due to side discharges which are set up by its radiations.

In conclusion I desire to thank Dr. R. S. Willows for the kindly interest he has taken throughout the course of this work.

Cass Institute,
Jan. 1909.

Discussion.

The Secretary read a letter from Dr. R. S. Willows stating that the experiments of the Author were the only ones that connected the extinction of the spark with any other feature of the discharge. The most obvious explanation seemed to be that on account of the ionizing action of the rays the potential never rose high enough to cause luminescence. This explanation was apparently ruled out by the fact that Röntgen rays do not cause extinction, although the ionization they produce is much greater than that produced by the radium used. It was thought that the intermittence of the rays was their cause of failure, but it was found that Lenard rays, also produced intermittently, caused extinction. In spite of this difference in the action of Lenard and Röntgen rays, the Author's experiments showing that the discharge in order to be sensitive must be intermittent, render it probable that the cause of the failure of the Röntgen rays to produce extinction must be looked for in their discontinuous character. Neither the Author nor he (Dr. Willows) could obtain a sensitive discharge with a positive point: it was therefore of interest to note that Lebedinsky had recently obtained this by connecting in parallel with a Leyden jar, thus producing the Author's condition of a periodic discharge of fairly long period.

Prof. C. H. Lees expressed his interest in the experiments, and referring to the side discharge to the radium, said that it was possible that a discharge towards the radium might be dispersed by the air without giving an actual current from the radium.

Mr. A. Campbell asked what was the frequency of the oscillation in the sensitive brush.

The Author, in reply to Prof. Lees, said that on several occasions he had detected a slight current from the radium to earth, but the results were so irregular that he omitted them from the paper. With reference to the question of frequency of the discharge producing the sensitive brush, he had compared it with that of a discharge in a hydrogen tube in which the discharge was produced by an induction-coil the primary circuit of which was interrupted by a tuning-fork making 86 $\sim$ per second. The number of images visible in the rotating mirror was in the case of the sensitive brush double that given by the tube, so that the frequency was a very low one.

XLV. *On the Least Moment of Inertia of an Angle-Bar Section.* *By* H. S. Rowell *.

The determination of the least moment of inertia of an angle-bar section is important in the design of a strut by the Euler-Rankine equations.

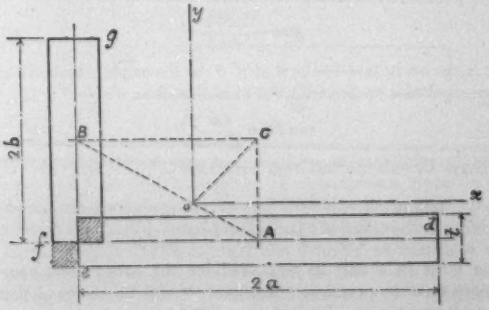
If k be the least radius of gyration of the section, $2a$ and $2b$ the lengths of the legs as shown in fig. 1. Then, very approximately,

$$\frac{k^2}{b^2} = 0{\cdot}17 + \frac{6{\cdot}4\left(\frac{a}{b}-1\right)}{100\left(\frac{a}{b}-1\right)^2+50} \quad . \quad . \quad . \quad . \quad (I)$$

This formula is a very good approximation indeed when $1<\frac{a}{b}<2$. When $4>\frac{a}{b}>2$ the following linear form is better

$$\frac{k^2}{b^2} = 0{\cdot}21 - {\cdot}022\left(\frac{a}{b}-2\right) \quad . \quad . \quad . \quad . \quad (II)$$

Fig. 1.



These formulæ were obtained in the following manner:—The principal axes of inertia through the centre of gravity were determined to a good degree of approximation by assuming that the section may be replaced by the two

* Read March 12, 1909.

rectangles *de* and *fg* (see fig. 1). The centre of gravity of these two rectangles may be readily found by the following simple construction. From A and B the centres of the rectangles draw perpendiculars AC and BC (as in fig. 1) meeting in C, from C draw Co bisecting the angle ACB (using 45° set square) and meeting AB in *o*. Then *o* is the C of G. If we take coordinate axes *ox* and *oy* through *o* parallel to the legs, we may easily show that the coordinates of A and B relatively to these axes are

$$\left(\frac{ab}{a+b}, -\frac{b^2}{a+b}\right) \text{and} \left(\frac{-a^2}{a+b}, \frac{ab}{a+b}\right)$$

respectively.

Then the moment of inertia of the system about *ox* is readily found after algebraical reduction to be

$$\alpha = \frac{2t}{3(a+b)}\left\{b^3(4a+b) + \frac{t^2}{4}(a+b)a\right\}. \quad . \quad . \quad (1)$$

By symmetry the moment of inertia about *oy* is

$$\beta = \frac{2t}{3(a+b)}\left\{a^3(a+4b) + \frac{t^2}{4}(a+b)b\right\}. \quad . \quad . \quad (2)$$

The product of inertia of the system about these axes is

$$\phi = -\frac{2a^2b^2t}{a+b}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

It is shown in text-books that if θ be the angle which either principal axis makes with the axis of x, then

$$\tan 2\theta = \frac{2\phi}{\beta - \alpha},$$

whence by substitution from equations (1), (2), and (3)

$$\tan 2\theta = \frac{6}{\frac{b}{a}\left(\frac{b}{a}+4\right) - \frac{a}{b}\left(\frac{a}{b}+4\right) + \frac{t^2}{4ab}\left(\frac{a}{b}-\frac{b}{a}\right)}. \quad . \quad (4)$$

The term in t^2 may be neglected for the proportions commonly used in practice. Moreover, it will be observed that a small error in the determination of θ is unimportant since by definition in the neighbourhood of a principal axis

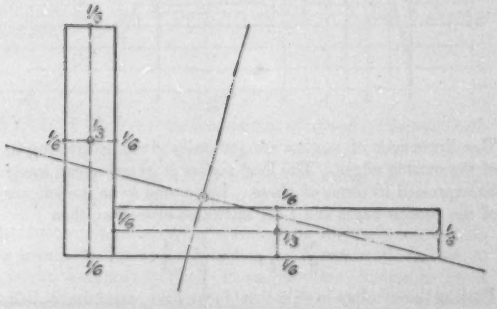
$$\frac{dI_1}{d\theta} = \frac{dI_2}{d\theta} = 0,$$

I_1 and I_2 being the principal moments of inertia.

The remainder of the work was chiefly done by graphical methods. Sections were taken from the standard forms of manufacturers, the centre of gravity was formed by the construction above given, a correction being made for the two small squares shown shaded in fig. 1. This correction amounts to $\frac{0{\cdot}088\,t^2}{a+b}$ in the direction Co remote from C and is very small. The principal axes were then drawn in according to the values found from equation (4).

The section was divided into the two rectangles shown in fig. 2, and each rectangle replaced by its equimomental

Fig. 2.



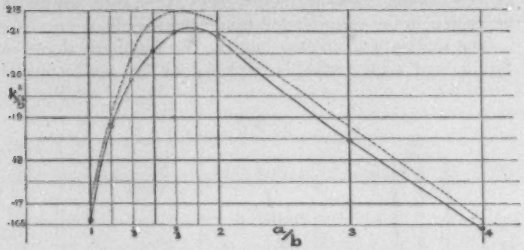
system of particles, viz., $\frac{1}{6}$ of its area at mid-point of each side and $\frac{1}{3}$ of area at centre. Perpendiculars were drawn from each particle to the axis of least inertia, and the square of each one multiplied into the mass of its respective particle. In this way it was found that the moment of inertia of the actual section was greater than that of the centre lines (both being of equal mass) by about 5 or 6 per cent. The least radius of gyration of the centre lines was then determined for various values of $\frac{a}{b}$ ranging from 1 to 4.

The values of $\frac{k^2}{b^2}$ so found were then plotted against $\frac{a}{b}$ (this is seen to be a rational proceeding from considerations

of similitude), and the full-line curve of fig. 3 was thus obtained.

The dotted-line curve shown in the same figure is that given by the empirical formulæ I. and II., and while allowing somewhat for the thickness of the section, it always errs on the side of safety from about one to two or three per cent.

Fig. 3.



The dimensions of angles are generally given as the lengths of the outside edges. The least radius of gyration can easily be expressed in terms of these. Let a_1 and b_1 be the lengths of the outside edges and t the thickness of section, then

$$a = \frac{a_1}{2} - \frac{t}{4}, \quad b = \frac{b_1}{2} - \frac{t}{4}.$$

Putting these values in equation (1) we have, assuming $b_1 = 6t$, a more handy practical form, viz.

$$\frac{k^2}{b_1^2} = \cdot 037 + \frac{1 \cdot 35\left(\frac{a_1}{b_1} - 1\right)}{100\left(\frac{a_1}{b_1} - 1\right)^2 + 50}.$$

The greatest value of $\frac{a_1}{b_1}$ in the British Standard Sections $= 2\frac{1}{2}$, and this last equation should not be used where $\frac{a_1}{b_1}$ exceeds this.

There was much tedious work involved in the above computations, which would not have been completed but for the kind encouragement of Professors Perry and Harrison.

Discussion.

Dr. Russell expressed his interest in the problem. He suggested that the phrase "second moment of an area" was preferable to "moment of inertia of an area." He asked the Author whether he had compared the numbers obtained by his formulæ with the numbers given in manufacturers' catalogues. He noticed that manufacturers introduce corrections for tapering flanges, rounded corners, etc., and that the numbers they gave for the minimum radius of gyration varied largely with the thickness of the web. Large factors of safety were used in practice, but it was highly desirable that the theoretical numbers should be as accurate as possible. Unfortunately the Author had made some approximations which were not really necessary, and these made it difficult to know what weight to attach to his results. Personally he thought that the best way of finding the second moments of angle-bar or other sections was to use an Amsler's integrator. These integrators had been extensively used for many years, more particularly by naval architects, and their accuracy was not inferior to that of graphical methods.

XLVI. *A Note on the Production of Steady Electric Oscillations in Closed Circuits and a Method of Testing Radiotelegraphic Receivers.* *By* J. A. Fleming, *M.A., D.Sc., F.R.S., and* G. B. Dyke, *B.Sc.**

In testing radiotelegraphic detectors the difficulty is generally to obtain facilities for working in actual stations and at various distances. Thus, if an inventor desires to know whether an improvement which he has made in oscillation detectors is an advance on anything yet done, he must be able to test this receiver at a station in correspondence with others at various and at considerable distances, and even then quantitative measurements are difficult, or impossible, to obtain on account of the continually varying atmospheric conditions which, as is well known, introduce an element of difficulty in connexion with long-distance radiotelegraphy, or else on account of the limited or continually changing distance between the sending and receiving station.

The first-named author of this paper has therefore been seeking for some years past for a method of testing receivers

* Read March 26, 1909.

within very moderate distances which can afford all the advantages to be obtained by working over long distances without any of the disadvantages.

This has now been achieved by the use of closed electric circuits or magnetic oscillators instead of electric oscillators. In a paper read before the Physical Society on October 25th, 1907 (see Phil. Mag. Dec. 1907 or Proc. Phys. Soc. Lond. vol. xxi. p. 47), "On Magnetic Oscillators as Radiators in Wireless Telegraphy," by J. A. Fleming, the author gave two formulæ: one for the radiation in watts from a linear oscillator of the Hertzian type of length l, and the other from a square closed circuit of area S, on the assumption that the oscillations were persistent oscillations having a root-mean square value a and a frequency N. These formulæ were as follows:—

$W = 87 \times 10^{-30} l^2 a^2 N^2$ (for the open or electric oscillator).
$W = 4 \times 10^{-35} S^2 a^2 N^4$ (for the closed or magnetic oscillator)

In the above formulæ W stands for the radiation in watts, l for the length of the linear oscillator, and S for the area of the closed or magnetic oscillator.

These formulæ show that in the case of the open or electric oscillator the power radiated varies as the square of the current-strength and as the square of the frequency, whereas in the case of the closed or magnetic oscillator it varies as the square of the current, but as the fourth power of the frequency. Hence, for any such frequencies as are used in radiotelegraphy and for such dimensions as are generally possible, an open or linear oscillator has much greater radiative power than a closed oscillator of about the same linear dimensions. Accordingly, if two closed-circuit oscillators are placed at a certain distance apart and oscillations set up in one of them, and the other one used as a receiving-circuit, the current in the receiving-circuit can be made extremely feeble when the oscillators are separated by not more than a few hundred yards, and we can avail ourselves of such a means to provide what is the equivalent to two radiotelegraphic stations with open or linear oscillators separated by many hundreds of miles. The convenience, therefore, of the closed or magnetic oscillator is very great because it is possible to set up in an ordinary building, such

as a College or Technical Institution, two square circuits, both under cover and within a reasonable distance of each other, which can be equivalent to two radiotelegraphic stations separated by several hundred miles.

In order to supply the necessary conditions for quantitative work, one of these closed circuits must be made the seat of perfectly constant oscillations, damped or undamped, by preference damped oscillations. The appliances which have now been in use in the radiotelegraphic research Laboratory at University College, London, for many years past for this purpose are as follows :—

The source of electromotive force may be an induction-coil or transformer. If a transformer, it must then be operated by a current from some supply circuit or from an alternator. If an induction-coil, it is preferably operated by large secondary cells, the primary current being interrupted by some good form of mercury coal-gas brake. We have used for this purpose with great advantage the Béclère mercury coal-gas brake, in which a jet of mercury is raised by a rotary pump and squirted against the copper plate in an atmosphere of coal-gas. The mercury brakes that are generally sold for use with paraffin oil as the insulating medium for the mercury, are very messy in use and give great trouble by necessitating constant purification of the mercury and are by no means constant in action. Any mercury turbine brake, however, can be converted into a coal-gas brake by making the vessel gas-tight and employing, instead of paraffin oil, an atmosphere of coal-gas supplied from a small rubber bag under slight pressure. We have also used with even greater advantage for some time a mercury turbine brake by Schall, thus converted into a coal-gas brake which will work for hours at a time for many months without the slightest attention. The next element in the oscillatory circuit is the spark-gap. To obtain perfectly constant results, it is necessary to cause a jet of air to impinge upon the spark-gap to destroy the arcing which otherwise would take place, and, as shown in another paper, the result of this air-blast is to remove the causes of the irregularity in the discharge current (see the following paper on "The Effect of an Air-Blast upon the Spark-Discharge of a Condenser charged

by an Induction-Coil or Transformer," by J. A. Fleming and H. W. Richardson).

It is also desirable to enclose the spark-balls in a cast-iron chamber for silencing purposes.

The form of spark-gap therefore used by the authors consists of a cast-iron chamber S (see fig. 1) closed by a lid with a glazed peephole in it, the distance of the spark-balls being adjustable by a screw, and a glass jet J being arranged so as to cause a steady jet of air from a small Lennox blower under a pressure of 16 to 20 inches of water, to impinge upon the spark-gap. An aperture is left in the cast-iron chamber, through which this air escapes. By using a small spark-gap not more than

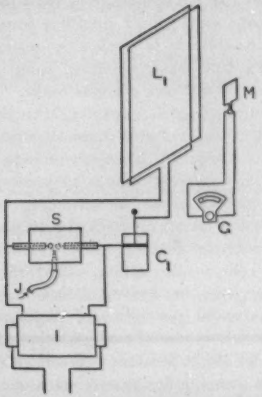


Fig. 1.—Transmitting Circuits.

3 mm. in length, and a suitable pressure of air, it is possible to maintain oscillations of great constancy in a circuit which includes the spark-gap S, a suitable condenser C_1, and the closed circuit L_1 which constitutes the radiator. The radiator is preferably made by winding 8 or 10 turns of stranded insulated wire upon a square wooden frame, which may be anything from 2 feet to 8 or 10 feet in side. The condenser may be an ordinary leyden-jar, or preferably an oil-con-

denser consisting of metal plates placed in anhydrous paraffin oil. The high-frequency capacity of this condenser can then be measured accurately, and also the inductance of the radiative circuit and the frequency of the sparks can be ascertained by a spark-counter, as described previously by one of us. The mean-square value of the current in the oscillatory circuit can be determined by a hot-wire or thermo-electric ammeter inserted in it, or in a circuit M inductively coupled to it, and it will be found that if the above arrangements are adopted, the mean square value of the discharge current in the oscillatory circuit can be kept extremely constant for hours together. Signals can also be automatically sent by interrupting the primary circuit of the transformer or induction-coil by a key operated by a punched tape. In this manner, a succession of Morse signals may be sent, or long and short signals of any kind for the purpose of testing the transmission of any particular words or letters. If the spark-chamber is made of thick cast-iron the apparatus will be nearly noiseless, and may therefore be set up in a Laboratory without disturbing other workers, which is not the case when

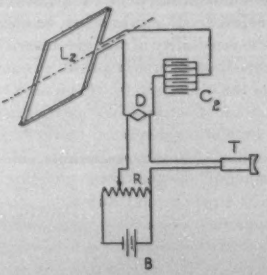


Fig. 2.—Receiving Circuits.

an open oscillatory spark is employed. At a distance, say of 50 to 150 feet or more, another square circuit may be set up consisting of a similar square coil of insulated wire L_2 (see fig. 2) and a condenser C_2, which is preferably a condenser of variable capacity for tuning purposes. A convenient form is one consisting of fixed semicircular plates and a number of

movable semicircular plates fixed on a shaft, which can be rotated, so as to bring the second set of plates more or less in between the first set, the vessel being filled with a highly insulating oil. Two of such closed oscillatory circuits can be set up at a distance, say of 50, 100, or 200 feet within a large building, and even the interposition of brick walls makes no difference, provided they do not contain metal girders.

To test, then, a radiotelegraphic detector of any kind, it is necessary to be certain that the detector *per se*, when unconnected to the oscillatory receiving-circuit, is not directly affected by the spark at the distance at which the sending and receiving circuits are set up; but this can easily be done, and then any particular type of oscillation-detector, D (see fig. 2), whether of the current actuated type or the potential actuated type, can be tested as to sensibility by inserting it, either in series with the condenser of the receiving-circuit, or in parallel with the condenser of the receiving circuit. The detector D is associated with a telephone T and a battery B shunted by resistance R as usual. The use of the closed circuits has this great advantage, that being directive radiators and absorbers, it is possible, by a displacement of the planes of the magnetic oscillators with reference to one another, to obtain a quantitative measure of the sensibility of any given oscillation-detector. Thus, for instance, it is generally possible, but not always, to find a position for the closed circuit of the receiver, such that at a certain distance no effect can be detected in that receiving circuit by any oscillation-detector, however sensitive. We may then call this the zero position. If the receiving circuit is moved out of the zero position by turning it through a certain angle round any axis, it will begin to be affected by the distant transmitting circuit, and a quantitative measure of any oscillation-detector can be obtained by noting the angle through which the receiving circuit must be turned, so that good audible signals may just be obtained. Whether the action of the sending on the receiving circuit is due to true electromagnetic radiation or to ordinary electromagnetic induction seems immaterial. The result in either case is that the receiving circuit is the seat of feeble electrical oscillations and the oscillation detector has to detect these if it can.

Another method is to maintain the receiving-circuit in its position of maximum effect, but to upset the tuning of the receiving-circuit by varying the capacity of the condenser or the inductance of the circuit. This varies the mean-square value of the received current and from the resonance-curve enables us to get the measure of the current or potential-difference which the particular oscillation-detector under test will just not detect. The authors have been employing arrangements of this kind very successfully for a long time past in investigations connected with improvements in the Fleming oscillation-valve.

It has been found convenient to denote the relative telegraphic value of detectors by stating the angle in degrees through which the receiving-coil has to be rotated from the zero position that good audible signals can be obtained on the telephone. If a note is made of the value in amperes or milliamperes of the current in the closed transmitting-circuit, this can always be recovered, and if the spark-length and spark-frequency are the same, we can always be sure that the sending circuit is in a constant and similar condition when comparative tests are made.

The closed receiving-coil is conveniently made by winding silk-covered copper wire, No. 16 S.W.G., on a square

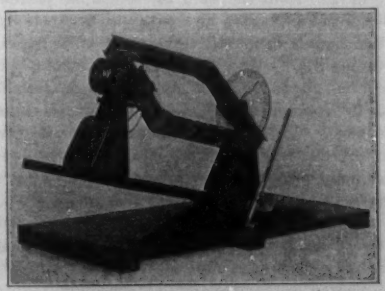


Fig. 3.—Radiotelegraphic Detector Tester.

mahogany frame, which can be revolved on pivots carried on a baseboard which can itself be set at any required angle (see fig. 3). A divided circle and pointer attached to the

frame serve to show the angle through which the frame is rotated. In general appearance it resembles an instrument used in Physical Laboratories under the name of an earth-inductor, for obtaining small induced currents by means of the rotation of a coil in the terrestrial magnetic field.

If a closed transmitting-circuit is set up at a distance, then it is generally possible to find a position for the receiving-coil, such that it will not detect any signals when coupled with a condenser and tuned and associated with a highly sensitive receiver. On turning the coil through a certain angle the signals will be heard. If a very sensitive oscillation-detector is employed, then there may be no position of absolutely null reception, but there will be a position of minimum reception. Thus, for instance, in a certain case, with a coil used at the Pender Electrical Laboratory, some Fleming oscillation-valves of a new type were found to be so sensitive to oscillations, that no position in which the receiving-circuit could be placed was so completely a position of zero mutual induction that these valves, when used with a telephone, gave no signals from a tuned transmitter. Such valves were called zero valves. Others, on the contrary, could not detect signals until the coil had been turned through 5°, 10°, or 20° from the zero or minimum position. A magnetic detector inserted in series with the coil could not detect the signals from the transmitter until the coil was turned through 15°. An electrolytic detector of a particular make required a rotation of 40°, and a carborundum detector required 45° rotation of the coil to give audible signals on the telephone. These measurements are not given as absolute and final measurements of the relative sensibility of all magnetic, electrolytic, or crystal detectors, but merely as examples of the ease with which the sensibility of these special samples of receivers could be tested for order of sensibility. The instrument has proved of great use in a research being conducted now in connexion with improvements in ionized gas radiotelegraphic detectors.

XLVII. *The Effect of an Air-Blast upon the Spark Discharge of a Condenser charged by an Induction Coil or Transformer.* By J. A. Fleming, *M.A., D.Sc., F.R.S., Professor of Electrical Engineering in University College, London, and* H. W. Richardson, *B.Sc.**

When the oscillatory discharge of a condenser is caused to take place across a spark-gap in the usual manner by charging the condenser by means of an induction-coil or transformer, the intermittent spark which takes place between the spark-balls is a complex effect. It consists partly of the true oscillatory discharge of the condenser and partly of an electric arc, unidirectional or alternating, which is superimposed on the true condenser oscillatory spark. If a hot-wire ammeter or other means of measuring the effective or mean-square value of the discharge current is inserted in the condenser circuit, this current will generally be found to be irregular, and if a radiative circuit is coupled to the condenser circuit as in radiotelegraphy, the radiation from it will be found to consist of trains of waves whose initial amplitude is also variable. This irregularity is a source of difficulty in making radiotelegraphic or laboratory measurements of current, decrement, wave-length, &c., when originated by condenser discharges. The reason is that the moment the condenser begins to discharge, and the first so-called pilot spark takes place between the balls, the resistance of the spark-gap falls, and an arc discharge from the induction-coil or transformer commences across the gap. Until this arc is extinguished the condenser cannot again become charged to any high voltage, and the voltage to which it is charged will depend upon the state in which the ball surfaces are left as regards temperature and smoothness, since these are factors in determining the spark potential, and also on the condition of the air-space as regards conductivity. Accordingly, to produce a uniform oscillatory discharge this true arc-discharge must be either prevented or

* Read March 26, 1909.

arrested at once, and the spark between the balls should arise wholly from energy which comes out of the condenser, and not from energy coming directly from the transformer or coil. When moderate power is being employed this arc-discharge can be best annulled by a blast of air thrown on the spark-gap. This has the effect of blowing away the arc, but does not stop the condenser oscillatory discharge. For a long time past the utility of this air-blast in connexion with practical radiotelegraphy has been noted by one of us (J. A. Fleming), but its use in purely scientific measurements is an advantage, as shown in the following paper.

If a large induction-coil has its secondary terminals connected to a pair of spark-balls, or to the outside pair of a series of balls, so arranged that each gap is not more than a millimetre in width, and if a condenser having a capacity say of 0·005 mfd. is connected across the outer balls, then when the coil is in action intermittent sparks pass at the gap. If these sparks are examined in a revolving mirror or photographed on a moving plate, they present themselves as bright images set at fairly equal distances. If a jet of air under a pressure of 16 or 18 inches of water is thrown on the gap by a glass nozzle, the images are then seen to have a ragged tail or aureole which is blown away from the spark, and this tail is generally reddish in colour and easily distinguished from the bright condenser spark. The tail is the image of the arc-discharge superimposed on the oscillatory spark. To determine the effect of this air-blast the following experiment was tried. A 10-inch induction-coil had its secondary circuit connected to brass spark-balls 3 cms. in diameter set with a gap of 1 mm., and the balls were also connected to a rectangular circuit of round copper wire, the diameter of the wire being 0·162 cm. and the sides of the rectangle respectively 142·1 cms. and 34·17 cms. The ordinary or steady resistance of this rectangle is 0·046 ohm and its high frequency resistance to currents of a frequency of the order of $1{\cdot}25 \times 10^6$ is 0·31 ohm. The inductance of this circuit (calculated) is 5012 cms. In series with this circuit was placed a condenser consisting of metal plates immersed

in paraffin oil, the capacity of which was 0·002645 of a microfarad.

In contiguity to the long side of the above rectangle was placed the bar of a Fleming Cymometer, two such instruments being used in the experiments, called respectively No. 2 and No. 3. The cymometer circuit can have two short fine wires of constantan each about 5 cms. long, inserted in it at pleasure, against one of which a bismuth-iron thermo-junction is attached. By passing measured small continuous currents through this fine wire and connecting the ends of the thermo-junction to a low resistance single-pivot Paul galvanometer, the arrangement can be calibrated as a hot-wire ammeter to indicate directly the mean-square value of the oscillations, which when passed through the fine wire cause a certain deflexion of the galvanometer attached to the ends of the thermo-junction. The other fine wire can be inserted as an added resistance in the circuit of the cymometer. The cymometer consists of a circuit including a spiral wire having inductance L and a condenser of capacity C, which can be continuously varied in the same proportion, so that the oscillation constant ($\sqrt{CL}$) of the circuit can be given any value between certain limits. The cymometer was employed to take a resonance curve of the spark circuit by the usual Bjerknes-Drude method, (i.) when the spark-balls were not subjected to the air-blast, and (ii.) when the jet of air was thrown between them.

An extremely steady jet of air for this purpose can be obtained by the use of a small Lennox blower, which is a fan driven by an electric motor taking a current of about 1 ampere from any electric-lamp supply circuit. The resonance curve is obtained by plotting the values of the root mean-square current (a) expressed as a fraction of the maximum current (A) induced in the cymometer circuit by the damped oscillations in the spark-ball circuit corresponding to various values of the natural frequency n or oscillation constant for any setting of the variable inductance and capacity of the cymometer circuit. Then, if A is the maximum value of this secondary current in the cymometer corresponding to a certain natural frequency N or oscillation constant when

resonance between the circuits exists, and if δ_1 and δ_2 are the logarithmic decrements per semiperiod of the oscillations in the spark and cymometer circuits respectively, we have by Bjerknes' formula

$$\delta_1 + \delta_2 = D = \pi \left(1 - \frac{n}{N}\right) \sqrt{\frac{a^2}{A^2 - a^2}}$$

provided that such values of n are selected that n does not differ from N by more than say 5 per cent. If then a known resistance R is added to the cymometer circuit, thus increasing its decrement by a known amount δ_2', and a fresh set of observations taken, another resonance curve can be plotted which should lie wholly outside of the first when their maximum ordinates of both curves are taken as unity. From the formula given by Bjerknes we know then that

$$A^2 (\delta_1 + \delta_2) \delta_2 = A_1^2 (\delta_1 + \delta_2 + \delta_2') (\delta_2 + \delta_2'),$$

where A and A_1 are the mean-square values of the maximum or resonance currents in the two cases.

The actual observed quantity which gives us a is the deflexion of the needle of the galvanometer in connexion with the thermocouple pressed against the resistance wire inserted in the cymometer circuit. If there is no air-blast on the balls this deflexion hardly ever remains steady, the needle wanders to and fro over the scale, and the observer can at best but take a mean reading. The result is that the points plotted for the resonance curve do not lie well on a smooth curve, and it is particularly difficult to plot the important part of the curve near the maximum value. This difficulty has been experienced by other observers. Thus, Mr. R. A. Houstoun has recently made a series of measurements of spark resistance and decrement with spark-balls of various metals (see Proc. Roy. Soc. Edin. 1908, vol. xxviii. p. 369, or 'The Electrician,' vol. 62, p. 636); and he remarks that even with great care taken, measurements of logarithmic decrement of oscillation made with the same spark-gap and circuit do not agree well together, the resistance of the spark-gap appears to vary irregularly. Again, Messrs. J. E. Taylor and W. Duddell in some radiotelegraphic measurements made in 1905 (see Journ. Inst. Elect. Eng. vol. 35, p. 321, 1905)

also complained of the difficulty of obtaining good measurements with a spark-transmitter. These difficulties have their origin in the actions taking place in the spark-gap. If, however, a steady blast of air at a suitable pressure is thrown between the spark-balls, this irregularity is greatly reduced provided the spark-gap is not long. The deflexion of the galvanometer becomes greater and much more steady, and accurate observation of the values of the current a corresponding to known values of n becomes much facilitated.

With the above described arrangements a resonance curve can be easily taken as follows:—The jet of air from the Lennox blower conveyed by a rubber pipe ending on a glass nozzle is allowed to play between the spark-balls, and these are connected together by a condenser of known capacity in series with an inductance which has either been measured or predetermined. This inductance is preferably formed of round copper wire and may be rectangular in form, and its high frequency resistance and inductance can then be calculated by known formulæ.

The cymometer is then placed alongside this rectangle so as to have induced oscillations created in it, and by means of the hot-wire ammeter inserted in its circuit the mean-square value a of the cymometer current is taken for various settings of the cymometer circuit, which give it various assigned natural frequencies n. If A and N are the maximum values obtained we then calculate the value of the sum of the decrements of the spark and cymometer circuits from the Bjerknes-Drude formula given above *. Since the resonance curve is not symmetrical with respect to its maximum ordinate, the best mode of procedure is as follows:—Having taken a series of values of the cymometer current a without the extra damping resistance, and those of the current a_1 when the resistance is inserted in the cymometer circuit, and observed the maximum values A and A_1 of the quantities, for various values of n we plot two curves, each to abscissæ n/N, but one having ordinates a/A and the other ordinates a_1/A_1. The latter curve should lie outside the former, but

* See 'The Principles of Electric Wave Telegraphy,' by J. A. Fleming. Longmans & Co., pp. 221–223.

should have the same value for its maximum ordinate, viz., unity. We then draw a number of horizontal lines across the hump of the resonance curves, as shown in fig. 1, with such ordinates that the greatest length intercepted by the

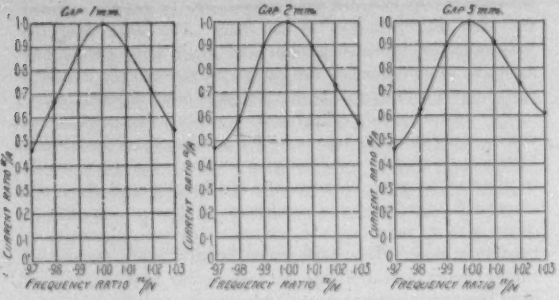


Fig. 1.—Resonance Curves taken with air-blast on spark-gap.

curves is not more than 0·06 on the same scale that N is taken as unity. Let the half length of any such horizontal intercept be called x and the corresponding ordinate a/A be denoted by y, we then calculate the value of

$$D = \pi x \sqrt{\frac{y^2}{1-y^2}},$$

and this gives us the value of the sum of the decrements of the spark and cymometer circuits. It is well to calulate D from several, say four or five, measurements of x and y for intercepts of different lengths, and then take the mean value for D. In the same way from the outer resonance curve we can calculate the value of D_1. The difference between D and D_1 is the increment in the decrement due to the added resistance wire, and if R is its high frequency resistance and if the frequency of resonance is N and the corresponding inductance of the cymometer is L, we should have

$$D_1 - D = \frac{R}{4NL} = \delta_2',$$

as a check on the observations.

The following details of one set of experiments will show the advantage gained by the use of the blower. The experiments were made with cymometer No. 2. The primary circuit had an inductance of 5012 cms. and a capacity of 0·002645 mfd., and was charged by a 10 in. induction-coil worked with a coal-gas mercury turbine break. The spark-balls were brass balls 3 cms. in diameter set with spark-gaps of length of 1, 2, and 3 mm. respectively in various experiments. The resistance added in the cymometer circuit was a very fine constantan wire of which the resistance was 7·1 ohms, and a similar wire against which a thermojunction pressed has a resistance of 5 0 ohms.

When the cymometer circuit was adjusted to be in resonance with the primary circuit the resonance frequency N was found to be $1{\cdot}25 \times 10^6$ and the corresponding inductance of the cymometer was 5500 cms. Hence for the added resistance, we have $R = 7{\cdot}1 \times 10^9$ c.g.s., $L = 5500$ cms, and $n = 1{\cdot}25 \times 10^6$. Therefore

$$\frac{R}{4nL} = \frac{7{\cdot}1 \times 10^9}{4 \times 1{\cdot}25 \times 10^6 \times 5500} = 0{\cdot}0258,$$

and for the thermojunction

$$\frac{R}{4nL} = \frac{5 \times 10^9}{4 \times 1{\cdot}25 \times 10^6 \times 5500} = 0{\cdot}0181.$$

The thermojunction ammeter having been calibrated with direct currents so that from the deflexions of the Paul single pivot galvanometer the R.M.S. value of the oscillations passing through the wire which gave any observed deflexion could be obtained, a series of observations was taken by varying the setting of the cymometer slowly and continuously when it was placed in loose inductive coupling with the primary circuit so as to alter its oscillation constant $O = \sqrt{CL}$, or the product of its inductance L and capacity C, from which the natural frequency n of the circuit is at once obtained by the formula $n = 1/2\pi\sqrt{CL}$. At the same time the R.M.S. value of the current a in the cymometer was read off by the thermo-ammeter and the maximum value A also taken and the resonance frequency N. These observations then give the means of calculating D or the sum of the decrements of the primary circuit δ_1 and that of the cymometer δ_2, this last including

that due to the resistance of the hot-wire ammeter. A resonance curve was then drawn and a series of horizontal intercepts measured off near the peak of the curve, and the ordinates of these intercepts also read off on the curve (see fig. 1). The half of the length of the intercept is taken as the mean-value of $1-\frac{n}{N}$, and the corresponding ordinate expressed as a fraction of the maximum ordinate of the curve is taken as a/A, and from this last the function $\pi\sqrt{\frac{a^2}{A^2-a^2}}$ is calculated and the product of these two quantities, viz.,

$$\left(1-\frac{n}{N}\right)\pi\sqrt{\frac{a^2}{A^2-a^2}}$$

gives us $D=\delta_1+\delta_2$, or the sum of the decrements of the primary and secondary circuits.

The resonance curves are plotted to such a scale that the maximum ordinate representing A is unity, and the abscissa of that ordinate representing N is also taken as unity.

The complete set of observations for the 1 mm. spark-gap when subjected to the air-blast was as follows:—

TABLE I.

1 mm. spark-gap with air-blast.

Resonance frequency = $N = 1{\cdot}25\times10^6$.

Oscillation constant of Cymometer $O=\sqrt{CL}$.	Corresponding Frequency of Cymometer circuit $=n$.	Cymometer current in amps. $=a$.	Ratio n/N.	Corresponding actual frequency n.	Corresponding current from Resonance Curve a.	Ratio a/A.
4·15	$1{\cdot}205\times10^6$	0·046	0·96	$1{\cdot}2\times10^6$	0·041 amps.	·343
4·1	1·22 ,,	0·074	0·97	1·211 ,,	0·056 ,,	·469
4·05	1·235 ,,	0·10	0·98	1·224 ,,	0·0803 ,,	·672
4·025	1·242 ,,	0·113	0·99	1·238 ,,	0·106 ,,	·886
4·000	1·250 ,,	0·1195	1·00	1·250 ,,	0·1195 ,,	1·000
3·975	1·257 ,,	0·1146	1·01	1·262 ,,	0·107 ,,	·895
3·95	1·264 ,,	0·1025	1·02	1·275 ,,	0·0865 ,,	·724
3·925	1·272 ,,	0·086	1·03	1·281 ,,	0·0656 ,,	·549
3·9	1·282 ,,	0·077				
3·85	1·295 ,,	0·0558				

The maximum cymometer current = A = 0·1195 ampere without the added resistance in circuit, and the maximum current with the added resistance of 7·1 ohms = A_1 = 0·0635.

From the above observations and curve-measurements we calculate the following Table II. :—

TABLE II.—1 mm. spark-gap with air-blast on gap.

Cymometer Current as per cent. of max. current = 100 a/A.	Value of $\pi\sqrt{\frac{a^2}{A^2-a^2}}$.	Mean Value of $1-\frac{n}{N}$.	Value of $D=\delta_1+\delta_s$.
95	9·58	·0067	·0643
90	6·47	·0098	·0635
85	5·09	·0126	·0642
80	4·18	·0152	·0636
75	3·58	·0177	·0635
70	3·08	·0205	·0632
		Mean Value of D = ·0637	

The values of D obtained from the various measurements of the resonance curves are seen to be in very fair agreement. We then calculate the value of δ_2 the cymometer decrement from the formula

$$A^2 D\delta_2 = A_1^2(D+0{\cdot}0258)(\delta_2+0{\cdot}0258),$$

where A = ·1195, A_1 = ·0635, D = ·0637, and hence δ_2 = ·017 = decrement of cymometer including that due to the resistance of thermojunction wire.

Hence $\delta_1 = D-\delta_2 = 0637-{\cdot}017 = {\cdot}0467$.

Accordingly the total high frequency resistance of the primary circuit = $R=\frac{4N^2L_1\delta_1}{10^9}$, where L_1 is the inductance = 5012 cms. and N is the natural frequency = $1{\cdot}25\times10^6$. Hence we have R = 1·17 ohms, and since the high frequency copper resistance of the rectangular circuit = 0·31 ohm, we find that the spark-gap resistance for this 1 mm. spark is 0·86 ohm.

Observations made with 2 mm. and 3 mm. spark-gaps with air-blast in each case gave results of which the final calculations and measurements are embodied in Tables III. and IV.

TABLE III.—2 mm. spark-gap with air-blast.

Cymometer Current as per cent. of max. current 100 a/A.	Value of $\pi\sqrt{\frac{a^2}{A^2-a^2}}$.	Mean Value of $1-\frac{n}{N}$ from Curve.	Calculated Value of $D=\delta_1+\delta_2$.
95	9·58	·0069	·0662
90	6·47	·0010	·0647
85	5·09	·0127	·0647
80	4·18	·0144	·0609
75	3·58	·01695	·0605
70	3·08	·0192	·0595
		Mean Value =	·0627

Maximum Cymometer Current = A = ·101 ampere.
Maximum Cymometer Current
when resistance is added = A_1 = 0·046 ampere.
Calculated Value of δ_1 = ·0443.
" " δ_2 = ·0184.
" " of spark-gap resisistance = 0·80 ohm.

TABLE IV.—3 mm. spark-gap with air-blast.

Cymometer Current as per cent. of max. current 100 a/A.	Value of $\pi\sqrt{\frac{a^2}{A^2-a^2}}$.	Mean Value of $1-n/N$ from Curve.	Calculated Value of $D=\delta_1+\delta_2$.
95	9·58	·0063	·0604
90	6·47	·0095	·0615
85	5·09	·0117	·0595
80	4·18	·0138	·0579
75	3·58	0161	·0577
70	3·08	·0187	·0577
		Mean Value =	·0591

Maximum Cymometer Current = A = 0·1195 ampere.
Maximum Cymometer Current
with added resistance = A_1 = 0·0652 ampere.
Calculated Value of δ_1 = ·0397.
" " δ_2 = ·0194.
" " of spark-gap resistance = 0·68 ohm.

The same experiments were then repeated with the same spark-gap lengths but without air-blast. It was found to be more difficult to obtain good points for resonance curves, but the final results are as shown in the figures in Table V.

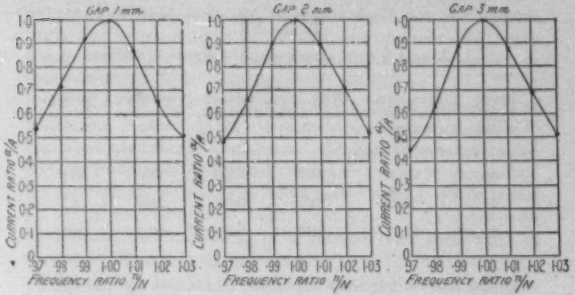


Fig. 2.—Resonance Curves taken without air-blast on spark-gap.

To economise printing we shall write X for the quantity $100\,a/A$, and Y for $\pi\sqrt{\frac{a^2}{A^2-a^2}}$, and Z for the mean value of $1-n/N$.

Then since the values of X and Y are the same in all cases they apply to each spark-length, and need not be repeated if it be understood that the six rows of figures correspond respectively to X = 95, 90, 85, 80, 75, and 70 per cent. We accordingly compress the final results for the 1, 2, and 3 mm. spark-gap unblown on in one Table V.

TABLE V.—1, 2, and 3 mm. spark-gaps without air-blast.

$1-n/N=Z.$			$D=\delta_1+\delta_2.$		
1 mm.	2 mm.	3 mm.	1 mm.	2 mm.	3 mm.
·0072	·008	·0067	·069	·0766	·0642
·00975	·01	·0092	·0631	·0647	·0595
·0017	·0124	·0116	·0595	·0631	·0591
·0141	·0146	·0139	·0592	·0612	·0582
·0164	·0172	·0162	·0587	·0617	·0581
·0187	·0193	·0185	·0576	·0595	·0570
Mean Values.....			·0613	·0645	·0593

The values of the maximum cymometer current in amperes without the resistance in its circuit (=A) and with added resistance (=A_1), and the calculated values of the decrements and spark-gap resistances are given in Table VI.

TABLE VI.

1, 2, and 3 mm. spark-gaps without air-blast.

Spark-length in mm.	Max. Cymometer Current in amperes =A.	With resistance added =A_1.	Primary decrement δ_1.	Secondary decrement δ_2.	Spark resistance in ohms.
1.....	·0952	·052	·0423	·019	.75
2.....	·1202	·067	·0447	·0198	·81
3.....	·119	·066	·0383	·0216	·65

If we collect together the same quantities for the spark-gaps when blown on we have results as in Table VII.

TABLE VII.

1, 2, and 3 mm. spark-gaps with air-blast.

Spark-length in mm.	Max. Cymometer Current =A.	With resistance added =A_1.	Primary decrement δ_1.	Secondary decrement δ_2.	Spark-resistance in ohms.
1.....	·1195	·0635	·0467	·017	·86
2.....	·101	·046	·0443	·018	·80
3.....	·1195	·0652	·0397	·0194	·68

It will be seen, therefore, that when the air-blast is not applied the various values of $D=\delta_1+\delta_2$ calculated from the resonance curves are more irregular and deviate more from the mean, this being the result of the difficulty of obtaining correct galvanometer readings to delineate a good resonance curve.

The observations also appear to show that the air-blast has not much influence upon the primary decrement or upon the spark-resistance, or at most tends to slightly increase the

spark-resistance for very short sparks. The good effect of the air-blast is seen best when applied to short sparks, which tends to show that it is of assistance in destroying the arcing then occurring. This arc, however, is unable to persist with longer gaps; that is to say, it blows itself out, and accordingly for spark-lengths of 3 mm. or upwards and when using an ordinary 10 in. induction-coil with mercury break as interruptor, the blast is of no special advantage.

Another set of experiments was conducted, the object of which was to ascertain the effect of the air-blast upon multiple and upon very short spark-gaps. At one time some radio-telegraphists were of opinion that an advantage was gained by dividing up a spark-gap into smaller spark-gaps in series. To test this opinion more carefully, a series of six spark-balls were made, each consisting of a pair of brass balls 2 cm. in diameter adjustable as to distance by a screw of 0·5 mm. pitch, having a divided head. These spark-balls were arranged so that they could be put in series with one another, the series forming the spark-gap in an oscillatory circuit having an inductance of 125,000 cms. and a capacity of 0·0045 mfd. A number of glass jets were arranged so that an air-blast could be directed against each pair of spark-balls across the gap. In the oscillatory circuit a hot wire ammeter was placed, so arranged that a reading could be taken of the ammeter first with a single spark-gap of a definite length, and then with a series of 2, 3, 4, 5, and 6 gaps, each adjusted to be equal in length, the sum of them all being equal to that of the single gap employed. Thus, for instance, the effect of a single gap 0·5 mm. could be compared with that of five gaps in series each 0·1 mm., both with the air-blast and without the air-blast against the gaps. As in the case of very short gaps, the phenomenon of multiple sparks exists; that is to say, each interruption of the induction-coil in the primary circuit gives rise not merely to one discharge in the oscillatory circuit, but to a series of discharges, because a discharge takes place between the balls corresponding to the length of the spark-gap employed, and whilst the electromotive force in the secondary circuit of the coil increases and endures, during this time many sparks may take place.

Whether this occurs or not can be determined by means

of a revolving mirror. If the image of an oscillatory discharge produced by an induction-coil is examined in a revolving mirror, then if there is only one spark corresponding to each interruption of the primary coil, a series of widely separated sharp images of the spark will be seen, but if the phenomenon of multiple discharge is taking place, then the images are each seen to consist of 4, 5, 6, or more small sparks rapidly succeeding each other. Also, if there is any sensible arcing at the spark-gap it is at once seen in the image in the revolving mirror in the form of a trail of light of a different colour to that of the true oscillatory discharge-spark. On examining the spark-discharge with and without the air-blast for certain short spark distances, the effect of the air-blast is easily detected, because it is seen to blow away this trail which accompanies the image of the spark-discharge.

In order to avoid misinterpreting the results, it was necessary to be sure that when using say one single spark of 0·5 mm. in length and comparing it with the effect of 5 spark-gaps of 0·1 mm. in series, any increase in the current at the discharge circuit was not due to an alteration in the number of discharges.

As far as could be observed this was not the case when sufficiently short spark-gaps were employed.

The following Table VIII. (p. 679) embodies the results of the measurements. Employing single spark-gaps ranging in length from 0·1 mm. to 3 mm., observations were taken of the mean-square value of the current in the discharge circuit, both with and without the air-blast, and these values are recorded in the first two columns. It will be seen that up to a certain length of gap (about 2 mm. in the case of these experiments) the air-blast had a very decided effect in increasing the mean-square value of the discharge current, but beyond that point it seems to have the effect of diminishing it. On the other hand, if a single gap, say of 0·2 mm., is broken up into two gaps in series each of 0·1 mm. and so on for the other spark-lengths, it is found that dividing the spark-gap into two parts also increases the discharge current up to about 0·8 mm., and after that the current diminishes. If the double air-gap is blown upon, the current is increased

Table VIII.

Comparisons of the Effect of breaking up a single Spark-gap into several Spark-gaps in series.

Total Gap.	In 1 Gap.		Separated into 2 Gaps.		Separated into 3 Gaps.		Separated into 4 Gaps.		Separated into 5 Gaps.		Separated into 6 Gaps.	
	No Air Blast.	With Air Blast.	No Air Blast.	With Air Blast.	No Air Blast.	With Air Blast.	No Air Blast.	With Air Blast.	No Air Blast.	With Air Blast.	No Air Blast.	With Air Blast.
mm.	amp.	amp.										
·1	·54	·66										
·2	·684	·816	·728	·816								
·3	·628	·912	...	...	·71	·896	...	...	·88	·88		
·4	·628	1·04	·684	1·029	...	...	·88	1·042				
·5	·71	1·095	...	...	...	...	...	...	1·34	1·34		
·6	·755	1·253	·728	1·258	·88	1·142	·937	1·042	1·029	1·029		
·8	·84	1·284	·84	1·228	...	...	·914	1·042	...	...	1·017	1·017
1·0	·896	1·35	·914	1·35	...	...	...	...	·896	·896		
1·2	·975	1·287	·975	1·35	1·042	1·142	·896	·896	...	...	·755	·755
1·5	1·142	1·142	...	...	1·029	1·029						
2·0	1·2	1·2	1·042	1·042	...	...	...	...	·755	·755		
3·0	1·258	1·042	1·017	1·017	·937	·937	...	...	...	...	9·5	9·5

as compared with the same two gaps not blown upon, but is not increased as compared with a single gap blown upon. The same is true when the spark-gap is broken up respectively into 3, 4, 5, and 6 gaps. The effect of dividing the gap up to a certain point is to increase the discharge current, but the effect of the air-blast on the multiple gaps becomes less and less marked in proportion as the number of gaps increases, so that the effect of dividing up a gap, say 0·6 mm., into five gaps of 0·12 mm. each, is to increase the discharge current almost as much as by subjecting the single gap of 0·6 mm. to the air-blast. This increase appears to be due to the suppression of the arc-discharge, as shown by the appearance of the images of the spark in the revolving mirror, and hence the arcing is suppressed by the separation of the gap into multiple gaps, almost as effectively as by blowing upon the single gap. On the other hand, separating the spark-gap into more than five separate gaps seems to result in diminishing the total discharge current beyond a certain very short length of spark. The figures in the table, however, do not show what is observed in practice, namely, the greater steadiness of the discharge current under the operation of the air-blast; and the conclusion, therefore, is that in any experiments in which great constancy is required in the discharge current in the condenser circuit, a great advantage is obtained by using a short spark-gap, and by subjecting the discharge spark to an air-blast, as this both increases the charging voltage of the condenser and steadies the discharge current by abolishing the arc, which would otherwise take place even with an ordinary induction-coil.

A similar set of experiments was tried with a larger alternating current plant and a high tension transformer, constituting one of the wireless telegraph sets of the Radiotelegraphic Laboratory, University College. The alternating current was supplied from a 5-k.w. alternator and raised in pressure by a transformer discharging across a spark-gap between two iron balls, the spark-gap being shunted by a condenser in series with the primary circuit of an oscillation transformer, the secondary circuit of which was inserted between an antenna and an earth connexion, the antenna circuit being

tuned to the condenser circuit. As no suitable hot-wire ammeter was available for measuring the large current in the condenser circuit, readings were taken by connecting the terminals of a hot-wire voltmeter to points on the earth-wire connexion of the antenna, consisting of a copper strip. These points being selected a few inches apart so as to give a convenient reading on the low reading hot-wire voltmeter. These readings, however, are approximately proportional to the currents flowing in the condenser circuit. Spark-gaps were then adjusted from 0·5 mm. to 3 mm. in length, and the reading of the voltmeter taken, both when the spark-gap was blown upon and when it was not blown upon a by a jet of air.

The following table shows the readings:—

TABLE IX.

Spark Lengths.	Voltmeter Reading.	
	No Air Blast.	With Air Blast.
3·0 mm.	2·4	2·7
2·5 „	2·4	2·7
2 „	1·85	2·2
1·5 „	1·7	1·7
1·0 „	1·65	2·0
·5 „	·75	1·0
1·0 „	2·0	2·25
1·5 „	2·1	2·4
2·0 „	2·05	2·35
3·0 „	2·3	2·45

The Table shows in every case an increase in the current circulating in the condenser circuit when the spark-gap is subjected to the air-blast, but does not indicate the much greater steadiness of the voltmeter-needle which is seen with the air-blast. Without the air-blast the voltage was by no means constant, and the readings given in the first column are therefore a mean reading from which the extreme readings may differ by 10 per cent.

The final result of the experiments is to show that when using spark-gaps, 1, 2, or 3 mm. in width, in a condenser circuit, for the purpose of exciting oscillations, much greater uniformity in the discharge current can be obtained if the spark-gap is subjected to an air-blast as described.

Discussion.

Mr. W. Duddell congratulated the Authors and remarked that the papers contained a great deal of useful material. The use of the earth inductor for testing receivers was a feature of the first paper. He asked Dr. Fleming to what extent the effects obtained in the receiving circuit were due to true radiation, and why they had used coal-gas instead of alcohol in their interrupter. Referring to the Authors' method of determining the current in the transmitter, he asked if there was any objection to putting an ammeter in the circuit and reading the current directly. He should also like some more information about the cases in which it was found impossible to obtain a position of the earth inductor giving silence in the telephone. Was it because the distance apart was not sufficiently great or was some physical impossibility involved. With regard to the second paper he had always found it possible to get a uniform discharge by using an alternating current of suitable frequency in the primary. He asked the Authors if it was the arc or the spark which was blown out by the air and whether the part blown out had a spectrum different from the rest.

Dr. W. H. Eccles asked how much of the energy absorbed by the receiver was due to radiation and how much to electromagnetic induction. He had obtained results, depending on electromagnetic induction, similar to those described by the Authors by using very much smaller apparatus, but he had discontinued his experiments because in practice the whole of the energy received was due to true radiation. He pointed out that a receiver adjusted and tested in a laboratory was never in proper adjustment for actual work. He suggested that the reason it was sometimes impossible to get a position of silence arose from stray radiation falling upon the receiver.

Dr. Erskine-Murray pointed out that detectors varied greatly in resistance and that therefore a telephone of suitable resistance should be selected in each test.

Dr. A. Russell thought that Prof. Fleming and Mr. Dyke's method of testing radio-telegraphic receivers would be a great help in judging their relative values. He much appreciated the clear distinction drawn between the function of the spark and the arc as this cleared up some of his difficulties. Although the air-blast of the Lennox blower was doubtless beneficial by preventing arcing, he thought that the fact that the dielectric coefficient of the glass nozzle used was greater than unity might have accelerated the sparking. He mentioned some of the difficulties encountered in computing sparking voltages when there were

two gaps in the circuit. In this case it had to be remembered that the sparking voltages at the moment of the discharge are not equal and opposite. When the potentials of the electrodes are known, however, and they are spherical in shape, the sparking voltages with two gaps in series can be calculated with fair accuracy.

Dr. R. S. Willows pointed out that by blowing out the spark the resistance of the path was increased and the rate of charge of energy thereby altered. The fact that greater regularity and greater energy could be obtained could be easily demonstrated by using an electrodeless discharge-bulb. Referring to the fact that it was necessary to know the self-induction of one of the circuits, he asked the Authors what form of circuit had been chosen and how its self-induction had been calculated. In one of the experiments a resistance r had been added, and he remarked that attention should be directed to its effective resistance under rapidly alternating currents, as this might depend upon whether the added resistance was a pure metal or an alloy.

Mr. L. H. Walter agreed with Dr. Erskine Murray that it was necessary to choose a suitable telephone when making a test. Although the electrolytic detector was supposed to be more sensitive than the magnetic form, it was possible to choose a telephone of such a resistance as to make the magnetic detector appear the more sensitive.

Dr. Fleming in reply said that it was impossible to state precisely what proportion of the current produced in the receiving circuit was due to true radiation from the closed transmitting circuit, and how much was due to electromagnetic radiation, but from his point of view it did not matter. All that was necessary was that a feeble oscillatory current should be produced in the receiving circuit which should be capable of being varied by turning the receiving circuit through a certain angle, and whether this was due to actual detachment of energy from the transmitting circuit, or to the mere movement of lines of magnetic force backwards and forwards through space, seemed immaterial. The oscillation detector in any case is a mere detector of oscillations.

In reply to Mr. Duddell, he said that there was no objection to putting an ammeter in the transmitting circuit provided it was a low resistance instrument and did not produce any sensible damping of the oscillations.

As regards the existence of an exact zero point, this seemed to be a question of distance from the transmitter. It had been usual at University College to work with two coils about 60 feet apart, and at that distance some very sensitive oscillation valves detected sounds which might be due to the action directly upon the valve or upon the connecting wires, but by going to larger distance, it was possible to get complete silence at the telephone.

With respect to the use of coal-gas or alcohol, in the interrupter, coal-gas had proved itself to be incontestably superior to alcohol.

As regards the action of the air-blast, it appears tolerably certain that the part of the discharge which is blown away is that due to energy coming directly out of the induction-coil or transformer.

In reply to the remarks of Dr. Eccles, he said that they had found it necessary to work at a certain distance from the transmitter, but that when this was done the order of sensitiveness in which oscillation-detectors were arranged by the apparatus shown was also the order in which they were found to be sensitive when employed in actual radio-telegraphic work. It is necessary not to work the transmitter and receiver too near to one another, otherwise there are direct effects on wire connections, rheostats, etc., which obscure the real effects.

In reply to Dr. Willows, Dr. Fleming said that the reason for choosing the rectangular form of circuit was because the inductance could be readily calculated from formulæ given in well-known text-books.

With respect to the remarks of Mr. Walter and Dr. Erskine Murray, Dr. Fleming agreed that it was necessary to choose a telephone of suitable resistance, and that the results taken with different telephones would not be the same.

XLVIII. *On the Action between Metals and Acids and the Conditions under which Mercury causes Evolution of Hydrogen.*

By S. W. J. Smith, *M.A., D.Sc., Lecturer on Physics, Imperial College of Science and Technology* *.

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§ 1. *Introductory.*—Under ordinary conditions metals like mercury, silver and copper are unable to displace hydrogen from solutions of acids (dilute or concentrated) with appreciable and easily demonstrable evolution of the gas.

An attempt is here made to show how, in the case of

* Read March 26, 1909.

mercury, the fluidity of the metal at the ordinary temperatures may be utilized to exhibit the cause of the inability and to supply a method by which it may be overcome.

The effects with hydrochloric and sulphuric acids only have been examined, but there is no apparent reason why solutions of other acids should fail to yield results of the same kind.

§ 2. *Electrolytic Solution Pressure.*—A method of regarding the question whether a metal will dissolve in an acid solution has been suggested by Nernst and developed by himself, Ostwald and others. The general idea of this method has become familiar, and provides a means of forming a more suggestive picture of what actually happens than is obtained from purely thermochemical considerations (*Cf. e. g.* Thomsen, 'Thermochemistry,' pp. 349 to 356, &c.), unsatisfactory in other ways as well.

In order to present a consecutive account of the present experiments, this view is restated below in a form as far as possible free from hypotheses not absolutely necessary*.

§ 3. *A conception of the Interaction of Metals and Acids.*—Imagine that a metal M comes suddenly into contact with an air-free solution of an acid HX. It is known that in general equilibrium will be impossible; a certain quantity of H will be precipitated upon the surface of M and an equivalent quantity of M will dissolve.

It may be that this interchange will take place only to an indefinitely small amount, but we can safely say that no salt MX is absolutely insoluble and that every metal has at least some tendency to go into solution. It is the relative magnitude of this tendency which is to be regarded as the characteristic variable distinguishing Hg, for example, from metals like Fe and Zn.

On account of the electrostatic resisting forces which would arise, ions of H cannot escape spontaneously from a solution of HX, nor can ions of M escape from the metal M; but with M and HX in contact the conditions alter. The assumed tendencies of M and H to spread beyond their original boundaries can now become effective without

* *Cf.* Nernst, 'Theoretical Chemistry,' 2nd Engl. edit. p. 724.

development of electrostatic resisting forces; because equivalents of H and M can pass across the common surface of acid and metal without change in the electric charge on either side. Neutral molecules of H can escape from solution while ions of M enter.

It is a necessary conclusion from a consideration of this kind (abstract thermodynamics furnishes many other examples) that solution of the metal and precipitation of hydrogen must begin at the interface (assuming no other change possible), whether accompanied by loss or gain of heat from the rest of the system. The familiar parallel is the fact that a gas will always expand spontaneously from higher pressure to lower if the external constraints permit—the work which it does being performed at the expense, if necessary, of its own internal energy.

It would appear, therefore, that every metal must be able to displace to some extent the hydrogen of an acid solution. If in any case the action is imperceptible, it must be because the tendency of the metal to enter solution is very small. The entry of an insignificant amount is then sufficient to balance the tendency. After this, further solution of the metal with escape of hydrogen will be a process like the compressing of one quantity of gas by the expansion of another—the constraints of the two quantities of gas being such that mutual expansion and contraction is the only change possible.

In such a case equilibrium is reached when the work which could be done by any further expansion of the second gas would be less than the work required to increase the compression of the first. Similarly, the replacement of hydrogen by a metal will cease at a point defined by the condition that further escape of hydrogen would produce less available work than would be required to cause the equivalent quantity of metal to enter solution.

§ 4. *Symbolic expression of the argument of* § 3.—Assuming as a rough approximation that the process is reversible and takes place isothermally, the conception may be expressed symbolically as follows.

The work done (diminution of available energy) when one equivalent weight of hydrogen escapes from solution may be

written in the form $\mu_h^s - \mu_h^n$, in which μ_h^s is a function of the strength of the solution (increasing with the concentration), and μ_h^n depends upon the pressure at which the hydrogen escapes (increasing with the pressure),

Similarly the work done (gain of available energy) when an equivalent of the metal enters solution may be expressed as $\mu_m^s - \mu_m^n$, in which μ_m^s is a function of the amount of metal already dissolved per ccm. near the interface, and μ_m^n is a physical constant of M (at the temperature of the interaction and for a given curvature of surface if the metal is fluid), which may be large or small according to the nature of the metal.

The relation necessary for equilibrium is

$$\mu_h^s - \mu_h^n = \mu_m^s - \mu_m^n. \quad . \quad . \quad . \quad . \quad . \quad (i.^*)$$

The amount of M which enters solution (and of H which escapes) before equilibrium is attained will be determined by the value of

$$c = f(\mu_m^s) = f(\mu_h^s - \mu_h^n + \mu_m^n),$$

where c is the concentration of the salt MX near the surface which is necessary in order that μ_m^s may acquire the value required to satisfy (i.). If no such value of c can arise equilibrium is impossible (unless owing to secondary effects other conditions supervene), and interaction will continue until the supply of acid or of metal is exhausted.

§ 5. *Possible effect of Surface Tension.*—The above statement of the conditions of equilibrium neglects possible variation of the surface energy during the interaction. In many cases there may be relatively little variation; but in one at least, where the metal is mercury and the interface consequently separates two liquids, changes of surface tension

* In the strictly reversible system M : MX : HX : H_{Pt}, the corresponding equation of equilibrium (no electromotive force) is approximately of the form

$$a T \log c_{HX} - E_h = b T \log c_{MX} - E_m,$$

where E_h and E_m are 'electromotive constants' of hydrogen (at given pressure) and the metal, at the temperature of equilibrium T.

are easy to detect. In this case the surface energy can be seen to diminish with increase in the concentration of the mercury salt in solution.

Consequently there is now a greater diminution of available energy than is represented by the left-hand member of (i.), when an equivalent of H is replaced in solution by an equivalent of M. Hence μ_m^s must attain a greater value in the solution before equilibrium is reached than would be required to satisfy (i.).

Thus if $-\gamma$ is the decrease of surface tension which would accompany the exchange of equivalents of H and M across unit surface when equilibrium is attained, we should have

$$\mu_h^s - \mu_h^n + \gamma = \mu_m^s - \mu_m^n, \quad . \quad . \quad . \quad \text{(i. } a\text{)}$$

and the equilibrium concentration of the salt of M in solution would now be

$$c = f(\mu_h^s - \mu_h^n + \mu_m^n + \gamma).$$

§ 6. *The Problem for Experiment.*—If the equation (i. *a*) represents the condition of equilibrium between mercury and an acid (assuming that the only reaction possible is of the type

$$\mathrm{Hg} + \mathrm{HX} = \mathrm{HgX} + \mathrm{H})$$

it is clear that when the substances come into contact a certain amount of hydrogen must be displaced. Otherwise the right-hand member of the equation can never become equal to the left.

If (as is likely in the case of Hg) the quantity μ_m^n is very small, the quantity $\mu_m^s - \mu_m^n$, (being at constant temperature of the form $k \log c/c_0$, where c_0 is very small), may acquire a considerable value, even when c is small, *i. e.* when only a small quantity of hydrogen has been displaced.

Thus equilibrium may be reached and displacement of hydrogen cease (neglecting diffusion effects) before the amount separated per unit surface has become perceptible.

If, however, some means could be found of removing the mercury salt as fast as it was formed the reaction would continue, and thus the displacement of hydrogen might be rendered evident.

§ 7. *Detrimental effect of Oxygen in the Surface-layer.*—The simplest way of obtaining an experimental answer to the question whether the direct displacement of hydrogen by mercury ever occurs is not immediately obvious. The purest mercury in contact with the air will become coated with a film of condensed oxygen—possibly a minute layer of oxide. Hence, even if the acid with which it may be brought into contact is free from dissolved oxygen, the interaction contemplated in the equations above may be prevented.

When an equivalent of hydrogen forsakes the acid solution in the presence of oxygen, the loss of available energy can be greater than before because, instead of separating as gas, the hydrogen can now become part of a molecule of water. Hence the amount of metal which must dissolve before equilibrium is reached will much exceed that required to give μ_m^s the value sufficient to satisfy (i. *a*) above. The equilibrium value of μ_m^s in the present case may be written

$$\mu_m^s = \mu_h^s - \mu_h^{OH_2} + \gamma + \mu_m^n \quad . \quad . \quad . \quad . \quad (ii.)$$

In which the quantity $\mu_h^{OH_2}$ is much less than the corresponding term μ_h^n of equation (i. *a*).

It is known from electrical measurements that the earlier stages of this reaction proceed with great rapidity. In the presence of a sufficient quantity of oxygen, μ_m^s may thus almost at once reach a greater value than that required to prevent the evolution of hydrogen*.

It is therefore essential to experiment with a mercury surface as far as possible oxygen-free, or to devise a means of removing from near any mercury surface as much as is desired of the mercury salt in solution.

§ 8. *Possible methods of eliminating this effect.*—Imagine two masses of mercury A and B immersed in the same acid. Let the surface of A be one not originally oxygen-free and surrounded in consequence by solution containing dissolved mercury salt. Suppose that the state of the solution around A has become practically steady without appreciable diffusion of the mercury salt into the region round B. Let the surface

* The effect of the presence of oxygen can also be presented in a thermochemical form, omitted here for the sake of space.

of B be one originally free from oxygen, and suppose that no interaction has yet taken place between it and the acid. Left to itself this mercury might interact with the acid, some of it displacing some of the hydrogen; but suppose that, instead, it is brought into contact with the mass A at one point or more without considerable change in the extent of either surface in contact with the acid.

Well-known electrochemical phenomena leave no doubt as to what will happen. Mercury will immediately begin to deposit on A and to enter the solution round B. Mercury-salt will in fact disappear and appear in equal quantities round A and B respectively until the concentration of the salt in solution round both is the same. This change will be effected by a displacement of anionic 'chains' in the solution from A towards B and by some analogous process (shift of electrons) in the mercury from B to A. The time taken by the process to complete itself will depend upon the length of the ionic chains. If these are short this time will be very small compared with the time taken by A to reach the steady state acquired before the contact.

From the point of view already described this process occurs in a way analogous to the expansion of a gas when the external constraints permit. Here the constraints virtually permit the expansion of the mercury salt from the space round A into the space round B until the concentration is the same in both. The electrical phenomena are incidents, not causes, of the flow of matter which takes place.

From whatever point of view the result may be regarded there is no doubt that metallic contact between A and B would reduce the amount of mercury salt in solution round the former. If B were very large compared with A, the amount of mercury left in solution round A would be very small. Similarly if A were connected with a succession of masses initially like B but small (each being removed before another took its place), the amount of salt in solution round A could be continuously lowered.

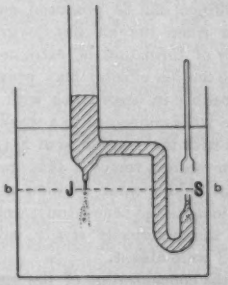
Although it appears possible to obtain oxygen-free surfaces of mercury (like B), they are usually in rapid motion and very difficult to observe. It is easy, however, to study a

surface (like A) subjected to metallic contact with a succession of surfaces much more nearly oxygen-free than itself.

§ 9. *Experimental realization.*—Mercury poured into a vertical glass tube drawn to a fine capillary at the lower end escapes in a narrow stream. Since the surface of the mercury per unit mass is much greater in this than in the tube, a considerable quantity of the mercury below the surface in the tube must enter the surface in the jet. Thus a surface film, upon the mercury originally, must become much thinner in the jet.

Suppose the stream to enter an acid solution as at J in the figure. It will possess much less oxygen per unit surface

Fig. 1.



than mercury at rest. The concentration of oxygen may fall below the amount sufficient to raise μ_m^e to the value required to satisfy (i. *a*), and therefore a small quantity of hydrogen may form.

The chance of formation of hydrogen will be greatest at the end of the jet where it breaks into drops and suddenly presents a new surface to the solution. If the jet is completely immersed this chance will be only a little greater than at the sides; but it will be much enhanced if the end of the jet just touches the surface of the solution (level of liquid at *b* in fig. 1).

Thus suppose the length of the completely immersed jet to be l and the velocity of efflux v. The "electrochemical" forces already described will tend to equalize the distribution of mercury salt in solution round the jet, although the various elements of the surface have been in contact with the solution for times varying between zero and l/v. The end of the jet after rupture is a surface bounded by that portion of the rest of the jet which has been longest in contact with the solution, and the concentration of mercury salt round the end will therefore be raised practically instantaneously to a considerable value.

If the jet breaks in the surface of the solution, however, the electrochemical short-circuit is reduced to a minimum because the sides of the jet are now practically out of contact with the solution.

Very little hydrogen can be produced on any element of the surface even when the jet has its greatest efficiency, and the difficulty of formation of extremely small bubbles (owing to surface tension effects) may prevent evolution of gas otherwise possible in accordance with (i. α). Suppose, however, that the jet is connected to a small mercury surface at rest in contact with the solution as at S (fig. 1). If there is less mercury in solution round J than round S, the metal will precipitate at S and enter solution at J. This action will continue as long as the concentration of Hg in solution round S exceeds the practically steady value which would exist round J if S were absent.

Consequently the concentration of the mercury salt round S must tend to diminish continuously until it is as small as that round J. But if, before this can happen, the amount of Hg in solution at S becomes smaller than that required to satisfy (i. α), a new reaction will begin. Direct action between the mercury of S and the acid will occur with evolution of hydrogen and formation of a new supply of mercury salt in solution.

This action will be continuous, for, in virtue of the continuous effect of J, Hg will be continuously removed at S. A steady state (neglecting secondary actions such as described below, § 12) will be reached when the rate of evolution of hydrogen at S is exactly equivalent to the rate of removal of

mercury from solution at S by J. For every equivalent of acid that disappears at S, an equivalent of the mercury salt will appear in solution at J.

§ 10. *Results.*—In the above way it is possible to conceive that mercury will decompose sulphuric or hydrochloric acid with evolution of hydrogen. Experiment justifies the conception, for, if either acid (of sufficient strength) is used in the vessel of fig. 1, a stream of hydrogen is evolved at S.

If the point of the capillary (at J) is below the surface of the acid the effect occurs most rapidly when the head of mercury just suffices to produce a short, approximately cylindrical, jet at J. (The ratio l/v for the jet has then a minimum value because, with increase of head, the jet length increases more rapidly than the velocity.)

The effect occurred with the strongest sulphuric acid available ('pure redistilled' s. g. about 1·84). If the acid is diluted the same evolution of gas can be obtained, but, when the density of the acid falls below about 1·25 the rate of evolution is very slow and occurs only when the jet breaks in the surface of the solution. Finally, when the density of the acid is below about 1·19, there is no perceptible evolution of gas in any position of the jet*.

The experience with hydrochloric acid is similar. With the most concentrated acid used (s. g. about 1·16) there was a fairly rapid evolution of gas when the jet broke in the surface. Gas ceased to come off when the density of the acid fell below about 1·09. The molecular concentrations of these limiting solutions are about the same—roughly 6 equivalent gram mols. per litre—although that of the HCl solution is slightly the smaller †. (*Cf.* § 13 below.)

§ 11. *Proof of Evolution of Hydrogen.*—The gas was not proved to be hydrogen by a direct test in every case; but only in the case most liable to suspicion, viz. when concentrated sulphuric acid was employed; and in one other, viz. when equal volumes of this acid and water were mixed. It

* In practice it is easier to adjust J and S if separate columns of mercury connected by a wire are used instead of the apparatus of fig. 1.

† Mr. J. S. G. Thomas has since made a more exact determination of the limiting concentrations. He finds them to be 6·25 gram equivalents per litre for H_2SO_4 and 5·75 gram equivalents per litre for HCl.

was proved to be hydrogen in the following way :—Collected in a small tube over the concentrated acid, it did not dissolve appreciably in the latter, nor, subsequently, in recently boiled distilled water by which the acid was displaced. It was therefore neither H_2S nor SO_2.

It was difficult to test the gas positively since with the arrangement used it took a considerable time to collect a few cubic millimetres. It was possible, however, to show that it underwent contraction on explosion with oxygen in the following way. A bubble of the gas, 4 mms. long, was collected at the top of a tube which ended in a capillary through which a fine platinum wire had been sealed. About 0·8 mm. of oxygen (prepared electrolytically) was added and a second fine wire was pushed into the collecting tube from below until it reached almost to the first. A spark was then passed, and the remaining gas was found to occupy about 2·5 mm. of the tube *. Thus the gas evolved behaved like hydrogen, the only constituent common to the two acids employed.

§ 12. *Secondary effects with Sulphuric Acid.*—Although, in the absence of oxygen, the simplest direct interaction between mercury and sulphuric acid is

$$Hg_2 + H_2SO_4 = Hg_2SO_4 + H_2, \quad . \quad . \quad . \quad \text{(I.)}$$

there are other possible interactions of which the next † in simplicity would be

$$4Hg_2 + 5H_2SO_4 = 4Hg_2SO_4 + H_2S + 4H_2O. \quad \text{(II.)}$$

Here every fifth molecule of the acid may be supposed to be reduced by the hydrogen resulting from the direct action between four molecules of the acid and mercury.

According to the view adopted in this paper the reaction I.

* Mr. W. F. Higgins kindly attempted to make a spectroscopic test of the gas, but various difficulties were encountered which it did not seem profitable to attempt to overcome since the gas had already been proved to be neither H_2S nor SO_2 nor oxygen.

† In the reaction representing the secondary reducing effect of hydrogen, viz.,

$$Hg_2 + (1 + a)H_2SO_4 = Hg_2SO_4 + (H_2 + aH_2SO_4)$$

the minimum value of a is 1/4.

can go on only so long as the concentration of mercury salt in solution does not exceed the value given by (i. *a*); but according to the same view the reaction II. can occur before and after this limit to reaction I. is passed. It probably does not occur to the exclusion of I. because it involves a greater rearrangement of the constituents of the reacting molecules than is involved in the displacement of hydrogen (*cf.* Thomsen, *l. c.* p. 354 *et passim*).

To take the case of concentrated sulphuric acid, which was carefully examined. The H_2S of reaction II. will interact with a further quantity of the acid precipitating sulphur. It will also in part precipitate the very nearly insoluble sulphide of mercury by interaction with the sulphate in solution. The former reaction may be represented by the equation

$$3Hg_2 + 4H_2SO_4 = 3Hg_2SO_4 + S + 4H_2O. \quad . \quad (II. a)$$

In this case, neglecting for simplicity the energy variation due to the decomposition of the SO_4 ion, the equilibrium value of μ^s_m may be represented qualitatively by

$$\mu^s_m = \frac{4}{3}\left(\mu^s_{\mathfrak{h}} - \mu^{H_2O}_{\mathfrak{h}}\right) + \gamma + \mu^n_m, \quad . \quad . \quad . \quad . \quad (iii.)$$

which shows that more mercury must now enter solution before equilibrium is attained than when hydrogen ceases to be evolved in accordance with (i. *a*). The reaction with precipitation of mercury sulphide leads to a similar result.

Thus we are led to infer that the production of sulphur and of sulphide of mercury may continue after the evolution of hydrogen has ceased.

This inference is fully confirmed by experiment:—

(*a*) When the diameter of S is small—less or not much greater than that of J—the mercury in solution round S is removed almost at once by the action of J and hydrogen simultaneously appears. Very soon, however, a yellowish-white cloud begins to form (particularly round the portions of S where the curvature is least and where the evolution of hydrogen is most noticeable). This cloud probably arises mainly from the decomposition of H_2S. In fact when the effectiveness of J was decreased by reducing the head some

of the last bubbles to escape seemed (when observed through a microscope) to be surrounded by a film exactly like that which is produced when bubbles of H_2S are passed into concentrated H_2SO_4. The formation of sulphide of mercury can also be detected after the jet has been in action for some time.

These reactions rapidly reduce the rate of evolution of hydrogen at S. The production of sulphur is a process which the action of J cannot reverse and the amount increases continuously, raising the electric resistance of the solution in the capillary and hence diminishing the effectiveness of J. The insoluble sulphide precipitated on the surface of the mercury further increases the circuit resistance. In addition this sulphide seems to prevent the evolution of hydrogen directly although the amount of mercury salt in solution may be very small. For instead of escaping, the hydrogen must now apparently interact with the sulphide in the surface layer producing H_2S and eventually sulphur. Thus after a time the only effect of J at S is to produce sulphur. If, however, the capillary at S is rinsed out by expelling a little of the mercury, the evolution of hydrogen begins again at the fresh surface of acid and mercury.

(*b*) When the diameter of S is large—for example, twenty times that of J—sulphur and sulphide only are formed, however efficient the jet may be. In this case the jet only slowly reduces the amount of mercury salt in solution round S, and the reactions depending on the formation of sulphuretted hydrogen begin before the evolution of hydrogen is possible and continue in such a way that it can never occur. In a qualitative sense it may be said that in this case the slower reaction (requiring greater molecular rearrangements) has time to prevent the first.

It is only when the acid is concentrated that these effects occur, for diluted sulphuric acid is not reduced to or by sulphuretted hydrogen. Thus even in the case (*b*) just mentioned, hydrogen is evolved freely when sulphuric acid solution of s. g. 1·5 is substituted for the concentrated.

§ 13. *Effects of Dilution of the Acids.*—The loss of available energy when an equivalent of hydrogen leaves the acid solution and is evolved as gas at atmospheric pressure falls

continuously* as the solution is diluted. Consequently the amount of mercury which can be in solution round the electrode without preventing the evolution of hydrogen becomes continuously less.

Before the hydrogen can escape it must reach a certain concentration (in the neutral state) in the mercury and in the solution at the surface layer. Disregarding the difficulty of formation of very minute bubbles, evolution just fails to occur when, electrode and solution being saturated, there would be no loss of available energy if an infinitesimally small quantity of hydrogen left the solution and the infinitesimal equivalent of mercury entered.

Thus if there is a limit below which J (however effective) cannot reduce the concentration of the mercury salt round S (§ 9), there is also a limit to the concentration of the acid which can be decomposed at S with evolution of gas. The experiments of § 10 show that this limit is reached at a concentration of about 6 equivalents per litre in the case of each acid.

Further, assuming that equally concentrated solutions of the acids and of their corresponding mercury salts are approximately equally dissociated, one would expect in virtue of equation (i.) that the limiting concentrations of the two acids would be approximately equal. This also agrees with experiment.

It happens, however, that the quantity γ in (i. a) is greater for HCl than for H_2SO_4. It might therefore be anticipated that if the corresponding dissociations were exactly equal, the limiting concentration for HCl would be rather less than for H_2SO_4; but although the results suggest the fulfilment of this anticipation, the data are not sufficiently accurate to

* In the case of concentrated sulphuric acid, dilution seems first to increase the rate of evolution of hydrogen. This result might be anticipated for two reasons. The first effect of dilution is to increase the conductivity and, probably, the ionic concentration of the acid. Thus the effectiveness of the jet and the value of μ_A^π will simultaneously rise. Again, some of the hydrogen which would otherwise escape will react with the concentrated acid while in the nascent state (§ 12). There should thus be a particular strength of sulphuric acid for which the rate of evolution of gas is a maximum.

make it worth while to attempt a quantitative proof—especially as there are other possible explanations.

§ 14. *Possible effects at the Jet.*—The explanation which has been given of the behaviour of the still mercury surface S supposes that the concentration of mercury salt round the jet J is small. In fact the behaviour of S is (by hypothesis) controlled by the rate of independent formation of mercury salt at J. Consequently when hydrogen is evolved at S the amount of mercury in solution round J should be insufficient to prevent the evolution of hydrogen at J.

If any such evolution takes place it is very difficult to detect. Mercury was allowed to run for a long time from a capillary tube fused into the top of a glass bulb which was filled with concentrated sulphuric acid and terminated below in a tube dipping into mercury. No trace of gas could be seen. The level of the acid was lowered until the jet broke in the surface, the space above being filled with air. In this case, the first impression is that there is a copious evolution of gas (*cf.* Paschen, *Wied. Ann.* vol. xli. p. 56, 1890). But this is an illusion. The greater part, if not all, of the gas which appears to form at the surface of the mercury drops is air dragged in from above. This effect is very pronounced in liquids of great viscosity. It is less conspicuous in sulphuric acid than it is in glycerine.

In consequence of this phenomenon it is impossible to tell by inspection whether any gas is evolved as the result of chemical action between the acid and the mercury. In a further experiment benzene (which had previously been shaken up with another sample of concentrated H_2SO_4 and then decanted) was substituted for the air above the acid. There was now a continuous circulation of drops of benzene within the acid like that of the air bubbles before. A considerable quantity of gas accumulated at the top of the apparatus; but it may have resulted from some chemical reaction in which the benzene took part. The evolution was even more noticeable when the benzene was replaced by pentane.

It has already been pointed out that the escape of hydrogen at the jet may be impossible although it takes place at S (§ 9). At the latter the effect is cumulative and all the hydrogen

is evolved at the same surface. Here minute bubbles can coalesce into larger ones in which the pressure is not greatly above that of the air. At the jet, however, the hydrogen which mercury replaces must first reach a certain concentration in every fresh element of the jet and of the solution surrounding it. Then it must overcome the resistance to the formation of a minute bubble before it can escape as gas.

That direct action between the jet and the acid might result in visible production of hydrogen, but for the counteractions just described, can be seen by allowing the jet to break in the surface of a concentrated solution of sulphate of copper. The surface of the mercury which collects on the bottom of the vessel containing the sulphate presents a tarnished appearance like that produced by the addition of copper. This result, it will be seen, is in close accord with the present point of view and suggests the need for qualification of the familiar statement: "The more electropositive metals, Cu . . . . . , precipitate the less electropositive metal Hg." It is probable that other metals, such as Pb and even Cd, could be precipitated by mercury, from solutions of their salts, in a similar way.

§ 15. *A kinetic representation of* §3.—The action between an acid and a metal is formulated in § 3 in a way which avoids the necessity of dealing with the kinetics of the process by which equilibrium is attained. The following is perhaps the simplest picture of what actually happens. The metal is assumed to be monovalent, but it is easy to see what change must be made when the valency is n.

Into the space (the "border layer") which separates acid and metal, Faraday tubes can stretch from the acid and from the metal respectively. Of these, a possible pair forming side by side will be (1) a tube stretching from the acid nearly across the border layer, with its negative end (an anion of the acid) in the solution and its positive end (an ion of hydrogen) near the metal, (2) a tube stretching from the metal, with its negative end on the metal and its positive end (a metal ion) almost reaching the solution.

These tubes forming simultaneously and close together will interact, yielding a molecule of the salt MX in solution and a neutral equivalent of H deposited on M.

The number of interactions of this kind taking place per sq. cm. of surface in the unit of time will depend upon the frequency with which such tubes form side by side, *i. e.*, upon the number of molecules of HX in solution per cc. on the one hand and upon some specific property of the metal (determining the rate at which tubes of the second kind form) on the other. For the "velocity" of the reaction

$$M + HX \longrightarrow H + MX,$$

we may therefore write

$$v = k\, c_h\, C_m,$$

where k is a constant at given temperature, c_h is the concentration of the hydrogen ions in solution, and C_m is a specific constant of the metal M.

As the result of this action, hydrogen will accumulate upon the metal and MX will be formed in solution. A reverse action, similar in kind to the first, now becomes conceivable. This reaction, which may be represented by

$$H + MX \longrightarrow M + HX,$$

will proceed with a velocity

$$v' = k'\, c_m\, C_h,$$

where c_m represents the ionic concentration of the metal in solution and C_h is a specific constant of hydrogen deposited at given pressure upon M.

If a steady state is reached, after a certain quantity of hydrogen has been displaced and the equivalent quantity of M has dissolved, it will be defined by the condition $v = v'$ or

$$k\, c_h / C_h = k'\, c_m / C_m.$$

We can thus deduce kinetically a result identical with that obtainable by application of the logarithmic formula of Nernst and contained as a particular case in the general equation of § 4.

Since there is no effective transfer of electricity across the border layer, a possible contact difference of potential between metal and solution would not affect the available work equation of § 4. Similarly it would not affect the final equation of equilibrium deduced kinetically. For if a

potential-difference existed it would have the same relative effect upon v' as upon v.

§ 16. *Summary of Conclusions.*—Pure mercury reacts with acid solutions with displacement of hydrogen in the same way as metals like zinc.

The reaction stops before a perceptible quantity of hydrogen is evolved because a very small quantity of mercury salt in solution is sufficient to cause it to cease.

The surface film of mercury which has been in contact with the air probably contains more than enough oxygen to oxidise to water all the hydrogen that would be displaced before the direct action ceased.

The amount of oxygen per sq. cm. of the surface film can be reduced to a very small quantity by allowing the mercury to escape from a containing tube in the form of a narrow jet.

But for certain counteracting influences due to the fact that the substance displaced is a gas, this jet might be used to obtain hydrogen from acid solutions of sufficient strength.

By the aid of the jet the direct action between mercury and the acid, with displacement of hydrogen, can be made continuous. The jet, when in direct communication with a mercury surface at rest in the same solution, prevents the concentration of the mercury salt, formed by the displacement of hydrogen, remaining or becoming large enough at the still surface to stop the evolution of gas. In consequence, hydrogen escapes freely and can be collected and analysed.

Owing to direct action at its own surface, the jet cannot reduce the concentration of mercury salt round the still surface below a certain limit. Further, the amount of mercury salt which is sufficient to stop the direct action at the still surface diminishes with the concentration of the acid solution used.

In consequence it will be impossible to decompose the acid by means of the jet when the strength of the solution falls below a certain limit. This limit might be expected to be about the same for different acids. It was found to be about the same for hydrochloric and sulphuric acids (roughly

6 gram equivalents per litre), although rather lower for the former than for the latter.

The various reactions which occur when metals are placed in contact with concentrated sulphuric acid are elucidated by the experiments described (§ 12).

I am very much indebted to Mr. J. S. G. Thomas, B.Sc, for frequent and valued help while performing the experiments I have described.

Discussion.

Dr. W. Watson congratulated the Author and remarked that in physical chemistry hypotheses were often used which were not based on experimental evidence. Dr. Smith had shown that his hypothesis corresponded with actual physical facts, and his theory could be looked upon with satisfaction.

In reply to a question by Mr. F. E. Smith, the Author explained that the process by which the concentration of the mercury salt in solution was reduced below the amount necessary to prevent evolution of hydrogen was endothermic.

XLIX. *Theory of the Alternate Current Generator.* By Thomas R. Lyle, *M.A., Sc.D., Professor of Natural Philosophy in the University of Melbourne* *.

It has been usual hitherto to ascribe the distortion of the wave-form of the current given by an alternate current-generator to:—

1. "Lack of uniformity and pulsation of the magnetic field, causing a distortion of the induced E.M.F. at open circuit as well as under load."
2. "Pulsation of the reactance causing higher harmonics under load."
3. "Pulsation of the resistance causing higher harmonics under load also" †.

And, as far as I have been able to find out, another cause has been overlooked, namely, the mutual reactions between armature and field, which when the generator is loaded is at least as important as any of the foregoing.

* Read April 23, 1909.

† Steinmetz, 'Alternating Current Phenomena.'

If such is the case it can only be explained by the fact that the theory of the simple alternator has not been completely worked out.

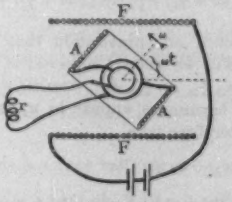
In the following paper this is done for an alternator with a uniform field by means of a new application of the vector method in which all the harmonics of a periodic function are dealt with simultaneously.

In the same is shown how to take account of hysteresis and eddy currents, and the theory of the action of dampers in reducing the heating in the field is also given.

The theory of the alternate current synchronous motor is also dealt with.

1. Let two coils be arranged as indicated in fig. 1, one of them, F, called the field coil, being fixed and having a battery of constant E.M.F. $= \eta$ in its circuit, the other, A, called the

Fig. 1.



armature coil, fitted in the usual way with slip rings for connexion to an external circuit and being rotated by power at a constant angular velocity ω round a fixed axis which is perpendicular to its own axis of figure and to the direction of the lines of force of F, and which passes through its own centre. It is required to determine completely the currents that flow in both A and F.

Let x and ξ be the currents at any instant in A and F respectively, and let the mutual inductance of A and F when their axes are coincident be m, and hence $m \cos \omega t$ at the time t. Also let r and l be the total resistance and self-inductance of the A circuit, and ρ, λ similar quantities for the F circuit.

Then, when the armature is being driven at constant

angular velocity ω, and x and ξ are flowing, the total number of lines linked on A is

$$lx + m\xi \cos \omega t,$$

and the number linked on F is

$$\lambda\xi + mx \cos \omega t.$$

Hence

$$\left.\begin{aligned} rx + \frac{d}{dt}\{lx + m\xi \cos \omega t\} &= 0 \\ \rho\xi + \frac{d}{dt}\{\lambda\xi + mx \cos \omega t\} &= \eta \end{aligned}\right\} \quad . \quad . \quad . \quad (\text{I.})$$

where η is the applied steady E.M.F. in the F circuit.

2. If we assume as the solution of these equations

$$x = \frac{x_0}{2} + x_1 \sin(\omega t + c_1) + x_2 \sin(2\omega t + c_2) + x_3 \sin(3\omega t + c_3) + \&c.,$$

$$\xi = \frac{\xi_0}{2} + \xi_1 \sin(\omega t + \gamma_1) + \xi_2 \sin(2\omega t + \gamma_2) + \xi_3 \sin(3\omega t + \gamma_2) + \&c.,$$

we can see at once on substitution that $\rho\xi_0 = 2\eta$, and that $x_0 = 0$, and it will be shown afterwards, § 15, that when $x_0 = 0$ then ξ_1, x_2, ξ_3, x_4, ξ_5, &c., vanish, or in words, when $x_0 = 0$ only odd harmonics appear in x, and only even ones in ξ. Let us therefore take

$$\left.\begin{aligned} x &= x_1 \sin(\omega t + c_1) + x_3 \sin(3\omega t + c_3) + x_5 \sin(5\omega t + c_5) + \&c. \\ \xi &= \frac{\xi_0}{2} + \xi_2 \sin(2\omega t + \gamma_2) + \xi_4 \sin(4\omega t + \gamma_4) + \&c. \end{aligned}\right\} \quad (\text{II.})$$

Now any harmonic in either x or ξ, for instance $c_q \sin(q\omega t + c_q)$, being completely specified by x_q, c_q, and q, can, when its order q is known, be represented by the vector drawn from the origin in any reference plane to the point in that plane whose polar coordinates are x_q, c_q, twice the constant term in ξ being represented in the same plane by the vector to the point ξ_0, $\frac{\pi}{2}$

The form of solution (II.) assumed may now be written

$$\left.\begin{aligned} x &= \mathfrak{a}_1 + \mathfrak{a}_3 + \mathfrak{a}_5 + \&c. \\ \xi &= \frac{\alpha_0}{2} + \alpha_2 + \alpha_4 + \alpha_6 + \&c. \end{aligned}\right\} \quad . \quad . \quad . \quad (\text{III.})$$

where $\mathfrak{a}_1$, $\mathfrak{a}_3$, &c., α_0, α_2, α_4, &c., are vectors whose orders are indicated by the subscribed numbers. Of these, one only, namely α_0, is known, as it is drawn to the point whose polar co-ordinates are ξ_0, $\frac{\pi}{2}$, where $\xi_0 = \frac{2\eta}{\rho}$. The others have to be determined.

Note a.—In the sequel it will sometimes happen that a vector, say a_q, originally assumed of order q, will be used to represent an harmonic of a different order, say $q+1$. In such a case it will be written $(\mathfrak{a}_q)_{q+1}$; thus

$$\mathfrak{a}_q = x_q \sin(q\omega t + c_q),$$

but

$$(\mathfrak{a}_q)_{q+1} = x_q \sin\{(q+1)\omega t + c_q\}.$$

Note b.—The length of a vector α will be written as $\bar{\alpha}$ (*i. e.* with the bar) ; thus $\bar{\mathfrak{a}}_3 = x_3$, unless in cases where no ambiguity can arise, when α simply will be written for the length of the vector α.

3. If we agree to indicate by ι^θ the operation of rotating any vector to which it is prefixed through an angle θ in the positive direction, then

$$\iota^\pi \alpha = -\alpha \quad \text{or} \quad \iota^\pi = -1,$$

and

$$\iota^\theta \alpha = (\cos\theta + \overset{\pi}{\iota^2} \sin\theta)\alpha \quad \text{or} \quad \iota^\theta = \cos\theta + \overset{\pi}{\iota^2} \sin\theta.$$

Also, if $t = \mathrm{D}\iota^f$, $t\alpha$ is the vector obtained by increasing α in length D times and then rotating the increased vector through an angle f in the positive direction.

Plane vector operators such as t are well known to be subject to the same rules as ordinary algebraical symbols.

Again, the sum of two operators $a_1\iota^{\theta_1}$, $a_2\iota^{\theta_2}$, can be expressed as a single operator $\mathrm{A}\iota^\psi$, say, that is

$$\mathrm{A}\iota^\psi \alpha = a_1\iota^{\theta_1}\alpha + a_2\iota^{\theta_2}\alpha,$$

where α is any vector.

Using the expression for ι^θ given above,

$$\mathrm{A}(\cos\psi + \overset{\pi}{\iota^2}\sin\psi)\alpha$$
$$= a_1(\cos\theta_1 + \overset{\pi}{\iota^2}\sin\theta_1)\alpha + a_2(\cos\theta_2 + \overset{\pi}{\iota^2}\sin\theta_2)\alpha,$$

so that

$$A\cos\psi = a_1\cos\theta_1 + a_2\cos\theta_2;$$

$$A\sin\psi = a_1\sin\theta_1 + a_2\sin\theta_2.$$

Hence

$$A^2 = a_1^2 + a_2^2 + 2a_1a_2\cos(\theta_1-\theta_2),$$

and

$$\tan\psi = \frac{a_1\sin\theta_1 + a_2\sin\theta_2}{a_1\cos\theta_1 + a_2\cos\theta_2}.$$

Again, if

$$\alpha_p = \xi_p\sin(p\omega t+\gamma_p),$$

then

$$\frac{d}{dt}(\alpha_p) = p\omega\iota^{\frac{\pi}{2}}\alpha_p,$$

seeing that

$$\frac{d}{dt}(\alpha_p) = p\omega\xi_p\sin\left(p\omega t+\gamma_p+\frac{\pi}{2}\right).$$

Hence for x and ξ as expressed in § 2

$$\frac{dx}{dt} = \omega\iota^{\frac{\pi}{2}}\Sigma q\mathfrak{a}_q, \qquad \frac{d\xi}{dt} = \omega\iota^{\frac{\pi}{2}}\Sigma q\alpha_p.$$

4. By means of the formula

$$2\sin a\cos b = \sin(a+b)+\sin(a-b)$$

it is easy to show that $2x\cos\omega t$, where x is the $\mathfrak{a}$ series of odd order vectors in § 2, is represented by the series of even order vectors of which the one of the pth order is the vector sum of $\mathfrak{a}_{p-1}$ and $\mathfrak{a}_{p+1}$, or that

$$2x\cos\omega t = (\mathfrak{a}_1)_0 + (\mathfrak{a}_1+\mathfrak{a}_3)_2 + (\mathfrak{a}_3+\mathfrak{a}_5)_4 + (\mathfrak{a}_5+\mathfrak{a}_7)_6 + \&c.,$$

$(\mathfrak{a}_1)_0$ being the resolved part of $\mathfrak{a}_1$ along the y axis, that is along the direction of vectors of zero order (see Note a, § 2).

Similarly

$$2\xi\cos\omega t = (\alpha_0+\alpha_2)_1 + (\alpha_2+\alpha_4)_3 + (\alpha_4+\alpha_6)_5 + \&c.$$

Again, by means of the formula

$$2\sin a\sin b = \cos(a-b)-\cos(a+b),$$

it is easy to show that

$$2\xi\sin\omega t = \iota^{-\frac{\pi}{2}}(\alpha_0-\alpha_2)_1 + \iota^{-\frac{\pi}{2}}(\alpha_2-\alpha_4)_3 + \iota^{-\frac{\pi}{2}}(\alpha_4-\alpha_6)_5 + \&c.$$

$$= \iota^{-\frac{\pi}{2}}\Sigma(\alpha_{q-1}-\alpha_{q+1})_q,$$

where q is odd with a similar result for the product $2x \sin \omega t$.

5. If we now substitute the vector expressions from §§ 2, 3, 4, in equations I. and equate separately to zero each set of vector terms of the same order we obtain the two series of vector equations

$$r\mathfrak{a}_q + q\omega\iota^{\frac{\pi}{2}}\left\{ l\mathfrak{a}_q + \frac{m}{2}(\alpha_{q-1} + \alpha_{q+1}) \right\} = 0, \qquad \text{(IV.)}$$

$$\rho\alpha_p + p\omega\iota^{\frac{\pi}{2}}\left\{ \lambda\alpha_p + \frac{m}{2}(\mathfrak{a}_{p-1} + \mathfrak{a}_{p+1}) \right\} = 0, \qquad \text{(V.)}$$

together with $\xi_0 = 2\eta/\rho$, where q is any odd number and p any even number.

From IV. we deduce a series of equations of the type

$$\alpha_{q-1} + 2\frac{q\omega l - r\iota^{\frac{\pi}{2}}}{q\omega m}\mathfrak{a}_q + \alpha_{q+1} = 0,$$

or

$$\alpha_{q-1} + t_q\mathfrak{a}_q + \alpha_{q+1} = 0,$$

where t_q is the operator $\mathrm{D}_q\iota^{-f_q}$, in which

$$\mathrm{D}_q \cos f_q = 2\frac{l}{m}, \quad \mathrm{D}_q \sin f_q = \frac{2r}{q\omega m},$$

that is

$$\mathrm{D}^2 = \frac{4}{m^2}\left(l^2 + \frac{r^2}{q^2\omega^2}\right), \quad \tan f_q = \frac{r}{q\omega l}.$$

Similarly from V. we deduce the series

$$\mathfrak{a}_{p-1} + \tau_p\alpha_p + \mathfrak{a}_{p+1} = 0$$

where τ_p is the operator $\Delta_p\iota^{-\phi_p}$, in which

$$\Delta_p \cos\phi_p = 2\frac{\lambda}{m}, \quad \Delta_p \sin\phi_p = \frac{2\rho}{p\omega m}$$

or

$$\Delta_p^2 = \frac{4}{m^2}\left(\lambda^2 + \frac{\rho^2}{p^2\omega^2}\right), \quad \tan\phi_p = \frac{\rho}{p\omega\lambda}.$$

Note that the vector equations in this paragraph are equations connecting the different vectors, considered purely as vectors, without any reference to the order of the harmonic they originally represented.

6. We have thus obtained the following infinite series of equations connecting the vectors used to represent x and ξ :—

$$\left.\begin{aligned} t_1\mathbf{a}_1+\alpha_2 &= -\alpha_0 \\ \mathbf{a}_1+\tau_2\alpha_2+\mathbf{a}_2 &= 0 \\ \alpha_2+t_3\mathbf{a}_3+\alpha_4 &= 0 \\ \mathbf{a}_3+\tau_4\alpha_4+\mathbf{a}_5 &= 0 \\ \alpha_4+t_5\mathbf{a}_5+\alpha_6 &= 0 \\ &\text{\&c., \&c.} \end{aligned}\right\} \qquad \text{(VI.)}$$

And as it is well known that algebraic methods are applicable to plane vector operators of the type here made use of, we obtain the following infinite determinant vector solution for $\mathbf{a}_1$, namely,

$$\mathbf{a}_1 = -\frac{\begin{vmatrix} \tau_2 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & t_3 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & \tau_4 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & t_5 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \tau_6 & 1 & \dots \\ & & & \text{\&c., \&c.} & & & \end{vmatrix}}{\begin{vmatrix} t_1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & \tau_2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & t_3 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & \tau_4 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & t_5 & 1 & \dots \\ 0 & 0 & 0 & 0 & 1 & \tau_6 & \dots \\ & & & \text{\&c., \&c.} & & & \end{vmatrix}} \alpha_0$$

or

$$\mathrm{P}_1\mathbf{a}_1 = -\Pi_2\alpha_0$$

where P_1 is the infinite determinant operator whose leading term is t_1, and Π_2 that whose leading term is τ_2.

7. P_1, Π_2, P_3, Π_4, &c., being the determinants whose leading terms are t_1, τ_2, t_3, τ_4, &c. respectively, we find at once by expanding that

$$\mathrm{P}_1 = t_1\Pi_2 - \mathrm{P}_3$$
$$\Pi_2 = \tau_2\mathrm{P}_3 - \Pi_4, \text{ \&c.},$$

hence

$$\frac{\mathrm{P}_1}{\Pi_2} = t_1 - \frac{\mathrm{P}_3}{\Pi_2} = t_1 - \frac{1}{\Pi_2/\mathrm{P}_3}$$
$$= t_1 - \frac{1}{\tau_2} - \frac{1}{t_3} - \frac{1}{\tau_4} - \text{\&c.}$$
$$= \mathrm{S}_1 \text{ (say)},$$

so that

$$\mathbf{a}_1 = -\frac{1}{\mathrm{S}_1}\alpha_0,$$

where S_1 is the infinite continued fraction operator whose leading term is t_1.

Again, if

$$\Sigma_2 = \tau_2 - \frac{1}{t_3} - \frac{1}{\tau_4} - \frac{1}{t_5} - \&c.$$

$$S_3 = t_3 - \frac{1}{\tau_4} - \frac{1}{t_5} - \frac{1}{\tau_6} - \&c.$$

$$\Sigma_4 = \tau_4 - \frac{1}{t_5} - \&c.,$$

and so on, we find in a similar way, or by making use of equations VI., that

$$\alpha_2 = -\frac{1}{\Sigma_2}\mathfrak{a}_1$$

$$\mathfrak{a}_3 = -\frac{1}{S_2}\alpha_2 \ \&c., \ \&c.$$

Hence for the complete solution we have

$$\alpha_0 = -S_1\mathfrak{a}_1 = S_1\Sigma_2\alpha_2 = -S_1\Sigma_2S_3\mathfrak{a}_3 = \&c.,$$

which gives $\mathfrak{a}_1$, $\mathfrak{a}_3$, $\mathfrak{a}_5$, &c., α_2, α_4, α_6, &c., in terms of the known vector α_0 provided the continued fraction operators S_1, Σ_2, S_3, &c., are determinate.

It is easily seen that they are determinate for when q becomes a large number

$$t_q = 2\frac{l}{m} = t_{q+2}$$

$$\tau_{q+1} = 2\frac{\lambda}{m} = \tau_{q+3} \text{ (see § 5)};$$

that is, S_1, Σ_2, S_3, Σ_4, &c. are recurring continued fraction operators, and the recurring elements when reached are simple numbers.

8. The form of solution obtained is one very easy of practical application. In computing the continued fraction operators just so many of their known t, τ elements (see § 5) need be taken account of as are necessary to give the required degree of approximation.

Moreover, in the case of a practical alternator, as the resistance of the field-magnet coils is negligible relative to

their reactance, all the τ elements are practically simple numbers independent of the field resistance, while for the t elements q is never a large number when t_{q+2} differs little from t_q. So that we could obtain S_q with considerable accuracy by assuming the recurring stage to be reached, and therefore

$$S_q = t_q - \frac{1}{\tau_{q+1} - \frac{1}{S_q}},$$

which gives the quadratic in S_q

$$S_q^2 - t_q S_q + \frac{t_q}{\tau_{q+1}} = 0,$$

from which S_q can be obtained by ordinary algebra.

[In solving this quadratic the two operators that come under the square-root symbol will have to be reduced by the addition theorem in § 3 to a single operator, $a\iota^b$ say, and the square root of this is $\sqrt{a}\iota^{\frac{1}{2}b}$.]

If S_q obtained in either of these ways be $= s_q \iota^{-b_q}$, then as

$$\Sigma_{q-1} = \tau_{q-1} - \frac{1}{S_q} = \tau_{q-1} - \frac{1}{s_q}\iota^{b_q},$$

Σ_{q-1} can be obtained by the addition theorem, and so on for S_{q-2}, Σ_{q-3}, &c., up to S_1. Let the results be written

$$S_1 = s_1\iota^{-b_1}, \quad \Sigma_2 = \sigma_2\iota^{-\beta_2}, \quad S_3 = s_3\iota^{-b_3}, \text{ &c.},$$

and in general

$$S_q = s_q\iota^{-b_q}, \quad \Sigma_p = \sigma_p\iota^{-\beta_p}.$$

9. As α_0 is the vector of length $2\eta/\rho$ lying along the axis of y (phase $=\pi/2$), and as

$$\mathfrak{a}_1 = -\frac{1}{S_1}(\alpha_0)_1 = -\frac{1}{s_1}\iota^{b_1}(\alpha_0)_1$$

$$\mathfrak{a}_2 = -\frac{2\eta}{s_1\rho}\sin\left(\omega t + \frac{\pi}{2} + \mathfrak{b}_1\right) = \frac{2\eta}{s_1\rho}\sin\left(\omega t - \frac{\pi}{2} + \mathfrak{b}_1\right).$$

Again, as

$$\alpha_2 = -\frac{1}{\Sigma_2}(\mathfrak{a}_1)_2 = -\frac{1}{\sigma_2}\iota^{\beta_2}(\mathfrak{a}_1)_2,$$

$$\alpha_3 = \frac{2\eta}{s_1\sigma_2\rho}\sin\left(2\omega t + \frac{\pi}{2} + \mathfrak{b}_1 + \beta_2\right);$$

similarly

$$\mathfrak{a}_3 = \frac{2\eta}{s_1\sigma_2 s_3\rho}\sin\left(3\omega t - \frac{\pi}{2} + \mathfrak{b}_1 + \beta_2 + \mathfrak{b}_3\right);$$

$$\alpha_4 = \frac{2\eta}{s_1\sigma_2 s_3\sigma_4\rho}\sin\left(4\omega t + \frac{\pi}{2} + \mathfrak{b}_1 + \beta_2 + \mathfrak{b}_3 + \beta_4\right);$$

$$\mathfrak{a}_5 = \frac{2\eta}{s_1\sigma_2 s_3\sigma_4 s_5\rho}\sin\left(5\omega t - \frac{\pi}{2} + \mathfrak{b}_1 + \beta_2 + \mathfrak{b}_3 + \beta_4 + \mathfrak{b}_5\right);$$

&c., &c.

And substituting these values in

$$x = \mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3 + \&c., \qquad \xi = \frac{\eta}{\rho} + \alpha_2 + \alpha_4 + \&c.,$$

we obtain the armature and field currents in the usual trigonometrical form of expression.

It is worth while drawing attention to the fact that the period of the alternating current induced in the field circuit is half that of the armature current, and that it contains all harmonics, both odd and even, relative to its own fundamental, and so its wave-form will in general be unsymmetrical with respect to the time-axis.

10. The total E.M.F. E generated in the armature circuit being equal to $-\frac{d}{dt}(m\xi\cos\omega t)$,

$$\mathrm{E} = -\frac{m}{2}\frac{d}{dt}\Sigma(\alpha_{q-1} + \alpha_{q+1})_q, \quad \text{(see § 4)}$$

$$= -\frac{\omega m}{2}\iota^{\frac{\pi}{2}}\Sigma q(\alpha_{q-1} + \alpha_{q+1})_q.$$

Also as $\qquad \alpha_{q-1} + \iota_q\mathfrak{a}_q + \alpha_{q+1} = 0$

$$\mathrm{E} = \frac{\omega m}{2}\iota^{\frac{\pi}{2}}\Sigma q\iota_q\mathfrak{a}_q.$$

And in either of these formulæ the trigonometrical expressions in § 9 for the vectors can be substituted.

In the first, however, it must be noted that both α_{q-1} and α_{q+1} are to be taken of order q (odd). Thus the fundamental harmonic of E is

$$= \frac{\omega m\eta}{\rho}\left\{\sin\omega t + \frac{1}{s_1\sigma_2}\sin(\omega t + \mathfrak{b}_1 + \beta_2)\right\}$$

from the first expression, or

$$= \frac{\omega m \eta}{\rho} \frac{D_1}{s_1} \sin(\omega t + b_1 - f_1)$$

from the second, as $t_1 = D_1 \iota^{-f_1}$.

Similarly the total alternating E.M.F. (H say) generated in the field circuit is given by either

$$H = -\frac{\omega m}{2} \iota^{\frac{\pi}{2}} \Sigma p (a_{p-1} + a_{p+1})_p$$

or

$$H = \frac{\omega m}{2} \iota^{\frac{\pi}{2}} \Sigma p \tau_p \alpha_p,$$

so that the fundamental harmonic of H is equal to either

$$\frac{2\omega m \eta}{\rho} \left[\frac{1}{s_1} \sin(2\omega t + b_1 + \pi) + \frac{1}{s_1 \sigma_2 s_2} \sin(2\omega t + b_1 + \beta_2 + b_2 + \pi) \right]$$

or

$$\frac{2\omega m \eta}{\rho} \frac{\Delta_2}{s_1 \sigma_2} \sin(2\omega t + b_1 + \beta_2 - \phi_2 + \pi).$$

11. The mean value of the product

$$\sin(a\omega t + \theta) \sin(b\omega t + \phi)$$

being zero when a and b are unequal and $\frac{1}{2}\cos(\theta - \phi)$ when a and b are equal, we find that the mean value of x^2 where $x = \Sigma a_q$ is

$$= \tfrac{1}{2} \Sigma a_q^2;$$

and the mean value of ξ^2, where $\xi = \frac{\alpha}{2} + \Sigma \alpha_p$, is

$$= \frac{\alpha_0^2}{4} + \tfrac{1}{2} \Sigma \alpha_p^2.$$

Again, for the same reason, if α and β be any two vectors *representing harmonics of the same order* and if $S\alpha\beta$ be the product of the lengths of α and β into the sine of the angle from α to β *measured in the positive direction*, then the mean value of the product

$$\iota^{\frac{\pi}{2}} \alpha \quad \text{into} \quad \beta$$

is

$$= \tfrac{1}{2} S\alpha\beta = -\tfrac{1}{2} S\beta\alpha.$$

Applying these principles to the determination of the

mean value $\overline{Ex}$ of the product of E and x, that is of the electrical power developed in the armature circuit, we find from the first expression for E in § 10 that

$$\overline{Ex} = -\frac{\omega m}{4}\Sigma qS(\alpha_{q-1}+\alpha_{q+1})\mathfrak{a}_q$$

$$= -\frac{\omega m}{4}\{S\alpha_0\mathfrak{a}_1 - S\mathfrak{a}_1\alpha_2 + 3S\alpha_2\mathfrak{a}_3 - 3S\mathfrak{a}_3\alpha_4 + 5S\alpha_4\mathfrak{a}_5 - 5S\mathfrak{a}_5\alpha_6 + \&c.\},$$

and from the second expression for E that

$$\overline{Ex} = \frac{\omega m}{4}\Sigma qS(i_q\mathfrak{a}_q \cdot \mathfrak{a}_q \cdot) = \frac{\omega m}{4}\Sigma qD_qS(\iota^{-f_q}\mathfrak{a}_q \cdot \mathfrak{a}_q \cdot)$$

$$= \frac{\omega m}{4}\Sigma\bar{\mathfrak{a}}_q^2 qD_q \sin f_q' = \tfrac{1}{2}r\Sigma\bar{\mathfrak{a}}_q$$

as $D_q \sin f_q' = \dfrac{2r}{q\omega m}$ (see § 5)

= the heat developed in the armature circuit.

Similarly the total electrical power $\overline{(H+\eta)\xi}$ developed in the field circuit is given by

$$\overline{(H+\eta)\xi} = \frac{\eta\alpha_0}{2} - \frac{\omega m}{4}\Sigma pS(\mathfrak{a}_{p-1}+\mathfrak{a}_{p+1})\alpha_p$$

$$= \rho\frac{\bar{\alpha}_0^2}{4} - \frac{\omega m}{4}\{2S\mathfrak{a}_1\alpha_2 - 2S\alpha_2\mathfrak{a}_3 + 4S\mathfrak{a}_3\alpha_4 - 4S\alpha_4\mathfrak{a}_5 + \&c.\}$$

or by

$$\overline{(H+\eta)\xi} = \frac{\eta\alpha_0}{2} + \frac{\omega m}{4}\Sigma pS(\tau_p\alpha_p \cdot \alpha_p \cdot)$$

which reduces to

$$\overline{(H+\eta)\xi} = \rho\frac{\bar{\alpha}_0^2}{4} + \frac{\rho}{2}(\bar{\alpha}_2^2 + \bar{\alpha}_4^2 + \bar{\alpha}_6^2 + \&c.)$$

= total heat developed in the field circuit

and is made up of two parts, the first $= \rho\dfrac{\alpha_0^2}{4}$ due to the direct exciting current and the second $= \dfrac{\rho}{2}(\bar{\alpha}_2^2 + \bar{\alpha}_3^2 + \bar{\alpha}_6^2 + \&c.)$ due to the induced alternating field-current.

Adding the first expressions for $\overline{Ex}$ and $\overline{(H+\eta)\xi}$ and cancelling $\eta\bar{\xi}$ against $\rho\dfrac{\alpha_0^2}{4}$ we find that

$$\overline{Ex} + \overline{H\xi} = -\frac{\omega m}{4}\{S\alpha_0\mathfrak{a}_1 + S\mathfrak{a}_1\alpha_2 + S\alpha_2\mathfrak{a}_3 + S\mathfrak{a}_3\alpha_4 + \&c.\}.$$

12. The torque exerted at any instant in driving the alternator is

$$= -x\xi \frac{d}{d(\omega t)} \{m \cos \omega t\} = mx\xi \sin \omega t$$

$$= \frac{m}{2} \Sigma \mathfrak{a}_q \times \iota^{-\frac{\pi}{2}} \Sigma (a_{q-1} - a_{q+1})_q \qquad \text{(see § 4)}$$

$$= -\frac{m}{2} \Sigma \mathfrak{a}_q \times \iota^{\frac{\pi}{2}} \Sigma (a_{q-1} - a_{q+1})_q.$$

Taking the mean value of this product we find that the mean driving torque T is given by

$$T = -\frac{m}{4} \{S\alpha_0 \mathfrak{a}_1 + S\mathfrak{a}_1 \alpha_2 + S\alpha_0 \mathfrak{a}_2 + S\mathfrak{a}_2 \alpha_4 + \&c.\},$$

which result, combined with the last one obtained in § 11, gives the power equation

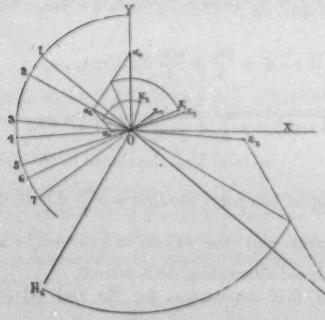
$$\omega T = \overline{E\dot{x}} + \overline{H\dot{\xi}}$$

as it ought.

13. The solution obtained can be represented geometrically in an interesting way as follows:—

Take two lines OX, OY, fig. 2, at right angles. Measure

Fig. 2.



off from OY in the positive direction the angles $YO1 = \mathfrak{b}_1$, $1O2 = \beta_2$, $2O3 = \mathfrak{b}_3$, $3O4 = \beta_4$, &c., where $\mathfrak{b}_1$, β_2, $\mathfrak{b}_3$, β_4, &c., are the angles determined in § 8.

In OY take $O\alpha_0 = 2\eta/\rho = 2\times$ exciting current. Produce 10 through O to $\mathfrak{a}_1$ so that $O\mathfrak{a}_1 = \frac{O\alpha_0}{s_1}$. In O2 take $O\alpha_2 = O\mathfrak{a}_1/\sigma_2$. Produce 3O through O to $\mathfrak{a}_3$ so that $O\mathfrak{a}_3 = \frac{O\alpha_2}{s_3}$, and so on where s_1, σ_2, s_3, σ_4, &c., are the quantities determined in § 8.

Then the vectors to $\mathfrak{a}_1$, $\mathfrak{a}_3$, $\mathfrak{a}_5$, &c., represent completely in amplitude and phase the different harmonics of the armature current, the subscribed numbers indicating the orders of the harmonics; and those to α_2, α_4, α_6, &c., represent completely in the same way the different harmonics of the induced alternating field current.

Again (see § 10), if we rotate the vector drawn to the middle point of $\alpha_0\alpha_2$ backwards through a right angle, we obtain the vector OE_1 that represents $\frac{1}{m\omega}$ into the first harmonic of the total E.M.F. E generated in the armature; and if we rotate backwards through $\pi/2$ the vector to the middle point of $\alpha_2\alpha_4$ we obtain the vector OE_3 that represents $\frac{1}{3\omega m}$ into the third harmonic of E; and similarly for the other harmonics of E.

In the same way, by rotating backwards through $\pi/2$ the vector to the middle point of $\mathfrak{a}_1\mathfrak{a}_3$, we obtain the vector OH_2 that represents $\frac{1}{2\omega m}$ into the fundamental harmonic of the E.M.F. H induced in the field circuit; and so on for the other harmonics of H.

Again, as $S\alpha\beta$ is = twice the area of the triangle whose sides are α and β and is positive if β follows α in rotation order in the diagram, the mean torque exerted on the generator by the driver (see § 12) is equal to $\frac{1}{2}m$ into the sum of the areas of the triangles $\alpha_0 O\mathfrak{a}_1$, $\mathfrak{a}_1 O\alpha_2$, $\alpha_2 O\mathfrak{a}_3$, $\mathfrak{a}_3 O\alpha_4$, &c., these triangles, in the case of any generator, being all taken as positive.

14. When, for any generator, the t, τ operators have been calculated for a particular load (see § 5), a geometrical solution can easily be obtained to a high degree of accuracy by aid of a ruler, scale, slide-rule, and protractor.

Thus if we neglect the harmonics α_4, α_9, α_{10}, &c., then, drawing any vector from the origin to represent α_7 we can construct for α_5 as $\alpha_5 = -t_7\alpha_7$ (see § 6). From α_6 we can construct for $\tau_6\alpha_6$ and as $\alpha_5 + \tau_6\alpha_3 + \alpha_7 = 0$ the triangle of vectors gives us α_3. Proceeding in this way, we obtain in succession α_4, α_9, α_2, α_1, α_0, which represent the harmonics of x and ξ correctly as regards relative phase and relative amplitude. But as α_0 must be equal to twice the exciting current we have a scale for our diagram, and hence obtain a complete solution.

The fact that for a practical alternator the τ operators may be taken as pure numbers (see §§ 8, 22), renders this method of solution both easy and expeditious.

15. If a source of constant E.M.F. be included in the armature circuit as well as in the field circuit of the simple alternator indicated in fig. 1, equations I. § 1 become

$$rx + \frac{d}{dt}(lx + m\xi\cos\omega t) = e$$

$$\rho\xi + \frac{d}{dt}(\lambda\xi + mx\cos\omega t) = \eta,$$

and both the armature and field currents will now contain harmonics of all orders, odd and even. In this case we assume that

$$x = \frac{\alpha_0}{2} + \alpha_1 + \alpha_2 + \alpha_3 + \&c.$$

$$\xi = \frac{\alpha_0}{2} + \alpha_1 + \alpha_2 + \alpha_3 + \&c.,$$

where α_0 is the vector to the point whose polar coordinates are $2e/r$, $\pi/2$, and α_0 is, as before, the vector to the point $2\eta/\rho$, $\pi/2$. The other vectors α_1, α_2, α_3, &c., α_1, α_2, α_3, &c., have to be determined.

On substituting for x and ξ in the above equations it will be found that the odd order vectors in x and the even order ones in ξ are determined by the same equations (IV. § 6) as when $e=0$, and are completely independent of the even order vectors in x and the odd order ones in ξ, these latter depending only on e and vanishing with e.

This being so, α_1, α_2, α_3, α_4, &c., are given by the solution already obtained, and α_1, α_2, α_3, α_4, &c., will be given by a

similar solution the equations expressing which may be written down from symmetry.

Thus the complete solution is given by

$$\alpha_0 = -S_1\mathfrak{a}_1 = S_1\Sigma_2\alpha_2 = -S_1\Sigma_2S_3\mathfrak{a}_3 = \&c.,$$
$$\mathfrak{a}_0 = -\Sigma_1\alpha_1 = \Sigma_1S_2\mathfrak{a}_2 = -\Sigma_1S_2\Sigma_3\alpha_3 = \&c.,$$

where α_0 is the vector to $2\eta/\rho$, $\pi/2$, as before,

and $\mathfrak{a}_0$ is the vector to $2e/r$, $\pi/2$.

Note.—In the former case ($e=0$) the t operators were all of odd and the τ ones of even orders. In this case the operators of either class are of both orders.

The translation from the above vector solution to the ordinary sine form follows as in § 10.

16. In the preceding solutions the magnetic fluxes have been assumed to be in phase with the magnetizing current-turns, and so iron loss due to hysteresis and eddy currents has been neglected. To take account of the latter the interpretation of the well-known relation $B=\mu H$ connecting steady magnetizing force and induction produced has to be modified.

The induction produced by $H=H_1\sin(\omega t+c_1)$ is known to be of the form

$$B = m_1H_1\sin(\omega t+c_1-\delta_1)+\text{higher harmonics},$$

and attending only to the fundamental harmonic in B, if H be represented as explained in § 2 by the vector h_1, and B by the vector b_1, then the above trigonometrical relation may be written

$$b_1 = \mu_1h_1$$

where μ_1 is the operator $m_1\iota^{-\delta_1}$.

In a former paper * by me was shown how these *permeability operators*, as they may be called, can be determined. They depend on the character of the iron and the thickness of the laminæ, on the amplitude and period of the fundamental harmonic of the induction oscillation they refer to, and to some extent on the wave form of the latter.

For the purposes of the following discussion we will

* "Variation of Magnetic Hysteresis with Frequency," Phil. Mag. Jan. 1905.

assume that when the magnetizing force

$$H = h_1 + h_3 + h_5 + \&c.$$

produces the induction

$$B = b_1 + b_3 + b_5 + \&c.,$$

then $b_1 = \mu_1 h_1$, $b_3 = \mu_3 h_3$, $b_5 = \mu_5 h_5$, &c., where the μ's are operators of the type given by

$$\mu_q = m_q \epsilon^{-\delta_q}.$$

[This assumption as regards all the harmonics of B but the fundamental is not strictly in accordance with what is known concerning the behaviour of laminated iron under periodic magnetizing forces, for b_3, b_5, &c., depend, at any rate for high values of b_1, more on b_1 than on h_3, h_5, &c. At the same time it is hoped that the following discussion may be of some value.]

In general if $H = \Sigma h_q$ produce $B = \Sigma b_q$, as the total iron loss per c.c. per cycle due to both hysteresis and eddy currents is

$$\frac{1}{4\pi}\int H dB = \frac{1}{4\pi}\int_0^T H \frac{dB}{dt} dt,$$

where T is the period, the total iron loss per c.c. per second is

$$= \frac{1}{4\pi} \cdot \text{ Average value of product } H\frac{dB}{dt}$$

$$= \frac{1}{4\pi} \cdot \text{ Av. product } \Sigma h_q \text{ into } \epsilon^{\frac{\pi}{2}} \Sigma q\omega b_q$$

$$= \frac{\omega}{8\pi}\{S b_1 h_1 + 3S b_3 h_3 + 5S b_5 h_5 + \&c.\}$$

(See § 11)

$$= \frac{\omega}{8\pi} \Sigma q b_q h_q \sin \delta_q.$$

Again, it is well known that if the steady magnetizing current-turns nx act on a magnetic circuit composed of different materials, the flux F produced is given by

$$F = \frac{4\pi n x}{\Sigma \frac{L}{A\mu}},$$

where the L, A's are the lengths and sectional areas of the

different portions of the circuit, and the μ's are the permeabilities of these portions for the particular flux densities in them.

If now the magnetizing current be an alternating one, that is if $x = x_1 \sin(\omega t + c_1) = \mathbf{a}_1$ (a vector) the same equation will give the corresponding harmonic of the flux produced, but the μ's are now the permeability operators, for the different portions of the circuit.

$\Sigma \frac{L}{A\mu}$ can, by the addition theorem in § 3, be reduced to a angle operator so that if the flux f_1 (vector) be produced in any magnetic circuit by the current-turns $n\mathbf{a}_1$ we have always a relation of the form

$$f_1 = G_1 n \mathbf{a}_1$$

where G_1 is an operator of the form $g_1 \epsilon^{-\delta}$ which can be determined.

Hence, following the assumption already made, if the magnetizing current-turns

$$nx = n(\mathbf{a}_1 + \mathbf{a}_3 + \mathbf{a}_5 + \&c.)$$

produce in a magnetic circuit the flux

$$F = f_1 + f_3 + f_5 + \&c.$$

then $$f_1 = nG_1\mathbf{a}_1,\ f_3 = nG_3\mathbf{a}_3,\ \&c.$$

where the G's are operators of the type given by $G_q = g_q \epsilon^{-\delta_q}$.

In the above the back E.M.F., e say, in the magnetizing coils due to variation of flux is

$$e = n\frac{dF}{dt} = n\omega\imath \overset{\pi}{2} \Sigma q f_q$$

and the power absorbed, that is the total iron loss per sec. in the magnetic circuit, is the mean value of ex, that is of the product

$$\Sigma \mathbf{a}_q \text{ into } n\omega\imath \overset{\pi}{2} \Sigma q f_q$$

which (see § 11) $$= \tfrac{1}{2} n\omega \Sigma (q S f_q . \mathbf{a}_q .)$$

$$= \tfrac{1}{2}\omega\Sigma\left(q\frac{f_q^2}{g_q}\sin\delta_q\right).$$

17. As an example let us determine the G operator for a magnetic circuit of uniform cross section = 100 cm.^{2}, made

up of 40 cm. length of laminated iron and two air-gaps each 1 mm. when B maximum = 5000 and the frequency 30.

In the paper already quoted we find for a sample of No. 26 iron well insulated between the laminæ when B_{max} = 5000 and frequency = 30 q.p. that

$$\mu = 2500\, \epsilon^{-50^\circ} \quad \text{(nearly)}.$$

Now
$$G = \frac{4\pi}{\Sigma \frac{l}{A\mu}}$$

and
$$\Sigma \frac{L}{A\mu} = \frac{1}{100}\left\{\frac{40}{2500}\epsilon^{50^\circ} + 2\right\}$$
$$= \frac{1}{10^3}\left\{16\epsilon^{50^\circ} + 200\right\}$$
$$= \frac{2106}{10^4}\epsilon^{3^\circ 18'}.$$

Hence
$$G = 5968\, \epsilon^{-3^\circ 18'}.$$

18. Returning to the alternator, if n be the number of armature turns, ν the number of field turns, and

$$x = \mathfrak{a}_1 + \mathfrak{a}_3 + \mathfrak{a}_5 + \&c., \quad \xi = \frac{\alpha_0}{2} + \alpha_2 + \alpha_4 + \&c.,$$

the armature and field currents respectively, the magnetizing current-turns M_x producing flux across the air-gap and through the armature in a direction axial to its windings are given by

$$M_x = nx + \nu\xi \cos \omega t,$$

and the current-turns M_y producing flux across the air-gap and through the armature in a direction parallel to the planes of the windings, and *behind* that of M_x by 90°, are given by

$$M_y = \nu\xi \sin \omega t,$$

or in vector notation (see § 4)

$$M_x = \Sigma\left[n\mathfrak{a}_q + \frac{\nu}{2}(\alpha_{q-1} + \alpha_{q+1})_q\right]$$

$$M_y = \frac{\nu}{2}\epsilon^{-\frac{\pi}{2}}\Sigma(\alpha_{q-1} - \alpha_{q+1})_q,$$

which produce the armature fluxes A_x and A_y given by

$$A_x = \Sigma G_q \left[n \mathfrak{a}_q + \frac{\nu}{2} (\alpha_{q-1} + \alpha_{q+1})_q \right]$$

$$A_y = \frac{\nu}{2} \iota^{-\frac{\pi}{2}} \Sigma G_q (\alpha_{q-1} - \alpha_{q+1})_q,$$

where $G_q = \sigma_q \iota^{-\delta_q}$ and q any odd number.

[Note that the directions of A_x and A_y are fixed in the armature.]

Now if magnetic leakage be otherwise taken account of, the flux in the stator must be continuous with that in the rotor so that the flux F looped on the field-windings at any instant is given by

$$F = A_x \cos \omega t + A_y \sin \omega t,$$

which by means of the relations in § 4 can be reduced to

$$F = \Sigma \left[\nu G_p \alpha_p + \frac{n}{2} (G_{p-1} \mathfrak{a}_{p-1} + G_{p+1} \mathfrak{a}_{p+1})_p \right]$$

where p is even and $2G_p = G_{p-1} + G_{p+1}$.

19. If l' be the self-inductance in the armature circuit either external to the armature or due to magnetic leakage in it, and if λ' be a similar quantity for the field circuit, the equations for the two circuits are

$$rx + l' \frac{dx}{dt} + n \frac{d}{dt} A_x = 0$$

$$\rho \xi + \lambda' \frac{d\xi}{dt} + \nu \frac{d}{dt} F = \eta$$

where r, ρ, η, x, ξ, have the same significations as in § 1.

Substituting in these equations from § 18, and then equating separately to zero each set of vector terms of the same order, we obtain the two series of vector equations,

$$r \mathfrak{a}_q + q \omega l' \iota^{\frac{\pi}{2}} \mathfrak{a}_q + nq\omega \iota^{\frac{\pi}{2}} G_q \left\{ n \mathfrak{a}_q + \frac{\nu}{2} (\alpha_{q-1} + \alpha_{q+1}) \right\} = 0$$

$$\rho \alpha_p + p\omega \lambda' \iota^{\frac{\pi}{2}} \alpha_p + \nu p \omega \iota^{\frac{\pi}{2}} \left\{ \nu G_p \alpha_p + \frac{n}{2} (G_{p-1} \mathfrak{a}_{p-1} + G_{p+1} \mathfrak{a}_{p+1}) \right\} = 0$$

with $\alpha_0 = \frac{2\eta}{\rho}$, where q is odd and p even.

These reduce at once to the two series,

$$\alpha_{q-1} + t_q G_q \mathfrak{a}_q + \alpha_{q+1} = 0$$

$$G_{p-1}\mathfrak{a}_{p-1} + \tau_p \alpha_p + G_{p+1}\mathfrak{a}_{p+1} = 0$$

or, after putting $\mathfrak{a}'_q$ for $G_q \mathfrak{a}_q$, to

$$\alpha_{q-1} + t_q \mathfrak{a}'_q + \alpha_{q+1} = 0$$

$$\mathfrak{a}'_{p-1} + \tau_p \alpha_p + \mathfrak{a}'_{p+1} = 0$$

equations of exactly the same form as those for the simple case but in which the t and τ operators are now given by

$$t_q = \frac{2}{n\nu G_q^2}\left\{ n^2 G_q + l' - \frac{\tau}{q\omega} \overset{\pi}{\iota}^2 \right\}$$

$$\tau_p = \frac{2}{n\nu}\left\{ \nu^2 G_p + \lambda' - \frac{\rho}{p\omega} \overset{\pi}{\iota}^2 \right\}.$$

These operators having been calculated from known data, the solution for $\mathfrak{a}'_1$, $\mathfrak{a}'_2$, $\mathfrak{a}'_3$, &c., α_2, α_4, &c., proceeds exactly as in the simple case, and as $\mathfrak{a}'_1 = G_1\mathfrak{a}_1$, $\mathfrak{a}'_2 = G_2\mathfrak{a}_2$, &c., $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{a}_3$, &c., can then be obtained.

20. In § 16 it was shown that the iron loss (*i. e.* energy dissipated per sec. in the iron) in a magnetic circuit is

$$\tfrac{1}{2}\omega \Sigma q \frac{f_q^2}{g_q} \sin \delta_q ;$$

hence the loss due to the flux A_x § 18 is

$$\tfrac{1}{2}\omega \Sigma q g_q \sin \delta_q \overline{n\mathfrak{a}_q + \tfrac{\nu}{2}(\alpha_{q-1} + \alpha_{q+1})}^2$$

and that due to the flux A_y is

$$\tfrac{1}{2}\omega \Sigma q g_q \sin \delta_q \frac{\nu^2}{4}\overline{(\alpha_{q-1} - \alpha_{q+1})}^2$$

Adding these we find that the total iron loss in the generator is

$$\tfrac{1}{2}\omega \Sigma q g_q \sin \delta_q \left\{ \overline{n\mathfrak{a}_q + \tfrac{\nu}{2}(\alpha_{q-1} + \alpha_{q+1})}^2 + \frac{\nu^2}{4}\overline{(\alpha_{q-1} - \alpha_{q+1})}^2 \right\}.$$

Expanding and remembering that

$$\overline{\alpha + \beta}^2 = \overline{\alpha}^2 + \overline{\beta}^2 + 2\overline{\alpha}\overline{\beta}\cos \overset{\wedge}{\alpha\beta},$$

and that

$$\alpha_{q-1} + \alpha_{q+1} = -t_q G_q \mathfrak{a}_q \qquad \text{(see § 19)}$$

we find that the total iron loss is equal to

$$\Sigma\left[\tfrac{1}{2}q\omega g_q \sin\delta_q\left\{\frac{\nu^2}{2}(\overline{\alpha}^2_{q-1}+\overline{\alpha}^2_{q+1})-n^2\overline{\mathfrak{a}}^2_q\right\}-\left\{r\sin^2\delta_q+q\omega l'\sin\delta_q\cos\delta_q\right\}\overline{\mathfrak{a}}^2_q\right]$$

where q is odd.

21. An approximate determination of the effect of iron loss on the performance of an alternator can be obtained by taking all the G operators for its magnetic circuit as equal to G_1, that is, equal to the one for the fundamental harmonic of the armature flux.

Making this simplification in the equations of § 19, we find that $\mathfrak{a}_1$, $\mathfrak{a}_3$, $\mathfrak{a}_5$, &c., α_0, α_2, &c., are connected by the two series of equations

$$\alpha_{q-1}+t_q\mathfrak{a}_q+\alpha_{q+1}=0$$

$$\mathfrak{a}_{p-1}+\tau_p\alpha_p+\mathfrak{a}_{p+1}=0$$

with $\alpha_0=2\eta/\rho$, in which

$$t_q=\frac{2}{n\nu\mathrm{G}}\left\{n^2\mathrm{G}+l'-\frac{r}{q\omega}\overset{\pi}{\iota^2}\right\}=\frac{2}{n\nu}\left\{n^2+\frac{l'}{g}\iota^\delta-\frac{r}{q\omega g}\overset{\pi}{\iota^{2+\delta}}\right\}$$

$$\tau_p=\frac{2}{n\nu\mathrm{G}}\left\{\nu^2\mathrm{G}+\lambda'-\frac{\rho}{p\omega}\overset{\pi}{\iota^2}\right\}=\frac{2}{n\nu}\left\{\nu^2+\frac{\lambda'}{g}\iota^\delta-\frac{\rho}{p\omega g}\overset{\pi}{\iota^{2+\delta}}\right\}$$

as $\mathrm{G}=g\iota^{-\delta}$.

Putting l for gn^2, λ for $g\nu^2$, m for $gn\nu$, and—remembering that δ is a small angle (see § 17)—unity for $\cos\delta$, we find that

$$t_q=\mathrm{D}_q\iota^{-f_q},\quad \tau_p=\Delta_p\iota^{-\phi_p},$$

where

$$\mathrm{D}_q^2=\frac{4}{m^2}\left\{(l+l')^2+\frac{r^2}{q^2\omega^2}+2\frac{lr}{q\omega}\sin\delta\right\}$$

$$\mathrm{D}_q\sin f_q=\frac{2r}{q\omega m}-\frac{2l'}{m}\sin\delta$$

$$\Delta_p^2=\frac{4}{m^2}\left\{(\lambda+\lambda')^2+\frac{\rho^2}{p^2\omega^2}+2\frac{\lambda\rho}{p\omega}\sin\delta\right\}$$

$$\Delta_p\sin\phi_p=\frac{2\rho}{p\omega m}-\frac{2\lambda'}{m}\sin\delta.$$

The t and τ operators having been calculated from these formulæ, the rest of the solution for this case follows in every particular the course for the simple case fully explained in §§ 8, 9.

22. In order to illustrate the practical application of the foregoing theory, I will determine the performance of a small two-pole alternator when carrying a rather heavy non-inductive load.

The details of the alternator are as follows :—

Armature { diameter 12 cm. / length 8 cm. / turns $n=100$. / resistance ·25 ohm.

Air-gap $=1$ mm.

Field { turns $\nu=400$. / resistance $\rho=3$ ohms. / exciter, three storage-cells ; $r=6{\cdot}6$ volts.

Frequency $100/\pi$, *i. e.* $\omega=200$.

Magnetic leakage $=5$ per cent.

Flux operator $G=5000\,\epsilon^{-\beta^2}$.

Let the external resistance in the armature circuit in the case in hand be 4·75 ohms so that $r=5$ ohms.

Hence (see § 21)

$$l=5\,.\,10^7, \qquad l'=\cdot05\,l=\cdot25\,.\,10^7$$
$$\lambda=8\,.\,10^8, \qquad \lambda'=\cdot05\lambda=4\,.\,10^7$$
$$m=2\,.\,10^8, \qquad r=5\,.\,10^9, \qquad \rho=3\,.\,10^9$$
$$\omega=200, \qquad \delta=3^\circ \quad \text{and}$$
$$\alpha_0=2\eta/\rho=\cdot44 \text{ (absolute)}.$$

Using these values for the constants and the formulæ in § 21 we obtain the t and τ operators which are given in the following table :—

$t_q=D_q\epsilon^{-f_q}$			$\tau_p=\Delta_p\epsilon^{-\phi_p}$		
q.	D_q.	f_q.	p.	Δ_p.	ϕ_p.
1	·592	24° 51′	2	8·4	0° 22′
3	·535	8 48	4	8·4	8′
5	·530	5 18	6	8·4	1′
7	·528	3 45	8	8·4	−1′
9	·527	2 54	10	8·4	−2′
11	·526	2 21	12	8·4	−3′
13	·526	1 57	14	8·4	−4′
15	·526	1 40	16	8·4	−5′
17	·526	1 28	18	8·4	−5′

[Note that the τ operators are all practically equal and simple numerical multipliers (see § 8).]

And from these, by the method explained in § 8 we obtain the S, Σ, continued fraction operators which are given in the following table:—

$S_q = s_{q\iota} - b_q$.			$\Sigma_p = \sigma_{p\iota} - \beta_p$.		
q.	s_q.	b_q.	p.	σ_p.	β_p.
1	·457	36° 32′	2	5·82	7° 46′
3	·366	15 32	4	5·63	5 8
5	·353	9 50	6	5·57	3 40
7	·350	7 9	8	5·55	2 54
9	·348	5 34	10	5·53	2 18
11	·346	4 30	12	5·51	1 51
13	·346	3 42	14	5·50	1 21

And from these as explained in § 9 we obtain the different harmonics of both the armature current and the induced field current. These also are given in tabular form:—

$x = \Sigma x_q \sin(q\omega t - \frac{\pi}{2} + c_q)$.			$\xi = 22 + \Sigma \xi_p \sin(p\omega t + \frac{\pi}{2} + \gamma_p)$.		
q.	x_q.	c_q.	p.	ξ_p.	γ_p.
1	·963	36° 32′	2	·166	44° 18′
3	·453	59 50	4	·080	64 58
5	·228	74 48	6	·041	78 28
7	·117	85 37	8	·021	88 31
9	·061	94 5	10	·011	96 23
11	·032	100 53	12	·006	102 44
13	·017	106 26	14	·003	107 47

The virtual armature current in amperes being equal to $10\sqrt{\frac{1}{2}\Sigma x_q^2}$ is 7·75 amps., and the terminal virtual voltage being 4·75 times this is 36·8 volts. The no-load voltage for the same exciting current is 62·2.

The virtual value of the alternating current induced in the field circuit in amperes being equal to $10\sqrt{\frac{1}{2}\Sigma \xi_p^2}$ is 1·34.

The copper losses are:—

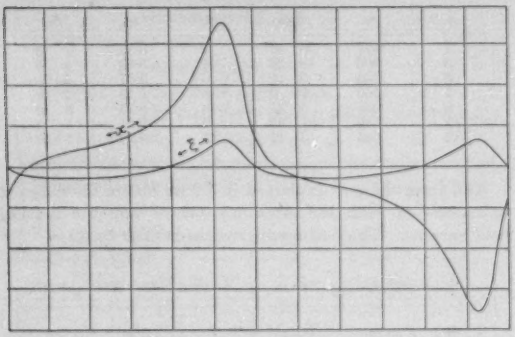
In the armature	15 watts.
In the field due to exciting current	12 „
In the field due to induced current	5·4 „

The total iron loss calculated by means of the formula in § 20 is 20 watts.

The total losses are therefore 52·4 watts, and as the output is 285 watts the efficiency is 84 per cent.

In fig. 3 are plotted the wave forms of both the armature

Fig. 3.—Theoretical armature current (x), and induced field-current (ξ), in a fully loaded alternator.



and induced field-currents determined above, correctly as regards both their relative amplitudes and phases.

23. In designing the field of an alternator attention should be given to the fact that the conductor has to carry not only the exciting current, but also the induced field-current, which, as we have seen, may at full load attain a relatively large value. In addition it should not be forgotten that in the field-magnet cores there is the associated alternating flux which causes some additional heat.

It is well known that in a case of excessive heating in the field, reduction of the heating is effected by the employment of heavy closed copper conductors, called dampers, embracing the field-magnet poles.

To explain this action, let us consider a two-pole machine on each field pole of which is a damper.

Neglecting magnetic leakage and iron loss, if ζ be the current in each damper, the magnetic flux through the armature windings is

$$g\{nx+(\nu\xi+2\zeta)\cos\omega t\},$$

and that through the field windings and the dampers is

$$g\{\nu\xi+2\zeta+nx\cos\omega t\},$$

so that the equations connecting x, ξ, and ζ are

$$\left.\begin{aligned} rx+gn\frac{d}{dt}\{nx+(\nu\xi+2\zeta)\cos\omega t\}&=0,\\ \rho\xi+g\nu\frac{d}{dt}\{\nu\xi+2\zeta+nx\cos\omega t\}&=\eta,\\ z\zeta+g\frac{d}{dt}\{\nu\xi+2\zeta+nx\cos\omega t\}&=0, \end{aligned}\right\}\quad\text{(VII.)}$$

where z is the resistance of each damper and the other symbols have the same significations as in the previous sections of this paper.

Obviously there is no constant term in ζ, and considering only the variable terms (harmonics) in ξ, we see at once that

$$\nu z\zeta=\rho\xi,$$

from which it follows that

$$\nu z+2\zeta=\nu\xi(1+\kappa)=\nu\xi' \text{ say},$$

where

$$\kappa=\frac{2\rho}{\nu^2 z},$$

and that

$$2z\zeta^2=\kappa\rho\xi^2.$$

The first two of equations VII. may now be written

$$rx+gn\frac{d}{dt}\{nx+\nu\xi'\cos\omega t\}=0,$$

$$\frac{\rho}{1+\kappa}\xi'+g\nu\frac{d}{dt}\{\nu\xi'+nx\cos\omega t\}=\frac{\eta}{1+\kappa},$$

η being divided by $1+\kappa$ as the constant term in ξ' is equal to the constant term in ξ (ζ having no constant term).

Now x and ξ' determined from these equations will be

very approximately the same as x and ξ determined from the equations in § 1 for the alternator without dampers; for α_0 the given vector is the same for both, as are also all the t operators.

The τ operators differ in that for ρ, $\rho/(1+\kappa)$ is substituted, but in § 8 and in note § 22 it is shown that the τ operators for generators as ordinarily constructed are practically independent of the value of ρ the field resistance.

Hence we see that ξ' is the alternating field current if the dampers are absent, ξ its value when the dampers are attached and these currents are connected by the relation

$$\xi' = (1+\kappa)\xi.$$

In addition as $2z\zeta^2 = \kappa\rho\xi^2$,

$$\rho\xi'^2 = \frac{2}{\kappa}z\zeta'^2 = \frac{1}{(1+\kappa)^2}\rho\xi'^2.$$

Hence if H′ be the copper loss in the field coils due to induced alternating current when the generator is without dampers, H the same when dampers are attached, and h the loss in the dampers when attached,

$$H = \frac{H'}{(1+\kappa)^2}, \qquad h = \frac{\kappa H'}{(1+\kappa)^2}, \qquad H+h = \frac{H'}{1+\kappa}.$$

If we assume that the mean length of a field turn is equal to the length of a damper turn, it is easy to show that κ is the ratio of the volume of copper in the dampers to the volume of copper in the field windings when there is no resistance external to the windings in the field circuit. If there is resistance external to the windings in the field circuit, κ is greater than the above volume ratio.

The magnetic flux in the field coils being equal to

$$g\{\nu\xi' + nx\cos\omega t\}$$

is practically unaffected by the presence of dampers, so that the iron loss in the field-magnets remains the same.

24. If a source of alternating E.M.F. $= E$ where

$$E = E_1\sin(\omega t + h_1) + E_2\sin(3\omega t + h_3) + \&c.,$$
$$= e_1 + e_3 + e_5 + \&c. \text{ (vectors)}$$

be included in the armature circuit, and if the armature

rotate in synchronism with this E.M.F., we have the case of the synchronous A.C. motor.

In this case the armature and field currents x and ξ are connected by the equations (see § 1)

$$rx + \frac{d}{dt}(lx + m\xi\cos\omega t) = \Sigma E_q \sin(q\omega t + h_q)$$

$$\rho\xi + \frac{d}{dt}(\lambda\xi + mx\cos\omega t) = \eta.$$

Assuming as in § 2 that

$$x = \mathfrak{a}_1 + \mathfrak{a}_3 + \mathfrak{a}_5 + \&c.,$$

$$\xi = \frac{\alpha_0}{2} + \alpha_2 + \alpha_4 + \&c.,$$

and proceeding exactly as in § 5, we obtain the infinite series of equations

$$\begin{aligned}
t_1\mathfrak{a}_1 + \alpha_2 &= -\alpha_0 - \kappa_1 \\
\mathfrak{a}_1 + \tau_2\alpha_2 + \mathfrak{a}_3 &= 0 \\
\alpha_2 + t_3\mathfrak{a}_3 + \alpha_4 &= -\kappa_3 \\
\mathfrak{a}_3 + \tau_4\alpha_4 + \mathfrak{a}_5 &= 0 \\
\alpha_4 + t_5\mathfrak{a}_5 + \alpha_6 &= -\kappa_5 \\
&\&c., \ \&c.,
\end{aligned}$$

in which

α_0 = the vector to the point $2\eta/\rho,\ \pi/2$, as before

and $$\kappa_q = \frac{2}{qm\omega}\,\iota^{\frac{\pi}{2}} e_q,$$

where Σe_q is the applied E.M.F., and the t and τ operators have the same values as in § 5.

Solving for $\mathfrak{a}_1$ we find that

$$P_1\mathfrak{a}_1 = -\Pi_2(\alpha_0 + \kappa_1)_1 - \Pi_4(\kappa_3)_1 - \Pi_6(\kappa_5)_1 - \&c.,$$

where P_1, Π_2, Π_4, &c., are the infinite determinant operators whose leading terms are t_1, τ_2, τ_4, τ_6, &c., respectively as in § 6.

Reducing to the continued fraction operators of § 7 we obtain

$$\mathfrak{a}_1 = -\frac{1}{S_1}(\alpha_0 + \kappa_1)_1 - \frac{1}{S_1\Sigma_2 S_2}(\kappa_3)_1 - \frac{1}{S_1\Sigma_2 S_2\Sigma_4 S_5}(\kappa_5)_1 - \&c.,$$

and using the equations

$$\alpha_2 = -t_1\mathfrak{a}_1 - \alpha_0 - \kappa_1\,;\ \mathfrak{a}_2 = -\tau_2\alpha_2 - \mathfrak{a}_1\,;\ \&c.,$$

the successive harmonics of the armature and field currents can be obtained.

25. If in the last example the E.M.F. inserted in the armature circuit be sinusoidal and equal to $\mathrm{E}\sin(\omega t + h) = e$ (a vector), the solution will (see equations in last paragraph) obviously be identical with that for the simple generator given in §§ 5 *et seq.*, when in the latter $\alpha_0 + \kappa$ is substituted for α_0 where

$$\kappa = \frac{2}{\omega m}\,\iota^{\frac{\pi}{2}}e,$$

and in this case it is important to know the condition which determines whether the machine will run as a motor and develop mechanical power.

In § 12 the *driving* torque T was shown to be given by

$$\mathrm{T} = -\frac{m}{4}\{\mathrm{S}\alpha_0\mathfrak{a}_1 + \mathrm{S}\mathfrak{a}_1\alpha_2 + \mathrm{S}\alpha_2\mathfrak{a}_3 + \mathrm{S}\mathfrak{a}_3\alpha_4 + \&c.\}$$

and T must be negative for a motor.

Now, as

$$\mathfrak{a}_1 = -\frac{1}{\mathrm{S}_1}(\alpha_0 + \kappa)_1 = -\frac{\iota^{\mathfrak{b}_1}}{s_1}(\alpha_0 + \kappa)_1 \text{ (see § 8)}$$

$$\mathrm{S}\alpha_0\mathfrak{a}_1 = -\frac{1}{s_1}\left\{\mathrm{S}\alpha_0 \,.\, \iota^{\mathfrak{b}_1}\alpha_0 + \frac{2}{\omega m}\mathrm{S}\alpha_0 \,.\, \iota^{\frac{\pi}{2}+\mathfrak{b}_1}e\right\}$$

$$= -\frac{1}{s_1}\left\{\bar{\alpha}_0^{\,2}\sin\mathfrak{b}_1 + \frac{2}{\omega m}\alpha_0 e\sin(\mathfrak{b}_1 + h)\right\}.$$

Again, as

$$\alpha_2 = -\frac{1}{\Sigma_2}\mathfrak{a}_1 = -\frac{\iota^{\beta_2}}{\sigma_2}\mathfrak{a}_1,$$

$$\mathrm{S}\mathfrak{a}_1\alpha_2 = -\frac{\sin\beta_2}{\sigma_2}\bar{\mathfrak{a}}_1^{\,2} = -\frac{\sin\beta_2}{s_1^{\,2}\sigma_2}\overline{(\alpha_0 + \kappa)}^{\,2}\,;$$

similarly

$$\mathrm{S}\alpha_2\mathfrak{a}_3 = -\frac{\sin\mathfrak{b}_2}{s_1^{\,2}\sigma_2^{\,2}s_2}\overline{(\alpha_0 + \kappa)}^{\,2},\ \&c.,\ \&c.\,;$$

but

$$\overline{(\alpha_0+\kappa)}^2 = \bar{\alpha}_0^2+\bar{\kappa}^2+2\bar{\alpha}_0\bar{\kappa}\cos\widehat{\alpha_0\kappa}$$

$$= \bar{\alpha}_0^2+\frac{4}{\omega^2m^2}\bar{e}^2+\frac{4}{\omega m}\bar{\alpha}_0\bar{e}\cos h\ ;$$

and if

$$B = \frac{\sin\beta_2}{s_1^3\sigma_2}+\frac{\sin b_3}{s_1^3\sigma_2^2s_3}+\frac{\sin\beta_4}{s_1^3\sigma_2^2s_3^2\sigma_4}+\&c.,$$

we find that

$$\frac{4T}{m} = \left\{\frac{\sin b_1}{s_1}+B\right\}\bar{\alpha}_0^2+\frac{2}{\omega m}\left\{\frac{\sin(b_1+h)}{s_1}+2B\cos h\right\}\bar{\alpha}_0\bar{e}+\frac{4}{\omega^2m^2}Bc^2,$$

which must be negative if the machine runs as a motor.

Now, s_1, σ_2, s_3, σ_4, &c., $\sin b_1$, $\sin\beta_2$, $\sin b_3$, are all essentially positive and therefore B is so. Also $\bar{\alpha}_0$ and $\bar{e}$ are essentially positive. So, in order that the machine may work as a motor, h, the phase angle of the applied E.M.F. must have such a value as to make the above expression for T negative.

The power supplied by the source e being the mean value of the product of e and x, that is of e and a_1, that is of e and $-\frac{1}{s_1}\epsilon^{b_1}(\alpha_0+\kappa)$ is

$$= -\frac{1}{2s_1}\bar{\alpha}_0\bar{e}\sin(h-b_1)+\frac{1}{2s_1}\bar{e}\bar{\kappa}\sin b_1 \text{ (see § 11)}$$

$$= -\frac{\bar{e}}{2s_1}\left\{\bar{\alpha}_0\sin(h-b_1)-\frac{2}{\omega m}\bar{e}\sin b_1\right\}.$$

It is interesting to note that the armature and alternating-field currents which flow when an E.M.F. $= E\sin(\omega t+h)$ is acting in the armature circuit, the angular velocity of the armature is ω, and the exciting current C, would be unchanged if the E.M.F. $E\sin(\omega t+h)$ be removed, the speed maintained, and the exciting field-current changed from

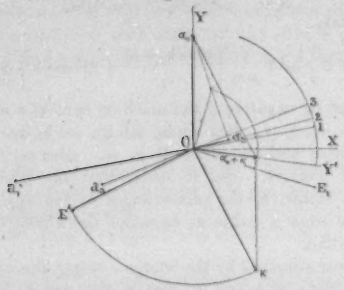
$$C \text{ to } \sqrt{C^2+\frac{2}{\omega m}CE\cos h+\frac{1}{\omega^2m^2}E^2}.$$

This follows immediately from the vector equations in § 31 connecting a_1, α_2, a_3, &c., with $\alpha_0+\kappa$ when κ_3, κ_5, κ_7, &c., are zero.

26. The case of the synchronous motor with sinusoidal applied E.M.F., discussed in the last paragraph, can easily be represented geometrically.

In fig. 4, let $O\alpha_0$, taken in the axis of Y be equal (as in § 13) to twice the steady exciting current of the machine. Draw

Fig. 4.



the vector OE′ to represent in amplitude and phase $2/\omega m$ times the applied E.M.F. ; that is, if the latter

$$= e' \sin(\omega t + h), \qquad OE' = 2e'/\omega m,$$

and the angle from OX to OE′ measured in the positive direction is $= h$.

Rotating OE′ forward through 90° gives us κ of § 25, and completing the parallelogram $\alpha_0 O \kappa$, its diagonal is $\alpha_0 + \kappa$ in the line OY′.

Knowing the motor circuits, we can determine s_1, $\mathfrak{b}_1$, σ_2, β_2, s_3, $\mathfrak{b}_3$, &c., and then construct for $\mathfrak{a}_1$, α_2, $\mathfrak{a}_3$, α_4, &c., exactly as in § 13, except that in this construction the vector $\alpha_0 + \kappa$ takes the place of α_0 in § 13 (see § 24).

Now the mechanical torque *developed by* the machine is (see § 12)

$$= \frac{m}{4}\{S\alpha_0\mathfrak{a}_1 + S\mathfrak{a}_1\alpha_2 + S\alpha_2\mathfrak{a}_2 + \&c.\}$$

$$= \frac{m}{2} \text{ into the sum of the areas, attending to signs, of the triangles } \alpha_0 O\mathfrak{a}_1,\ \mathfrak{a}_1 O\alpha_2,\ \alpha_2 O\mathfrak{a}_2,\ \&c.$$

But the triangles $\mathfrak{a}_1O\alpha_2$, $\alpha_2O\mathfrak{a}_3$, $\mathfrak{a}_3O\alpha_4$, &c., are all essentially negative [their sum is $= -\frac{1}{2}B\alpha_0 + \kappa^2$ of § 25], so that if the machine is to develop mechanical power and run as a motor, the phase of OE′ must be such that the area of the triangle $\alpha_0O\mathfrak{a}_1$ is positive (as it is in fig. 4) and numerically greater than the sum of $\mathfrak{a}_1O\alpha_2$, $\alpha_2O\mathfrak{a}_2$, &c.

The power supplied by the source is

$$= \frac{1}{2} \cdot \frac{\omega m}{2} \overline{OE'} \cdot \bar{\mathfrak{a}}_1 \cos \mathfrak{a}_1OE',$$

$$= \frac{\omega m}{4} \bar{\mathfrak{a}}_1 \bar{\kappa} \sin \mathfrak{a}_1O\kappa,$$

$$= \frac{\omega m}{2} \times \text{area of triangle } \mathfrak{a}_1O\kappa,$$

and the power developed by the motor $= \omega T_1 = \frac{\omega m}{2} \times$ sum of areas of the triangles $\alpha_0O\mathfrak{a}_1$, $\mathfrak{a}_1O\alpha_2$, $\alpha_2O\mathfrak{a}_2$, &c., attending to signs.

Hence the efficiency is equal to

$$\frac{\alpha_0O\mathfrak{a}_1 + \mathfrak{a}_1O\alpha_2 + \alpha_2O\mathfrak{a}_3 + \&c.}{\mathfrak{a}_1O\kappa}.$$

By rotating the vector from O tc the middle point of $\alpha_0\alpha_2$ backwards through 90° and doubling we obtain OE_1, which represents in amplitude and phase $2/m\omega$ times the first harmonic of the total E.M.F. of the motor (see § 13).

This vector can now be compared with OE′ which represents $2/m\omega$ times the applied E.M.F.

Fig. 4 easily explains how, by increasing the exciting current of an A.C. motor, the phase of the armature current is advanced relative to that of the applied E.M.F.

Discussion.

Mr. W. Duddell expressed his interest in Prof. Lyle's paper, and remarked that the results which he had obtained were in accord with experiments made by himself and Dr. Marchant some years ago.

Dr. Russell congratulated the Author on having obtained such instructive solutions of the differential equations which determine the value of the armature and field currents in a simple generator. The subject of armature reaction has been carefully studied by electrical

engineers, and the literature of the subject is quite extensive. He referred in particular to the study made by Professor Blondel in 1900 of the ripples in the exciting current of a two-phase and a three-phase alternator. It was well known that in a single-phase machine the frequency of the ripple superposed on the exciting current by the alternate magnetising and demagnetising effect of the armature current was double the frequency of the armature current. This ripple disappears at no load, but in practice the ripples at no load are often very marked owing to pulsations of the reluctance due to slots in the armature. The latter ripples are much less pronounced at full load as the load circuit acts like a damping-coil; but new ripples due to the armature reaction appear, causing a distortion of the wave-form. As a rule, electrical engineers assumed the existence of a sine wave of armature current and then investigated the ampere-turns to be added or subtracted from the field coils so as to neutralise the magnetising or demagnetising effect produced. In this connexion Blondel's two-reaction method was extensively used as it enabled approximate values to be rapidly obtained. The effect of the reaction on the wave-form of the machine, however, had been practically neglected, and Professor Lyle deserves great credit for his solution. In connexion with the parallel running of turbo-alternators it was important, and the theory deserved careful study by engineers. The speaker thought that "dampers" were mainly used to prevent phase-swinging. He agreed with Professor B. Hopkinson that a similar effect might often be more economically produced by putting more copper on the field windings. The difficulties that arose in connexion with perfecting the theory of synchronous motors arose mainly from the difficulty of taking hysteresis into account in the mathematical equations. He thought that Professor Lyle's work in this direction was most valuable.

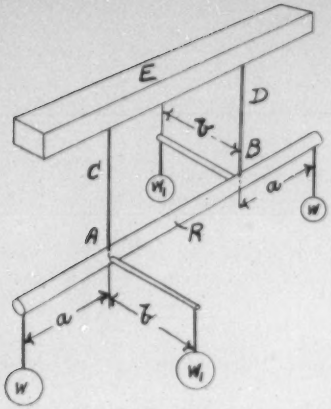


FIG. 1.

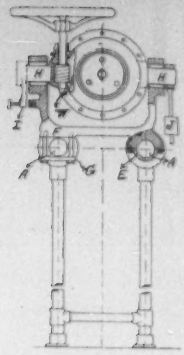
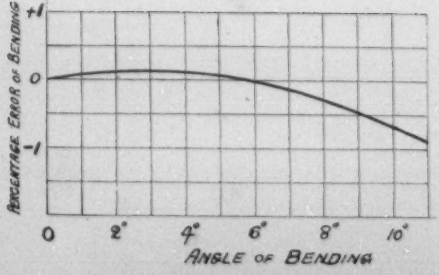
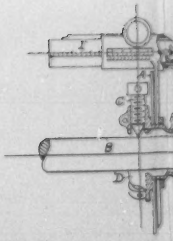


FIG 2.

FIG. 7.





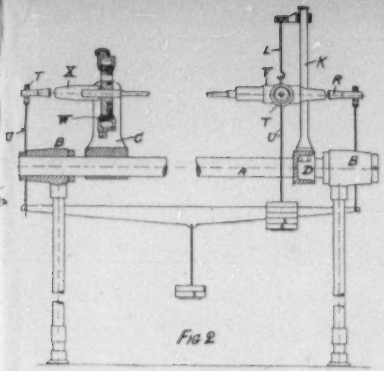


FIG 2

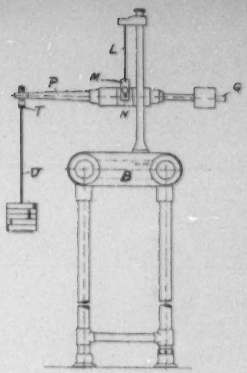


FIG 4.

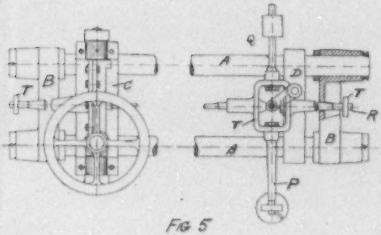


FIG 5

FIG 6

FIG. 9.

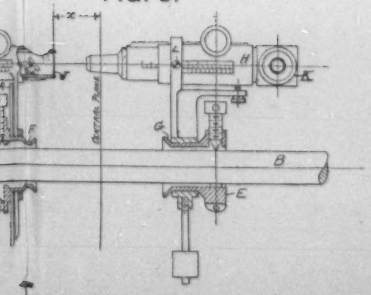


FIG. 10.

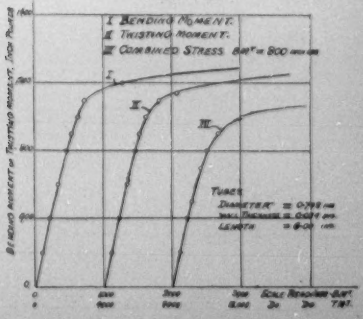
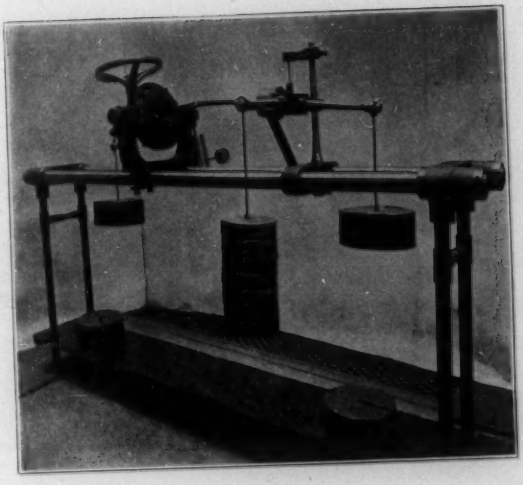


FIG. 8.



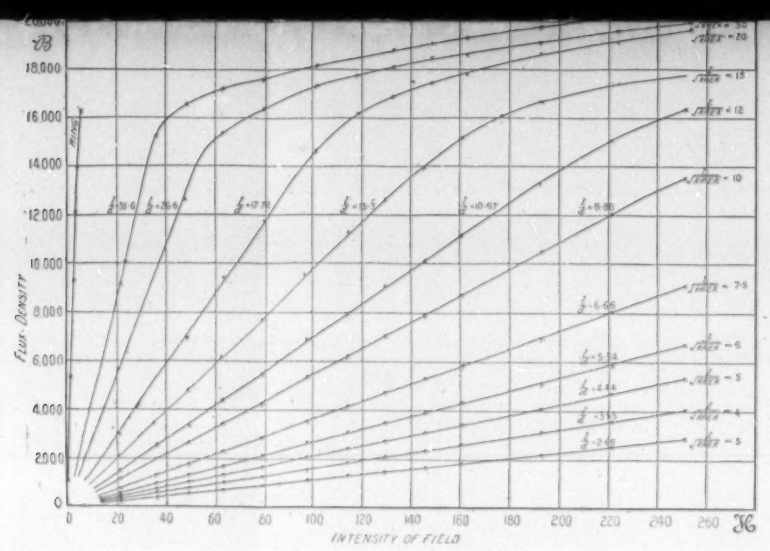


FIG. 4.

B-H CURVES FOR BARS OF VARIOUS FORMS OF SECTION. AREA OF SECTION 1 sq. centimetre. LENGTH 80 millimetres.

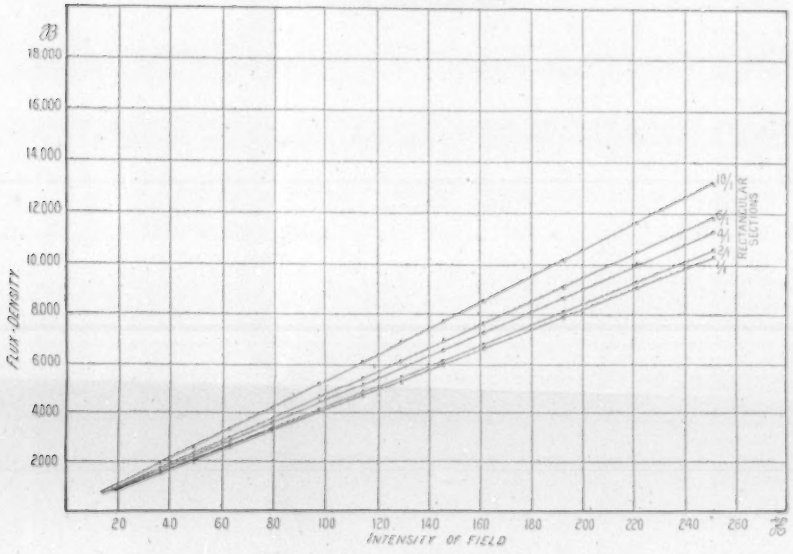
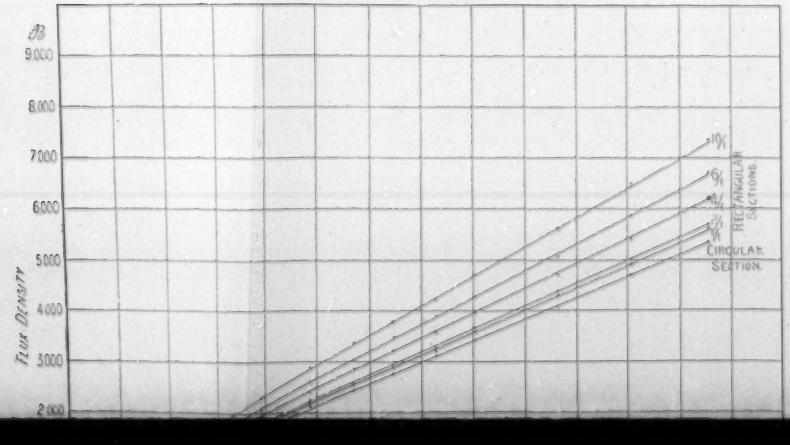


FIG. 7.

B-H CURVES FOR BARS OF VARIOUS FORMS OF SECTION. AREA OF SECTION 1 sq. centimetre. LENGTH 40 millimetres.



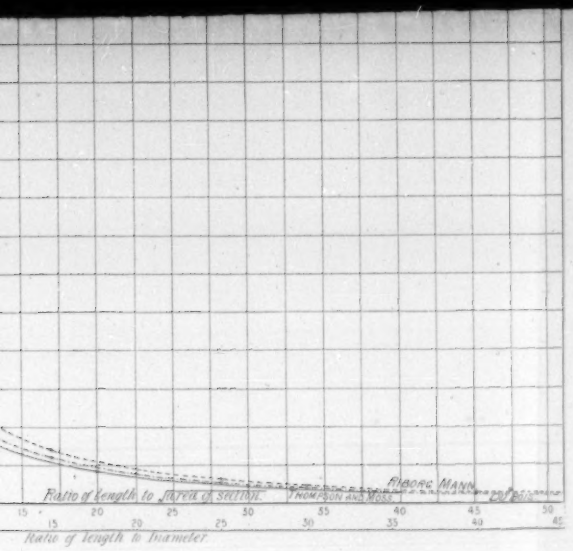


Fig. 5.

OF VARIOUS FORMS OF SECTION. AREA OF SECTION 1 sq. centimetre. LENGTH 60 millimetres.

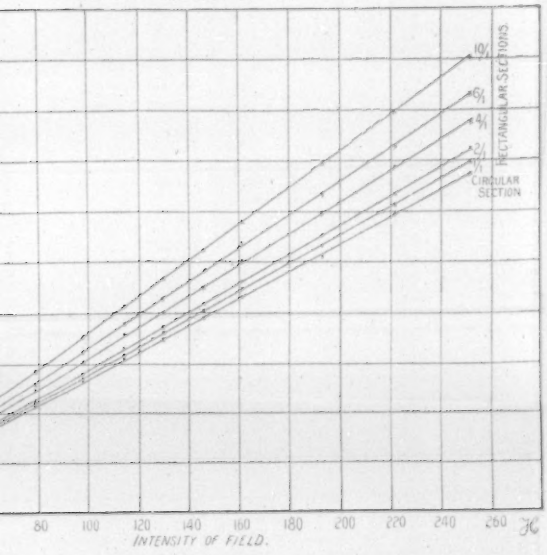
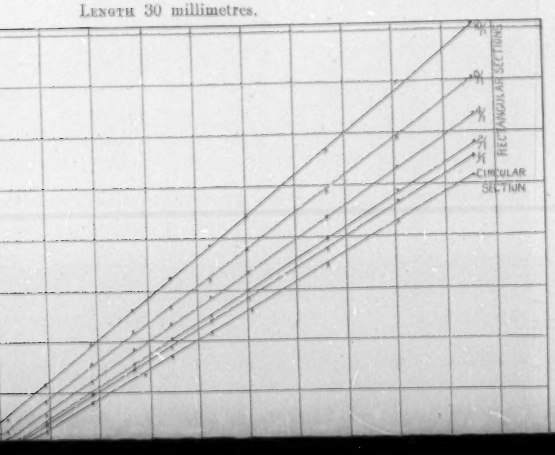


Fig. 8.

ARS OF VARIOUS FORMS OF SECTION. AREA OF SECTION 1 sq. centimetre. LENGTH 30 millimetres.



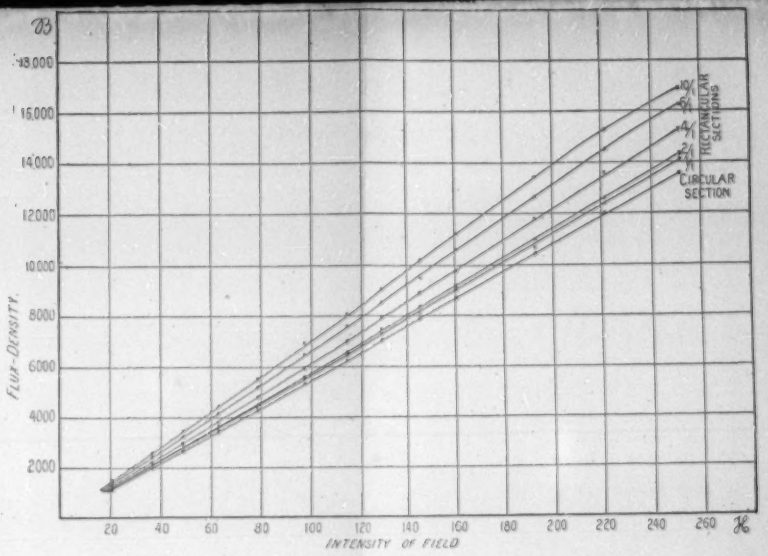


Fig. 6.

B-H CURVES FOR BARS OF VARIOUS FORMS OF SECTION. AREA OF SECTION 1 sq. centimetre. LENGTH 50 millimetres.

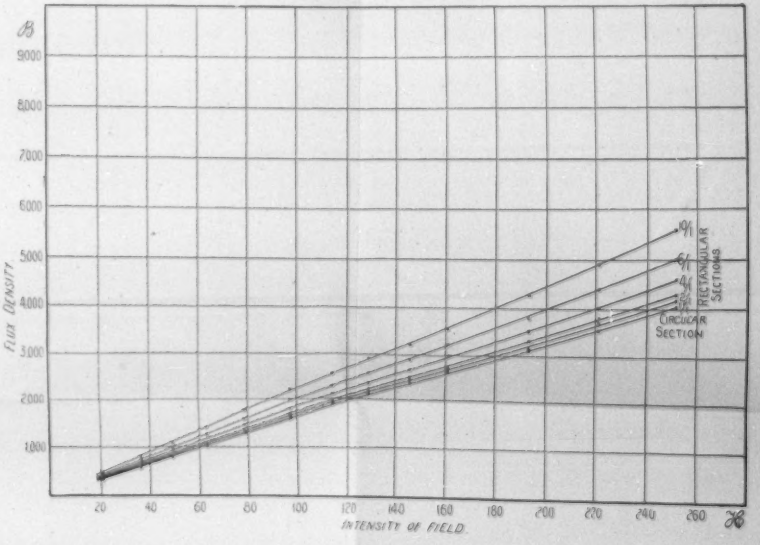
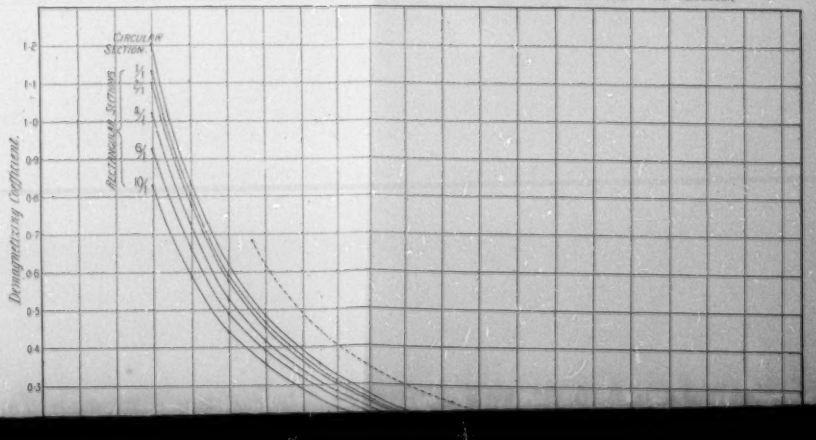
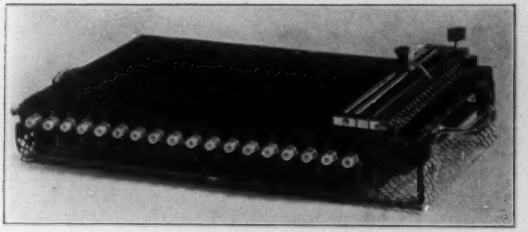
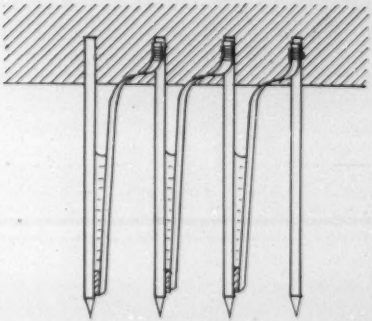


Fig. 9.

COEFFICIENTS OF DEMAGNETIZATION FOR BARS OF VARIOUS LENGTHS AND FORMS OF SECTION.







Scale – Full Size

L. *On a Want of Symmetry shown by Secondary X-Rays.* *By* W. H. Bragg, *M.A., F.R.S., Elder Professor of Mathematics and Physics in the University of Adelaide, and* J. L. Glasson *.

[From "Transactions of the Royal Society of South Australia," vol. xxxii., 1908.]

On the assumption that the Röntgen rays consist of æther pulses it has been shown by J. J. Thomson ('Conduct. of Electr. through Gases," p. 323) that it is possible to account for the existence of secondary Röntgen rays by assuming that the primary pulses set in motion electrons over which they pass, and cause them to become new centres of radiation. If the electron easily follows the guiding force of the primary pulse, then the secondary radiation resembles the primary in quality. But if the electron is hampered by attachments to other portions of the atom to which it belongs, then the new pulse has not the same quality as the old; the time of motion of the electron is dragged out, and the pulse produced is softer.

Now, if an electron becomes in this way a centre of radiation the intensity of the secondary effect must be symmetrical about the line of motion of the electron. In particular, the intensity of the secondary radiation must be symmetrical about a plane passing through the electron perpendicular to the primary ray, since this ray contains the line of motion referred to. This deduction forms an integral part of Thomson's theory of secondary Röntgen radiation, and its truth has been assumed in calculations intended to show that experimental results are in agreement with theory. Barkla proves the same deduction in a paper published in the Philosophical Magazine of February 1908.

Now it has recently been shown (Bragg and Madsen, Trans. Roy. Soc. S.A., May 1908) that the cathode radiations excited by γ rays show a very marked want of symmetry about the plane normal to the exciting ray; and again (Madsen, Trans. Roy. Soc. S.A., July 1908) that

* Read April 23, 1909.

there is a similar want of symmetry in respect to the secondary γ rays. The γ rays and X-rays resemble one another so closely in all their known properties, that it is fairly safe to assume any effect found to be true of the one kind to be true also of the other kind, though perhaps to a different degree. In this case, indeed, Cooksey ('Nature,' April 2, 1908) has already shown that the secondary cathode radiations excited by X-rays are not at all symmetrical about the normal plane, the emergence rays being greater than the incidence, as in the case of the γ rays.

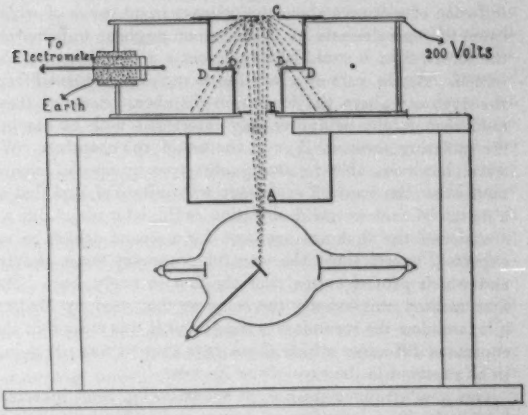
It remained, therefore, to examine the secondary X-rays excited by primary X-rays; and the experiments described in this paper were made with that object. We find that in general want of symmetry does exist, that it is sometimes very pronounced, and that is in keeping with expectation based on Madsen's study of the secondary γ rays. Hard γ rays show a very large difference between the quantities of emergence and incidence radiation; for soft γ rays the difference is smaller. Since X-rays are to be looked on as a very soft form of γ rays, the difference should be smaller still; and this is what we have found to be the case.

The general form of the apparatus which we have used is shown in fig. 1. Variations of the upper portion of it are shown in figs. 2 and 3. A small pencil of X-rays passed upwards through apertures in lead plates at A and B, and then along the axis of the ionization-chamber and out into the open. In our first experiments the upper part of the apparatus was arranged as in fig. 3. The primary rays did not pass through the effective part of the ionization-chamber, being separated therefrom by the cylindrical screen SS, which could be made of various thicknesses and various materials. But if a thin sheet of any substance was laid over the hole at B, secondary X-rays spread out therefrom, and some passed through the screen SS, and caused a deflexion in the electrometer. The difference between the deflexions (a) without and (b) with the sheet at B was taken as a measure of the emergence secondary X-ray radiation. When the sheet was removed from B, and the same or a similar sheet placed in the plane of the top of the screen so as to be struck from below by the primary rays, then the

measure of the incidence secondary radiation was obtained as the difference between the deflexions (*a*) without and (*c*) with the sheet so placed.

In this way it was easy to show that the expected want of symmetry actually existed, particularly with aluminium,

Fig. 1.



4 in.

Fig. 2.

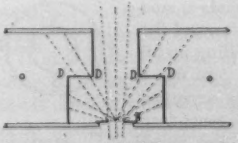
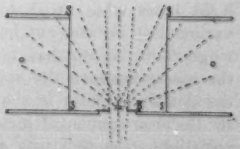


Fig. 3.

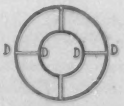


celluloid, or paper as the radiators, substances of small atomic weight. But the experiments were open to some extent to the objection that *a* was too large compared with *b*–*c*, and that possibly the excess of emergence over incidence

was an apparent effect due to actual variations of a under different circumstances. The current a was, in fact, due to several causes. There was a small natural ionization leak even when the X-rays were not acting; there was an effect due to primary X-rays which had penetrated the walls of the chamber, though they were made of zinc one-eighth of an inch thick. But the greatest part of a was due to a diffusion of soft rays about the primary beam, much of which came through the hole at B at such an angle as to penetrate the screen SS; it could be largely cut out by thickening the screen. Again, part of a was due to radiation returned from the open air above the ionization-chamber. Some of these radiations might be appreciably interfered with by placing the radiating sheet at B or at the top of the chamber. We were, however, able to satisfy ourselves by special experiments that the want of symmetry was quite real, and that as a matter of fact no valid objection could be made. But we abandoned the first arrangement for a second which, as we expected, would show the want of symmetry more clearly, and which proved better than the first in every way. The first method was exactly the same as that used by Madsen in examining the secondary γ rays; but it was clear that the enormous difference which these rays showed was not going to be repeated in the case of the X-rays.

Our new arrangement was, as shown in fig. 1, or, inverted, in fig. 2. Two cylinders of brass, each 2 in. long, but of different diameters—4 in. and 2 in.—were fixed to a connecting piece DD, shown in plan in fig. 4. The latter resembled a light brass wheel with four spokes, and various thin screens cut in the form of flat rings could be attached to it, filling up all the spaces between the spokes. In fig. 1 the double cylinder is shown as arranged for the measurement of incidence secondary radiations; the radiating sheet was placed at C, supported by a sheet of celluloid lying flat on the top of the cylinder. A hole was cut in the centre of the celluloid sheet big enough to allow the primary beam to pass through without touching the edges; and a fluorescent screen was used to make sure

Fig. 4.



that this was the case. The radiating sheets were of thin metal, about 1½ in. square. In fig. 2 the cylinder is shown as arranged for the measurement of emergence secondary radiations: it hardly requires further explanation.

We expected that this arrangement would show up the want of symmetry better than the former, because the portions of the emergence and incidence beams under comparison would be more nearly normal to the plate. Looking upon the radiations as material, we should naturally expect the intensity of the secondary radiation to decrease gradually as its direction increased in inclination to the forward direction of the primary ray. The emergence rays lie, in inclination, between 0° and 90°; the incidence between 90° and 180°. In our first arrangement we compared the emergence rays between about 40° and 90°, with the incidence rays between about 90° and 140°. There should be a larger ratio of emergence to incidence with the newer arrangement, since the emergence rays between about 30° and 50° would be compared with the incidence between about 130° and 150°. This proved to be the case; the improvement was considerable. Again, with the new arrangement, the current with no radiator in position became relatively far smaller. For example, when the radiator was Al, ·4 mm. thick, and the absorbing screen DD of tinfoil (two thin sheets), the currents with and without the radiator at B in fig. 1 caused deflexions of 86 and 26 mm. in ten seconds respectively; the currents with and without the radiator at B in fig. 2 were 220 and 35 respectively. There could be very little error, therefore, in taking the incidence and emergence radiations as 60 and 185 respectively; and the want of symmetry is beyond doubt.

It should be observed that the emergence radiation can never be shown to an unfair advantage in these experiments, and is often at a disadvantage, for the radiator, when placed as in fig. 2, cuts down the very primary rays to which the secondary radiation is due. It is not difficult to show that if the thickness of the radiator is so adjusted as to give the maximum emergence current (it can of course be too thick or too thin), then the ratio of this maximum to the maximum incidence current (which can be obtained simply by making

was an apparent effect due to actual variations of a under different circumstances. The current a was, in fact, due to several causes. There was a small natural ionization leak even when the X-rays were not acting; there was an effect due to primary X-rays which had penetrated the walls of the chamber, though they were made of zinc one-eighth of an inch thick. But the greatest part of a was due to a diffusion of soft rays about the primary beam, much of which came through the hole at B at such an angle as to penetrate the screen SS; it could be largely cut out by thickening the screen. Again, part of a was due to radiation returned from the open air above the ionization-chamber. Some of these radiations might be appreciably interfered with by placing the radiating sheet at B or at the top of the chamber. We were, however, able to satisfy ourselves by special experiments that the want of symmetry was quite real, and that as a matter of fact no valid objection could be made. But we abandoned the first arrangement for a second which, as we expected, would show the want of symmetry more clearly, and which proved better than the first in every way. The first method was exactly the same as that used by Madsen in examining the secondary γ rays; but it was clear that the enormous difference which these rays showed was not going to be repeated in the case of the X-rays.

Our new arrangement was, as shown in fig. 1, or, inverted, in fig. 2. Two cylinders of brass, each 2 in. long, but of different diameters—4 in. and 2 in.—were fixed to a connecting piece DD, shown in plan in fig. 4. The latter resembled a light brass wheel with four spokes, and various thin screens cut in the form of flat rings could be attached to it, filling up all the spaces between the spokes. In fig. 1 the double cylinder is shown as arranged for the measurement of incidence secondary radiations; the radiating sheet was placed at C, supported by a sheet of celluloid lying flat on the top of the cylinder. A hole was cut in the centre of the celluloid sheet big enough to allow the primary beam to pass through without touching the edges; and a fluorescent screen was used to make sure

Fig. 4.

that this was the case. The radiating sheets were of thin metal, about 1½ in. square. In fig. 2 the cylinder is shown as arranged for the measurement of emergence secondary radiations: it hardly requires further explanation.

We expected that this arrangement would show up the want of symmetry better than the former, because the portions of the emergence and incidence beams under comparison would be more nearly normal to the plate. Looking upon the radiations as material, we should naturally expect the intensity of the secondary radiation to decrease gradually as its direction increased in inclination to the forward direction of the primary ray. The emergence rays lie, in inclination, between 0° and 90°; the incidence between 90° and 180°. In our first arrangement we compared the emergence rays between about 40° and 90°, with the incidence rays between about 90° and 140°. There should be a larger ratio of emergence to incidence with the newer arrangement, since the emergence rays between about 30° and 50° would be compared with the incidence between about 130° and 150°. This proved to be the case; the improvement was considerable. Again, with the new arrangement, the current with no radiator in position became relatively far smaller. For example, when the radiator was Al, ·4 mm. thick, and the absorbing screen DD of tinfoil (two thin sheets), the currents with and without the radiator at B in fig. 1 caused deflexions of 86 and 26 mm. in ten seconds respectively; the currents with and without the radiator at B in fig. 2 were 220 and 35 respectively. There could be very little error, therefore, in taking the incidence and emergence radiations as 60 and 185 respectively; and the want of symmetry is beyond doubt.

It should be observed that the emergence radiation can never be shown to an unfair advantage in these experiments, and is often at a disadvantage, for the radiator, when placed as in fig. 2, cuts down the very primary rays to which the secondary radiation is due. It is not difficult to show that if the thickness of the radiator is so adjusted as to give the maximum emergence current (it can of course be too thick or too thin), then the ratio of this maximum to the maximum incidence current (which can be obtained simply by making

the radiator thick enough) is only $2/e$ of the true ratio of emergence to incidence; provided that the secondary rays are as penetrating as the primary, and that we are considering homogeneous radiations. But if, other conditions being the same, the secondary rays are less penetrating than the primary, then the ratio, as found, is more nearly correct, and is very nearly so when the secondary rays are much less penetrating than the primary, as, for example, when we are considering secondary cathode rays due to X- or γ rays.

We have made a large number of measurements by the method described above, using the following metal sheets as radiators:—Pt, weight per square cm., ·0150 gr.: Sn, ·0096 gr.; Cu, ·0083 gr.; Fe, ·0077 gr.; Al, ·105 gr.; celluloid, ·20 gr. As screens we have used various thicknesses of Sn, Cu, and Al.

The proportion of emergence to incidence radiation differs considerably for the different radiators, but is much the same for different screens or different thicknesses of screen, except that the proportion tends to increase slightly as the screen is made thicker; and the tendency is most pronounced in the case of those metals which give out a quantity of soft secondary radiation. For example, Fe and Cu show little difference between incidence and emergence radiations until the screen is so thick that only a small fraction of either of the radiations can pass through. The results vary somewhat with the state of the bulb; and since these variations are comparable with those which are met with on changing the nature of the screens, we are not now in a position to discuss smaller variations in detail. We must content ourselves with quoting a few results in order to show the want of symmetry, which is a persistent effect. When, for example, two tinfoils were used as screen (weight per square cm. of each, ·0056), we obtained the following figures, which represent movements of the scale in mm. during 10 secs.:—

Radiator	Sn.	Cu.	Fe.	Al.
Emergence Current	176	140	39	185
Incidence Current	122	119	15	60

With four tinfoils the figures were :—

Radiator	Sn.	Cu.	Fe.	Al.
Emergence Current.....	143	24	23	116
Incidence Current	87	1	0	34

Again, using a copper screen ·002 cm. thick, we found :—

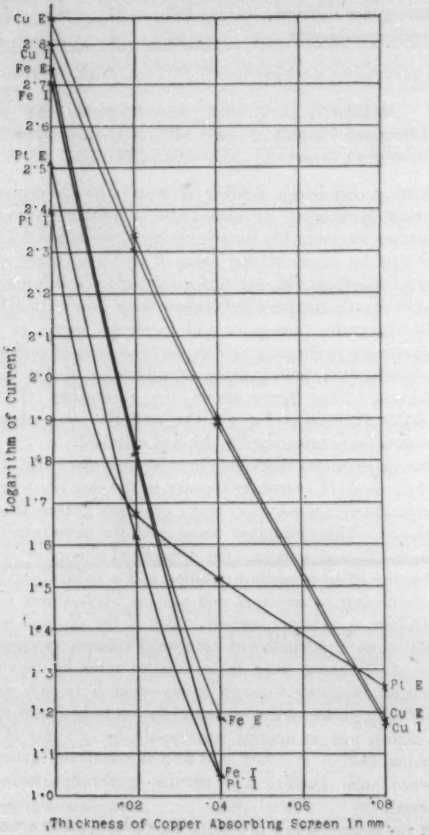
Radiator.....	Pt.	Sn.	Cu.	Fe.	Al.	Celluloid.
Emergence Current...	86	140	361	118	80	138
Incidence Current ...	65	104	364	118	32	93

Putting together a number of results for Cu screens of different thicknesses we obtain the logarithmic curves of absorption shown in the accompanying figures (figs. 5 and 6). It should be observed that some of the results thus shown were obtained at different times, so that too much must not be built upon a comparison between them ; only the relative positions of the emergence and incidence curves of each substance are sufficiently correct, and the form of each curve as showing the homogeneity or otherwise of the various radiations. One figure shows the emergence (E) and incidence (I) curves for Pt, Cu, and Fe; the other the corresponding curves for Sn, Al, and celluloid.

The experiments described in this paper show that a very marked want of symmetry occurs in the case of secondary X-rays, the emergence rays being generally greater than the incidence. This is another instance of the close parallelism between X- and γ rays. On a material theory of X- and γ rays the effect is easily explained, and is to be classed with the scattering to which β, and also, as lately shown clearly by Geiger, α rays are subject. But if the X- and γ rays consist of energy bundles of very small volume, as suggested by J. J. Thomson, then these bundles must be capable of deflexions in going through atoms—that is to say, swung out of their paths by the electrical forces to be found within the atoms, just as neutral pairs would be in virtue of their electrical fields. It seems hard to understand the distinction between such bundles and entities generally classed as material.

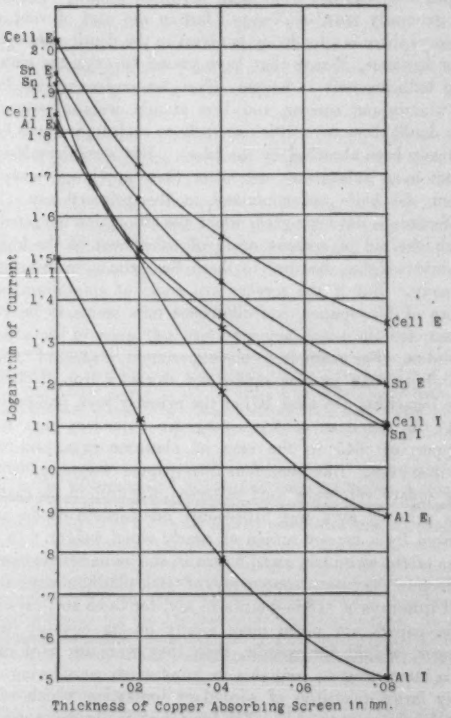
In the course of this investigation we have made a number

Fig. 5.


Cu E
Cu I
Fe E
Fe I
Pt E
Pt I
2·8
2·7
2·6
2·5
2·4
2·3
2·2
2·1
2·0
1·9
1·8
1·7
1·6
1·5
1·4
1·3
1·2
1·1
1·0
Logarithm of Current
·02
·04
·06
·08
Thickness of Copper Absorbing Screen in mm.
Pt E
Cu E
Cu I
Fe E
Fe I
Pt I

of experiments on the quantities and qualities of the secondary radiations. This subject has been fully treated by Barkla, some of whose recent papers have not yet reached us, and

Fig. 6.



any discussion we gave might be merely a duplication of part of his inquiry. There is, however, one point to which we should like to refer.

Very hard γ rays follow a density law of absorption, treating all atoms alike, except in respect to weight. Soft γ rays are not independent of atomic groupings of matter, and are far more strongly absorbed by heavy atoms than by light, after allowance has been made for weight. The same is generally true of X-rays; but in the case of very soft X-rays there is a tendency to revert to the density law again. For instance, X-rays that have passed through the glass of the bulb are soft to copper, silver, tin, and so on, but hard to aluminium, carbon, and low atomic weight generally. No doubt those rays which are soft to such light atoms have already been absorbed by the glass. But secondary X-rays from most substances are softer than anything emerging from the bulb and contained in the primary ray. The difference is not very great when the absorption is measured with the aid of screens made of substances of the higher atomic weights, because to these the primary rays are soft already. But if the screens are made of aluminium, still more of filter-paper, the difference now seems to be very great, for the secondary rays are soft even to low atomic weights. For example, in one experiment, a sheet of copper weighing ·018 gr. per square cm. caused a drop of ·401 in the logarithm (to base 10) of the primary rays, and only of ·447 in the case of the emergence secondary rays from copper, of ·645 in the case of platinum rays, and ·805 of iron rays. But when four filter-papers weighing ·02 gr. per square cm. were used as screen, the drop in the case of the primary rays was ·010—only one-fortieth of the drop caused by a copper screen of nearly equal weight. In the case of the secondary rays, however, the same screen caused a drop in the case of copper rays of ·100, platinum rays ·053, and iron rays of ·188—that is to say, for these soft rays the filter-papers are much more nearly on an equality with copper, weight for weight, than they were for hard rays. It is interesting to bear this in mind when considering the very large quantities of secondary ionization which some substances seem to give. The ionization is always measured in air, which of course consists of atoms not very different in weight from those contained in filter-papers. Consequently primary rays, and secondary rays which differ

very little from the primary, are very penetrating to air, and cause relatively small ionizations therein. But secondary rays from Cu and Fe are softened so much as to bring them within reach, so to speak, of air, which rapidly converts them into cathode rays, so that there is a very large ionization. For the cathode rays produced from these secondary rays have probably but little less energy than those produced from the primary; the speed of the cathode ray does not differ very greatly with the penetration of the primary X-ray, so far as experiments have shown. The very large secondary radiations, which some substances appear to give, therefore, owe their magnitude largely to the fact that the air in which they are measured is sometimes ten to twenty times as favourable to them as to the primary rays which produced them. In this way we may account to some extent for the startling results obtained by Crowther in the case of arsenic and bromine (Phil. Mag. Nov. 1907).

Discussion.

Prof. C. H. Lees said that Prof. Bragg had given a lucid account of his theories of γ and X rays. His researches would make physicists more careful in accepting the æther-pulse theory. He asked if it was likely that better means would be devised to discriminate between various forms of γ and X rays than dividing them into "hard" and "soft" radiations. He thought many discrepancies could be attributed to this want of discrimination.

Mr. C. A. Sadler pointed out that whatever lack of symmetry might exist in the emergence and incidence secondary X radiations from a plate of a substance which was a source of scattered primary radiation, Professor Bragg's own results conclusively proved that such lack of symmetry did not exist when the plate was a source of homogeneous radiation. If then it was a necessary condition of Professor Bragg's theory that such lack of symmetry should exist with secondary X radiations, we must either conclude that the theory here breaks down or that these homogeneous radiations are not X radiations as usually understood. It was to be noted also that the measured lack of symmetry (ignoring the lack of symmetry in the case of homogeneous beams, which can be shown to be only apparent) in the most pronounced cases was small compared with those obtained with γ rays.

Prof. Bragg, referring to the remarks of Prof. Lees, said that for precision the actual speed of all electrons ought to be measured. Instead of measuring the speed the penetrating power might be determined.

LI. *Transformations of X-Rays.* By CHARLES A. SADLER, *M.Sc., Oliver Lodge Fellow, University of Liverpool**.

WHEN a primary beam of Röntgen radiation falls upon any substance, secondary Röntgen rays are emitted, the character of which depends both upon the nature of the substance and upon the particular kind of primary beam used. With reference to the latter it has been found that variations in the intensity of the primary beam produce no perceptible change in the character of the resulting secondary, the sole controlling factor appearing to be the degree of "hardness" of the primary †.

It has been shown ‡ that if the radiating substance be an element of low atomic weight—as hydrogen, oxygen, or carbon—it emits a radiation similar in penetrating power to the primary producing it; its penetrating power varying with that of the producing primary. This type of radiation, which will be referred to as a "scattered" radiation, may be considered as produced by an acceleration of one or more electrons in the atom of the radiating substance, due to the action of forces in the primary pulse.

From an element of greater atomic weight than that of calcium (40)—and possibly from other elements of lower atomic weight under very penetrating primary beams—the emitted radiation has been shown to consist of scattered radiation, and superposed upon this, a type of radiation which is characteristic of the radiating element. The penetrating power of this radiation is a constant quantity peculiar to the substance and is independent of the penetrating power of the primary producing it. Moreover, this radiation appears to be entirely homogeneous in character, and experiments point to the conclusion that the radiating electrons producing this type of radiation are no longer appreciably under the influence of the forces in the primary pulse.

This homogeneous radiation is only produced when the

* Read April 23, 1909.

† Barkla, Phil. Mag. June 1906, pp. 812–828

‡ Barkla & Sadler, Phil. Mag. Oct. 1908 pp. 550–584.

penetrating power of the primary is greater than that of the homogeneous radiation characteristic of the radiator.

From the group of elements Cr–Ag the ionization produced by the homogeneous portion of the secondary radiation, emitted when any of its members is subjected to a sufficiently penetrating primary, is many times greater than that produced by the scattered portion—in the case of copper as radiator this ratio is as high as 150 : 1.

The homogeneous radiation from chromium is very "soft" much softer indeed than any ordinary primary beam, and from chromium down to silver and probably beyond, the penetrating power of the characteristic radiation increases with increase of atomic weight; the radiation from silver being many times more penetrating than that from chromium.

One of the chief difficulties experienced in the investigation of X-ray phenomena has been the heterogeneity of the primary beams hitherto available. Even where devices are adopted to ensure that the current through the X-ray bulb used as a source of primary rays is uni-directional and of nearly constant strength, the primary so obtained consists of a mixture of constituents of different penetrating power; so there remains the difficulty of ascertaining which particular constituents of the composite beam are principally concerned in producing the phenomena under investigation.

It was thought that useful information concerning the nature of X-rays might be obtained if the homogeneous rays previously mentioned were used to excite tertiary radiation in different substances.

Sagnac has shown that the tertiary X-rays from metals exited by secondary X-rays are more easily absorbed than the exciting rays.

Previous experiments* had shown that if two substances A and B be taken, each of which is found to emit a homogeneous radiation when a suitable primary beam falls upon it, the homogeneous radiation from A being more penetrating than that from B, then if a homogeneous beam from A be passed through a thin plate of B, tertiary radiation is

* Barkla & Sadler, Phil. Mag. Oct. 1908, pp. 550–584.

excited in B by the radiation from A, while if the process is reversed it is found that the radiation from B excites *no* tertiary radiation in A.

These phenomena were examined in greater detail in the experiments described below.

It was found that no trace of a homogeneous tertiary radiation could be detected from aluminium when subjected to any of the homogeneous radiations from the group of metals Cr–Ag, and the amount of scattered radiation produced was extremely small when compared with the secondary incident beam.

Advantage was taken of these facts, and the primary and secondary beams were passed through tubes of thick aluminium of rectangular cross-section. This enabled the apparatus to be arranged compactly with comparatively short distances between the anticathode of the X-ray bulb and the secondary radiator, and between the secondary and tertiary radiators respectively, a condition essential to secure that the ionization produced by the tertiary rays in a suitable ionization chamber should be sufficiently intense to ensure accurate readings in a reasonably short time, and that the direct tertiary radiation should produce an ionization large compared with that produced by stray secondary and tertiary rays from the surrounding air and neighbouring screens.

The general arrangement of the apparatus is indicated in Plan by fig. 1.

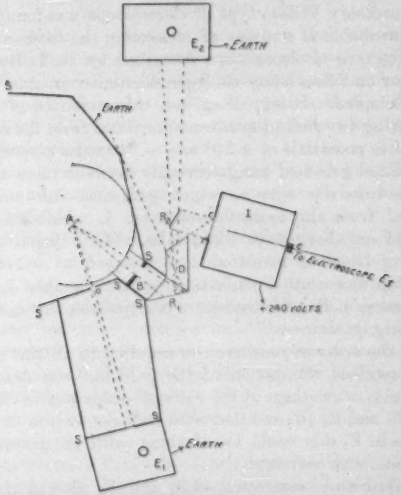
A rectangular brass tube B lined with aluminium ·2 cm. thick was fitted into an aperture in the lead screen SS surrounding the X-ray bulb emitting the primary rays, and the bulb was so placed relatively to this tube that the axis of the tube passed through the centre of the anticathode A normally.

The rays from A passing through this tube impinged on the radiator R_1 consisting of a rectangular plate of the metal, the secondary rays from which were to be studied. A portion of the secondary rays so produced passed down the rectangular brass tube D lined as before with ·2 cm. aluminium, and a plate of any substance placed in this secondary beam provided a source of tertiary rays.

The intensity of the primary beam was measured by

allowing a narrow pencil of rays proceeding from A through a small aperture O in the lead screen SS to enter an electroscope E_1 of the ordinary Wilson type through an opening

Fig. 1.



covered with tissue-paper and aluminium-foil in the wall of the electroscope.

The deflexions of the gold-leaf were observed by means of a microscope fitted with a scale in the eyepiece.

The intensity of the secondary beam was measured by means of a similar electroscope E_2 placed in the path of the secondary beam passing down the tube D (R_2 removed) as indicated in the Plan ; the size of the aperture in the wall of this electroscope being 3×2 cms.

It was found that by carefully shielding the electroscopes E_1 and E_2 from draughts and sudden changes of temperature very reliable readings could be obtained, the motion of the gold-leaf being absolutely dead-beat.

Preliminary experiments also showed that with a given radiator R_1 in position (R_2 being removed) the ratio of the deflexions in E_1 and E_2 when the bulb had reached a steady state rarely varied by more than 1 per cent. during a series of readings.

The ordinary Wilson type of electroscope was found to be quite unsuitable as a means of measuring the tertiary radiation; the type of electroscope described by R. T. Beatty in his paper on "Secondary Röntgen Radiation in Air" * was finally adopted. Briefly described, this consisted of a brass case having two sliding quadrants insulated from the case and charged to potentials of +240 and −240 volts respectively; an insulated gold-leaf hung vertically between them and was connected to the wire e, which projected into and was insulated from the ionization-chamber I, which itself was insulated and charged to +240 volts. An adjustable compensating-chamber insulated and charged to −240 volts eliminated the normal ionization in the chamber I. The sensitiveness of the electroscope was adjustable by means of the sliding quadrants.

With the order of sensitiveness required in all the experiments described, the motion of the gold-leaf was dead-beat, and a series of readings of the ratio of deflexions in electroscopes E_1 and E_2 showed that with deflexions up to 20 scale-divisions in E_2 this could be obtained with an accuracy of 2 per cent. with certainty.

A calibration of electroscopes E_1 and E_2 showed that the value of a deflexion of a scale-division was the same within 1 per cent. throughout the range of scale employed.

The sensitiveness of E_2 was adjusted and then maintained constant throughout a given series of readings, and in all readings a constant deflexion of E_2 was used, thus eliminating the effect of the slight change in value of a scale-division which was found to exist in different parts of the scale.

Preliminary experiments showed that the secondary radiation from air together with other stray effects were very small compared with the readings obtained when metallic radiators were employed.

* R. T. Beatty, Phil. Mag. Nov. 1907, pp. 604–614.

The portion of the radiator R_1 exposed to the primary beam was limited to that opposite to the tube D by means of a suitable stop placed in the tube B as indicated in the diagram.

Nature of the Tertiary Rays.

It was perhaps reasonable to expect that the tertiary radiation emitted by any member of the group Cr–Ag when subjected to a more penetrating homogeneous beam from the same group would be identical in character with that emitted as secondary radiation by the same substance when excited by a suitable primary.

Direct experiments were carried out to test how far this was true, and for this purpose pure iron was chosen as the tertiary radiator.

The radiator consisted of a small rectangle 3 cms. high by 2 cms. broad, formed of pure iron wire interlaced so as to expose as large a surface as possible to the exciting beam. This was placed in the position indicated by R_2, the centre of the radiator was at the intersection of the axis of the tube D with the normal from the centre of the aperture to the ionization-chamber I, the plane of the radiator making equal angles with these directions.

As a secondary radiator in the position R_1 a plate of pure copper was used.

The aperture YY of the ionization-chamber I in these experiments was 3 cms. high by 2 cms. broad, and the distance from the centre of the aperture to the centre of R_2, 4 cms.

Owing to the obliquity of some of the tertiary rays, it was evident that the absorption coefficients, obtained by studying the absorption by thin plates of different substances placed parallel to the aperture YY in the path of the beam, would be greater than would have been the case had it been possible to utilize a *pencil* of tertiary rays.

A control experiment conducted with a fairly soft primary beam falling upon the same iron radiator similarly situated before an aperture of the same size as that in the screen YY in an electroscope of the Wilson type placed in the secondary beam from the iron, gave an increase in the value of the

absorption coefficient by an aluminium plate ·00297 cm. thick of 6 per cent. over the value found when a narrow pencil of secondary radiation was used.

The absorption coefficients of the tertiary beam from iron were then determined by thin sheets of aluminium, iron, copper, and zinc. The values so obtained are compared with those obtained when the same absorbers were used with the secondary rays in the control experiment in the following table.

TABLE I.

Absorber.	Value of $\frac{\lambda}{\rho}$ for Secondary Rays.	Value of $\frac{\lambda}{\rho}$ for Tertiary Rays.
Al (·00297 cm.) ...	93·8	94·2
Fe (·00315 cm.) ...	69·1	69·1
Cu (·00298 cm.) ...	101·0	102·5
Zn (·00132 cm.) ...	119·2	120·0

It will thus be seen that within the limits of experimental errors the penetrating power of the tertiary beam is identical with that of a similar secondary beam from the same substance.

The Homogeneity of the characteristic Tertiary Radiation.

The tertiary beam from iron excited by the secondary homogeneous beam from copper was now cut down by thin aluminium sheets to test its homogeneity.

It was to be expected, even were the beam perfectly homogeneous, that after cutting down by a few plates the beam would appear slightly more penetrating, for the more oblique rays would suffer extinction to a greater extent than those passing through the absorber in a perpendicular direction.

It was not found possible to test for homogeneity to an exhaustive limit owing to the smallness of the readings in the later stages, but the results obtained show that for all

practical purposes the beam may be regarded as homogeneous.

The results are tabulated below:—

TABLE II.

Iron as Tertiary Radiator; Copper as Secondary Radiator.

Amount previously absorbed by Aluminium.	Subsequent absorption by a sheet of Al (·00297) cm. thick.
None.	55·6 per cent.
55·6 per cent.	55·7 „
80·3 „	55·5 „
94·0 „	55·0 „

Previous experiments have shown that associated with the homogeneous secondary radiation from a metal of the group Cr—Ag is a small proportion of scattered radiation, the relative ionizations produced by the homogeneous and scattered portions being about 150 : 1.

If the secondary beam from iron be absorbed by say ·0104 cm. aluminium, then while the homogeneous portion will be absorbed to the extent of about 90 per cent. the scattered portion will only be absorbed by about 30 per cent., giving an absorption of the whole beam of 89·6 per cent. (the absorption being measured by the relative diminution in ionization); so that the relative ionizations produced by the residual homogeneous and scattered portions respectively will now be as 21·4 : 1, and a sheet of ·0104 aluminium would now only absorb about 87·4 per cent. of the whole beam, and this increase in penetrating power would become more and more pronounced as further absorptions took place.

In a corresponding case of the tertiary beam from iron excited by copper radiation, the amount of scattered copper radiation in the beam if present at all will not be present to so great an extent since the copper radiation will not penetrate to anything like the same depth in the iron as an ordinary primary radiation, though on the other hand the ionization produced by beams of scattered primary radiation

and of scattered copper radiation conveying equal amounts of energy per second through unit volume of air, will not be equal; the scattered copper radiation being more easily absorbed will produce the greater ionization.

No direct evidence has yet been obtained that when the normal tertiary radiation from iron is excited by homogeneous radiation from copper, any of the copper radiation itself is scattered by the iron; its presence in the beam would be difficult to detect for a small percentage of copper radiation in the beam would produce a very minute change in its absorption coefficients. The figures given in Table II. therefore do not preclude the possibility of a small percentage of the ionization being due to scattered copper radiation.

Independence of the Penetrating Power of the Exciting Radiation.

Experiments were made to test whether the penetrating power of the tertiary radiation emitted by a substance depended in any way upon the penetrating power of the exciting secondary beams. In the first test, iron was chosen as the tertiary radiator, copper and arsenic as the secondary radiators. The radiation from arsenic being about twice as penetrating as that from copper.

In the second test chromium was taken as the tertiary radiator, and iron, copper, and arsenic as secondary radiators; the radiations from each of these substances being more penetrating than that from chromium, the radiation from arsenic being about four times as penetrating as that from iron.

The results are tabulated below:—

TABLE III.

Iron as Tertiary Radiator.

Secondary Radiator.	Absorption coefficient of the Secondary Radiation by Aluminium.	Percentage Absorption of the Tertiary Radiation by (·0297 cm.) Al.
Copper	128·9	55·6
Arsenic	60·7	55·4

TABLE IV.

Chromium as Tertiary Radiator.

Secondary Radiator.	Absorption coefficient of the Secondary Radiation by Aluminium.	Percentage absorption of the Tertiary Radiation by (·00297 cm.) Al.
Iron	239	75·1
Copper	128·9	75·3
Arsenic	60·7	75·1

From these results it will be seen that the penetrating power of the tertiary radiations is uninfluenced, so far as can be measured, by variations in the penetrating power of the exciting secondary beams; they further show that if any secondary radiation is scattered by the tertiary radiator, the ionization it produces is small compared with that produced by the tertiary radiation.

This result is analogous with that obtained with secondary beams *.

It has thus been shown that the characteristic tertiary radiation from iron is identical with the characteristic secondary radiation from iron in its penetrating power, its homogeneity, and in its independence of variations in the penetrating power of the exciting radiation.

Similar results were obtained with copper as a tertiary radiator.

Connexion between the Secondary and Tertiary Radiators.

A series of experiments was now undertaken in which the several members of the group of metals Cr–Ag were used as secondary radiators, the tertiary radiators being also chosen from among the earlier members of the same group.

By referring to column 1, Table V. it will be seen that the absorption coefficients by aluminium of the radiations from the metals of this group indicate a wide range of penetrating powers, so that it was reasonable to expect that for each tertiary radiator there would be at least some secondary

* Barkla & Sadler, Phil. Mag. Oct. 1908, pp. 550–584.

radiations sufficiently penetrating to cause it to emit its characteristic radiation.

The method of experimenting was as follows:—The secondary radiators successively placed in the position R_1 (see fig. 1) were subjected to the primary rays proceeding from the anticathode A, R_2 being temporarily removed. The ratio of the deflexions of the gold-leaves in the electroscopes E_2 and E_1 obtained from the microscope readings was determined in the case of each radiator; let r_1 be the value of this ratio and r_1' be the value of the ratio with no metallic radiator in the position R_1, the air in its neighbourhood acting as a source of secondary radiation.

Next, the particular tertiary radiator under examination was placed in the position R_2 (as previously defined), and the secondary radiators were again in turn subjected to the primary rays. Ionization was now produced in the chamber I and deflexions of the gold-leaf in electroscope E_3 were obtained. Let the value of the ratio of the deflexions of the gold-leaves in E_3 and E_1 be r_2, and when the tertiary radiator was removed and the air alone in its neighbourhood acted as a source of tertiary rays, let the corresponding value of the ratio be r_2'.

The tertiary radiator R_2 was then replaced and the ratio r_2 again found, and then finally removing R_2 the readings for the ratio r_1 were repeated.

Working with an X-ray bulb, having an auxiliary spark-gap, it was found that in the steady state the two sets of values for r_1 and likewise those for r_2 showed agreement to within 2 per cent.

In no case was the air-effect r_1' more than 1 per cent. of the value of r_1, in most cases less than $\frac{1}{4}$ of 1 per cent.; and in no case was the air-effect r_2' more than 2 per cent. of the value of r_2 when the characteristic homogeneous radiation was excited.

In those cases where the penetrating power of the secondary beam did not exceed that of the characteristic radiation from the tertiary radiator employed, the air-effect r_2' became of considerable importance, being as high as $\frac{1}{4}$ of r_2 in some cases. The author hopes to obtain more reliable data in these particular cases in the course of further experiments.

It will be seen later, however, that an accurate knowledge of these particular data is not essential in the present investigation.

It was estimated that if in all other cases $\frac{1}{4}$ of the air-effect was taken in addition to the normal leakage, and this subtracted in each case from the direct effect when the metallic radiator was in position, the resulting ratios finally obtained would be accurate to at least 2 per cent. If a denote the ratio of the normal leakage in any given time in the electroscope E_2 to the deflexion in E_1 in the same time, with a primary beam as during the actual experiment, then since the normal leakage is compensated for in E_2, the ratio of the ionizations in the electroscopes E_2 and E_2 due to the homogeneous radiations

$$= \frac{r_2 - \frac{1}{4} r_2'}{r_1 - \left[\frac{1}{4} r_1' + a\right]} = \mathrm{R} \text{ (say)}.$$

If the absorption coefficient of the homogeneous radiation from a given tertiary radiator by Al be denoted by λ_2, and that of the exciting secondary radiation by λ_1, then as long as $\lambda_1 =$ or is $> \lambda_2$ the value of R is quite small and approximately constant. As λ_1 decreases through the value λ_2 we get a rapid increase in the value of R, and for a comparatively small subsequent increase in penetrating power of the secondary beam, values of R 30 to 40 times as big as the previous steady values are obtained; this increase corresponding to the excitation of the characteristic tertiary radiation.

Let us consider the tertiary radiation emitted normally from a given radiator of area S upon which a uniform parallel beam of secondary rays is incident normally.

Let us define a quantity k, such that the fraction of the energy of the secondary beam passing normally through a thin layer δx of the tertiary radiator which is transformed into tertiary radiation is $k\delta x$. Thus, if I be a measure of the energy passing normally per second through unit area of the tertiary radiator at a depth x below the surface, the energy transformed per second in a layer $\delta x = \mathrm{I}k\delta x$.

Now Barkla * has shown that when this homogeneous

* Barkla, Phil. Mag. Feb. 1908, pp. 288-296.

type of radiation is excited, it is practically evenly distributed in all directions: consequently, the energy passing per second as tertiary radiation from the layer δx through the tissue-paper-covered window of an electroscope of the Wilson type (I being constant over the whole area of the radiator)

$$= \frac{\omega}{4\pi} \mathrm{S} \mathrm{I} k \delta x e^{-\lambda_2 x}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where λ_2 is the absorption coefficient of the tertiary radiation by the material of which the tertiary radiator is composed, and ω is the mean solid angle subtended by the aperture of the electroscope at all parts of the radiating area.

But if I_0 is a measure of the energy incident normally per second on unit area of the tertiary radiator at the surface,

$$\mathrm{I} = \mathrm{I}_0 e^{-\lambda_1 x},$$

where λ_1 is the absorption coefficient of the secondary beam by the material of which the tertiary radiator is composed, and the whole energy passing into the electroscope per second from the tertiary radiator

$$= \frac{\omega}{4\pi} \mathrm{S} \mathrm{I}_0 k \int_0^\infty e^{-(\lambda_1+\lambda_2)x} . dx \quad . \quad . \quad . \quad . \quad (2)$$

$$= \frac{\omega}{4\pi} \mathrm{S} \mathrm{I}_0 k \frac{1}{\lambda_1+\lambda_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If now we remove the tertiary radiator, and place the electroscope previously used in the path of the secondary beam, with the centre of its tissue-paper-covered window occupying the position previously occupied by the centre of the radiator, and with its plane perpendicular to the axis of the secondary beam, the energy passing per second through the window, if its area is A, is $\mathrm{I}_0 \mathrm{A}$.

Therefore the ratio of the energy in the tertiary beam passing per second into the electroscope in the first position to the energy of the secondary beam passing per second into the electroscope in its second position

$$= \frac{\omega}{4\pi} \frac{\mathrm{S}}{\mathrm{A}} \cdot \frac{k}{\lambda_1+\lambda_2}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

There is considerable evidence that the ionization produced in a given volume of air by a beam of Röntgen radiation is

approximately proportional to the absorption of that radiation by the air. It has been found * also that the ratio of the absorption coefficients of homogeneous beams of different penetrating powers by any two substances, *e. g.* carbon and aluminium, in which no radiation of the homogeneous type is excited by the beam under consideration, is a constant. Also it has been found that the absorption of a beam of Röntgen rays by any substance depends only upon the quantity of matter present and not upon its state of aggregation.

It is assumed on the basis of these results, that the ionizations produced in a given volume of air by homogeneous beams of different penetrating powers are proportional to the absorption coefficients of these beams by carbon or aluminium.

Making this assumption, we have the ratio of the ionizations produced in the electroscope by the tertiary and secondary beams

$$= \frac{\omega}{4\pi} \frac{S}{A} \cdot \frac{k}{\lambda_1 + \lambda_2} \cdot \frac{\beta}{\alpha} \cdot = \frac{\mathfrak{I}_2}{\mathfrak{I}_1} \text{ (say)} \quad . \quad . \quad . \quad (5)$$

where α and β are the absorption coefficients by aluminium of the secondary and tertiary beams respectively. From (5) we find that k

$$= \frac{\mathfrak{I}_2}{\mathfrak{I}_1} \cdot \frac{\alpha}{\beta} \cdot \frac{A}{S} \cdot \frac{4\pi}{\omega} \cdot (\lambda_1 + \lambda_2).$$

In the above calculations we have considered the special case of normal incidence and emergence, but, since we are dealing with radiators sufficiently thick to absorb the whole of the incident radiation, it is easy to show that the results (4), (5), (6) are quite general for oblique incidence and emergence, where the incident and emergent beams make equal angles with the normal to the radiating surface.

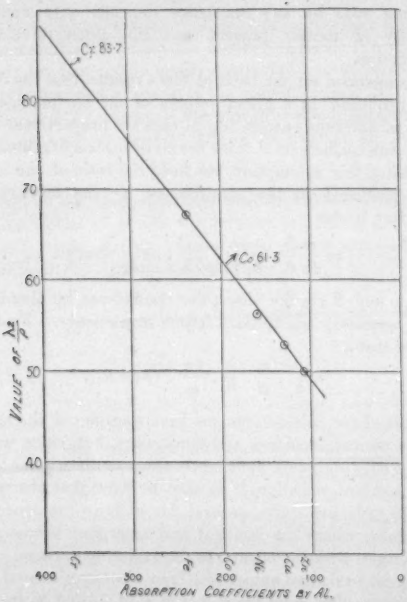
In order to deduce values of k from the experimental data, it is necessary to know the values of α, β, λ_1, and λ_2 for each of the tertiary radiators employed. α and β are readily obtained for the whole series and likewise λ_1 and λ_2 for Zn, Cu, Ni, and Fe. In the case of Cr and Co, being unable to obtain thin plates of these substances, the values for λ_1 and

* Barkla & Sadler, Phil. Mag. May 1909.

λ_2 could not be obtained directly but were deduced by the following method.

If the values of $\frac{\lambda_2}{\rho}$ (where λ_2 is the absorption coefficient by a substance of its own characteristic radiation and ρ its density) for the substances Fe, Ni, Cu, and Zn are plotted

Fig. 2.



as ordinates, against the absorption coefficients for the radiations from these substances by Al as abscissæ, it is found that the points obtained lie on a straight line, as shown in fig. 2.

The values of $\frac{\lambda_2}{\rho}$ for Co and Cr deduced from this graph are Co (61·3), Cr (83·7).

Moreover, a regular relationship is found to exist between the ratios of λ_1 for the radiations from consecutive members of the group Cr–Se for each of the absorbers Zn, Cu, Ni, and Fe. Taking the values of the absorption coefficients for any three of these absorbers for the radiations from the group Cr–Se, values for the coefficients for the fourth absorber could be deduced, starting with a value of λ_2 as a basis and using these observed relationships. In no case was the discrepancy between the experimental and deduced values greater than 2 per cent.

In a similar manner the remaining values of the absorption coefficients for Co and Cr were deduced, taking the values of λ_2 obtained above as a basis for calculation in each case.

TABLE V.

	Values of λ used in calculation of k.							Values of λ' used in calculation of the ratio $\frac{k}{k'}$.					
	ABSORBERS.												
RADIATORS.	Al.	Cr.	Fe.	Co.	Ni.	Cu.	Zn.	Cr.	Fe.	Co.	Ni.	Cu.	Zn.
Chromium...	367	544											
Iron	239	2500	514					2150					
Cobalt	193·2	2076	521	545				1785	102				
Nickel.....	159·5	1730	2440	584	482			1490	2090	134·0			
Copper	128·9	1478	2080	2560	537	474		1279	1798	2194	150		
Zinc	106·3	227	1715	2172	2275	497	361	1072	1485	1875	1936	120·1	
Arsenic	60·7	755	1040	1333	1422	1575	1464	666	915	1161	1232	1356	1260
Selenium ...	51·0	652	903	1159	1211	1340	1258	577	800	1014	1055	1157	1080
Silver.....	6·75					214	195·1					189·2	170·0

It will be seen later that it is desirable to know what fraction of the absorption of the exiting radiation by the material of the radiator is directly concerned in the transformations of energy which are taking place during the

passage of the secondary beam through the substance of the tertiary radiator.

It has been found* that if the values of λ_1 for the radiations from the group Cr to Ag for any absorber, such as silver, in which no homogeneous radiation is excited by the radiations from any member of the group, be plotted as ordinates against the corresponding values of λ_1 for aluminium, in which also no homogeneous radiation is excited, as abscissæ ; the points so obtained lie on straight lines, and these straight lines pass through a common point.

Similarly, if we plot as ordinates the values of λ_1 for any of the early members of the group Cr–Ag as absorber, in which the homogeneous type of radiation *is* excited by the radiation from those members of the group of higher atomic weight ; it has been found, that up to the point corresponding to the atomic weight of the absorber, the relation is also linear, and if the straight line be produced, it passes through the common point previously mentioned. The values of λ_1 for the radiation from the radiators of higher atomic weight are abnormally increased, the increase being attendant upon the excitation of tertiary radiation.

This increase is discussed in the paper on "The Absorption of X Rays" previously referred to.

In the actual experiments to determine the values of R with the various combinations of secondary and tertiary radiators, it was found necessary to alter the position of the quadrants of the tertiary electroscope from time to time, owing to a slight shift of zero, due to small progressive variations in the applied potentials. In this way small changes in the sensitiveness of the instrument were introduced. The sensitiveness could, however, be maintained constant during a period of three or four hours during any given day.

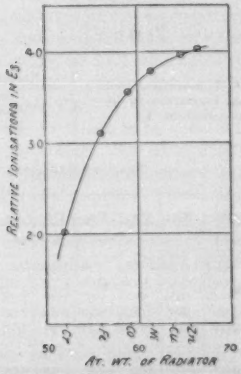
In order to obtain true relative values for the ionizations, the secondary radiation from arsenic was made to fall upon each of the tertiary radiators placed successively in the position R_2, the sensitiveness of the electroscope remaining constant during this series of observations. Values of R were obtained in this way for Zn, Cu, Ni, and Fe as tertiary radiators, the

* Barkla & Sadler, Phil. Mag. May 1909.

surfaces in each case being brightly polished, of the same size, and thick enough to totally absorb the incident beam. These values of R plotted as ordinates against the atomic weight of the radiators as abscissæ, were found to lie on a smooth curve (see fig. 3).

In the case of Cr and Co, being unable to obtain them pure in the form of flat plates, radiators were made in the following manner. A flat plate of aluminium, of the same size as the required radiator, was covered with a thin layer of

Fig. 3.



adhesive (previously tested and found to emit no detectable radiation of the homogeneous type), and then the metals, finely powdered, were pressed into this layer, the surface being covered with powdered metal. A sample radiator prepared with pure iron filings in this way emitted a tertiary radiation indistinguishable (by absorption tests) from that produced from a plate of the metal. This, however, is quite to be expected since practically the whole of the tertiary radiation from iron is obtained from a layer ·003 cm. thick.

But the value of R for a prepared plate of iron filings is a little less than that for a flat polished plate of iron of equal size placed in an identical position, owing to the irregularities

of the surface reducing the effective area. Three equal prepared plates of Co, Fe, and Cr and a polished plate of iron were therefore used, the powders being as nearly as possible equally fine, and the relative values of R obtained. From a comparision of the two values of R for iron, values of R for polished plates of Co and Cr could be deduced, and these were found when plotted to lie almost exactly upon the curve in fig. 3.

From the set of values thus obtained the true relative values of R for the complete series of experiments previously described were deduced and are given below in Table VI.

TABLE VI.

	Values of R calculated from the *total* Ionization in the Ionization-chamber I.						Values of R′ calculated from the *normal tertiary* Ionization alone in the Ionization-chamber I.					
	Tertiary Radiators.											
Secondary Radiators.	Cr.	Fe.	Co.	Ni.	Cu.	Zn.	Cr.	Fe.	Co.	Ni.	Cu.	Zn.
Cr	·076	·107	·109	·143	—	—	—	—	—	—	—	—
Fe	1·76	·076	·091	·118	·126	·124	1·68	—	—	—	—	—
Co	1·81	·42	·097	·118	·100	·106	1·73	·327	—	—	—	—
Ni	1·95	2·64	·466	·101	·084	·087	1·87	2·45	·366	—	—	—
Cu	1·92	2·64	3·00	·582	·093	·110	1·84	2·55	2·90	·472	—	—
Zn	2·02	2·88	3·37	3·53	·528	·115	1·94	2·79	3·27	3·42	·488	—
Ga ?	...	...	...	...	...	...	...	...	...	...	...	...
Ge ?	...	...	...	...	...	...	...	...	...	...	...	...
As	2·02	3·10	3·57	3·79	3·97	4·03	1·94	3·01	3·47	3·68	3·86	3·92
Se	1·86	2·87	3·11	3·47	3·61	3·86	1·78	2·78	3·01	3·36	3·50	3·75
...	...	...	...	...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...	...	...	...	...
Ag	—	—	—	—	1·35	1·40	—	—	—	—	1·24	1·29

However, it is found that when zinc is used as a tertiary radiator and the secondary beams from iron to zinc fall upon it successively, a small value R is obtained, even though none of the characteristic homogeneous radiation is excited by these secondary beams. The ionization in the chamber I in

these cases is due, partly to the scattering of the secondary radiation, partly to the production of tertiary radiation by the small percentage of scattered primary radiation in the secondary beam employed, and possibly in part to the production of a feeble type of very easily absorbed radiation much softer than the normal homogeneous radiation from zinc.

All evidence obtained up to the present points to the persistence of these residual effects even when the exciting secondary beam is sufficiently penetrating to produce the characteristic zinc radiation, and since in the calculation of k we only require a measure of the homogeneous tertiary radiation excited by the homogeneous secondary beam, these residual effects have been estimated and subtracted in each case.

The values of R finally obtained, which we may denote by R′, are given in Table VI.

The next step was to calculate in some one particular case an actual value of k from the data obtained by using a suitable pair of secondary and tertiary radiators. A flat polished plate of pure copper (3·04 cm. × 3·03 cm.) was mounted on a fine aluminium stem and placed in the position R_2. The ionization-chamber connected to the tertiary electroscope E_2 was removed and replaced by the electroscope E_2. The distance from the centre of the radiator to the centre of the tissue-paper-covered window of E_2 was 4·7 cm., the area of the window being the same as in previous experiments. The plane of the radiator being vertical with the normal to its surface bisecting the angle between the axis of the secondary beam and the normal to the aperture at its centre, of the electroscope E_2.

From the relative positions of the tertiary radiator and the window of E_2 (carefully determined) a value of ω was calculated and found to be approximately ·222.

The ratio of the ionizations in the electroscopes E_2 and E_1 was then determined, allowance being made for the air-effect in the manner previously explained, and the mean of several readings, none of which differed by more than 2 per cent. from the mean, was found to be ·115.

The tertiary radiator was then removed and the electroscope E_2 was placed with its window perpendicular to the axis of

the secondary beam, the centre of the window being in the position previously occupied by the centre of the tertiary radiator. Since the area of the window of E_2 was almost exactly equal in shape and size to the area of the tertiary radiator projected in a plane perpendicular to the axis of the secondary beam, the ionization will correspond to the mean intensity over the area of the tertiary radiator in the previous part of the experiment. The mean value of the ratio of the deflexions in E_2 and E_1 was now equal to 11·48.

Substituting the values of α, β, λ_1 and λ_2 from Table V., we find

$$k = \frac{\cdot 115}{11\cdot 48} \times \frac{60\cdot 7}{128\cdot 9} \times \frac{6\cdot 0}{9\cdot 2} \times \frac{4\pi}{\cdot 222} \times (1575+488) = 361$$

The value of λ_2 was increased by 3 per cent. to allow for the obliquity; the value of k was, however, affected by less than 1 per cent.

In the series of experiments previously mentioned, in the course of which a copper radiator was subjected to secondary beams from the group of metals Cr–Ag, we obtained values of R′ with arsenic as radiator, and in these cases

$$k = \frac{R'}{\mu} \cdot \frac{\alpha}{\beta} \cdot \frac{A}{S'} \cdot \frac{4\pi}{\omega'} (\lambda_1 + \lambda_2),$$

where μ is the ratio of the sensibility of the electroscope E_3 to that of E_2; S′ is the area of the tertiary radiator and ω' the mean solid angle subtended by the radiator R_2 at the window of the ionization-chamber I; and α, β, λ_1, and λ_2 as defined above and corrected for the obliquity of the rays. This correction was determined experimentally.

Equating these values of k, we find

$$\frac{4\pi A}{\mu \omega' S'} = \cdot 093,$$

and we may therefore write

$$k = \cdot 093 \times R' \times \frac{\alpha}{\beta} \times (\lambda_1 + \lambda_2).$$

The value of k calculated from the data given in Tables V. and VI. are given below.

TABLE VII.

	Values of k.						Values of $\frac{k}{\lambda'}$.					
	TERTIARY RADIATORS.											
SECONDARY RADIATORS.	Cr.	Fe.	Co.	Ni.	Cu.	Zn.	Cr.	Fe.	Co.	Ni.	Cu.	Zn.
Iron	378						·176					
Cobalt	275	31·5					·154	·309				
Nickel	216	526	38·4				·145	·252	·287			
Copper	162	403	670	43·2			·127	·225	·305	·288		
Zinc	117·5	307	528	657	43·0		·109	·207	·282	·340	·358	
Arsenic	50·5	137	227	288	390	412	·0757	·150	·195	·233	·288	·327
Selenium ...	36·0	96·0	151·0	197	267	300	·0624	·120	·149	·187	·231	·278
Silver					5·35	5·30					·0282	·0313

NOTE.—In the case of each tertiary radiator, when the secondary beam is only just more penetrating than the radiation characteristic of the tertiary radiator, the value of k is small and cannot be determined nearly as accurately as the later values; nor can the value of λ' corresponding to the initial small value of k be determined with accuracy. Consequently the initial values of $\frac{k}{\lambda'}$ are somewhat uncertain, and it is impossible to determine from them whether $\frac{k}{\lambda'}$ for any given tertiary radiator rises to a maximum, and then decreases in value, or whether it continuously decreases from the initial value.

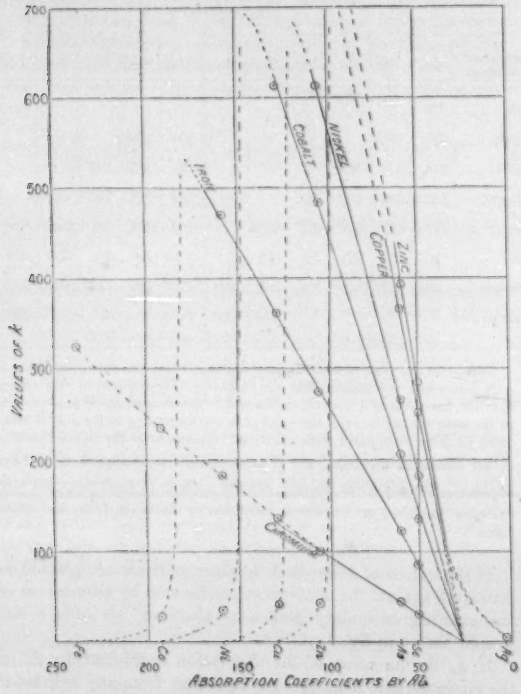
If the values of k for each tertiary radiator are plotted as ordinates against the absorption coefficients by aluminium of the exciting secondary beams as abscissæ, we obtain the graphs shown in fig. 4, p. 768.

If λ_2 be the value of the absorption coefficient by Al of the characteristic homogeneous radiation from any substance A and λ_1 that of the exciting radiation (also homogeneous), an examination of the graphs in fig. 4 leads to the following conclusions:—

(1) If λ_1 is equal to or greater than λ_2, the intensity of the homogeneous radiation emitted by A is inappreciable.

(2) If λ_1 is slightly less than λ_2 the intensity is measurable but still small.

Fig. 4.



(3) For a further slight decrease in λ_1 a very rapid increase in the intensity of the radiation from A takes place, and for a value of λ_1 not greatly less than λ_2 the intensity reaches a maximum value.

(4) Beyond this point, over a considerable range, as λ_1 decreases the intensity decreases as a linear function of λ_1. For this range the relationship may be represented by $k=a(\lambda_1-b)$, where a and b are constants. For the group of tertiary radiators employed, b has the same value for each member of the group, while a increases with the increase of the atomic weight of the tertiary radiator.

But the absorption by Al of the radiation from the group of metals Cr–Ag is proportional to the absorption by air of these radiations, as previously mentioned; and it has been shown that the intensity of the homogeneous radiation excited in a thin sheet of copper by a primary beam is proportional to the ionization produced in air by the same beam *: we may, therefore, over this range, write $k=a(ci-b)$, where i is a measure of the ionization produced in 1 c. c. of air by the exciting beam, and c the ratio of the absorption by 1 c.c. of air to the ionization produced in it in consequence of that absorption, a and b having the same values as before. (It will be seen from the Zn and Cu curves that there is a departure from this linear relationship for very penetrating beams.)

When a secondary homogeneous Röntgen beam falls normally upon a tertiary radiator, the amount of energy absorbed per sec. per unit area of its surface in a layer of thickness δx at a distance x below the surface is $I\lambda_1\delta x$, where I is the energy crossing unit area of the surface per second at the depth x below the surface and λ_1 is the absorption coefficient by the substance of the tertiary radiator of the secondary beam. If, however, we confine our attention to the absorption which is directly involved in the process of emission of tertiary rays, we must substitute λ' for λ_1, where λ' is the increase in the value of the absorption coefficient, and the amount of energy so absorbed per second in this layer is $I\lambda'\delta x$. [The values of λ' are given in Table V.]

But the amount of energy emitted as tertiary radiation per second from this layer in consequence of such absorption is, as we have seen, $Ik\delta x$.

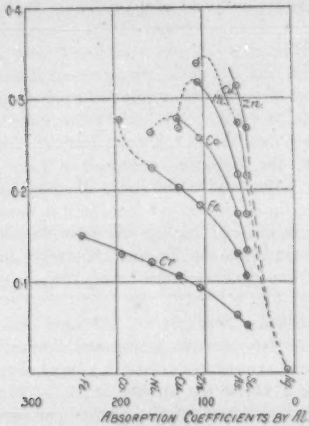
Therefore $\frac{k}{\lambda'}$ is a measure of that fraction of this special

* Barkla & Sadler, Phil. Mag. Oct. 1908, pp. 550–584.

absorption of energy which is re-emitted as energy of tertiary radiation.

The values of $\frac{k}{\lambda'}$ are given in Table VII., and the curves obtained by plotting these values as ordinates against the values of λ_1 by Al of the exciting radiation as abscissæ are shown in fig. 5.

Fig. 5.



For any given tertiary radiator the value of $\frac{k}{\lambda'}$ attains to a maximum for a value of λ_1 in the neighbourhood of that which gives a maximum value of k. As the exciting radiation becomes more penetrating this fraction diminishes slowly at first and then more rapidly.

Taking the case of copper as a tertiary radiator for example, the maximum value of the fraction which corresponds to a value of λ_1 about 89 is nearly 2/5, when λ_1 decreases to 51 the fraction has fallen to about 1/4, and for a value of $\lambda_1 = 6{\cdot}96$ to about 1/40.

At the present stage of the inquiry the cause of this rapid decrease in the value of $\frac{k}{\lambda'}$ can merely be a matter for

conjecture, but it is hoped that the data obtained in subsequent experiments will throw more light upon the phenomena*.

It will be observed that in determining these relationships the atomic weight of nickel is nowhere assumed, but the results obtained all point to the conclusion that the behaviour of nickel, whether as radiator or absorber of X-rays, is such that we should expect its atomic weight to lie between those of cobalt and copper.

The evidence for this, obtained during the course of these experiments, may be briefly summarized as follows :—

(*a*) The secondary homogeneous radiation from cobalt excites *no* homogeneous tertiary radiation in nickel.

(*b*) The secondary homogeneous radiation from nickel excites *no* homogeneous tertiary radiation in nickel.

(*c*) The secondary homogeneous radiation from nickel *does* excite homogeneous tertiary radiation in cobalt, and to an extent to be expected for the radiation from a substance of atomic weight about midway between those of cobalt and copper.

But it has been shown for the group of metals Cr–Ag that

(1) The penetrating power of the radiation characteristic of any member of the group increases with its atomic weight †.

(2) The radiation characteristic of a substance is only excited by a more penetrating radiation ‡.

If the atomic weight of nickel were greater than that of cobalt, then by (1) we should expect to find its characteristic radiation to be more penetrating than that of cobalt, and consequently by (2) that the radiation from nickel *should* excite in cobalt its characteristic radiation, while the radiation

* *Note.* Experiments now in progress indicate that when the exciting beam is more penetrating than the radiation characteristic of the tertiary radiator, part of the energy absorbed reappears as an easily absorbed corpuscular radiation, and this effect becomes more pronounced as the exciting beam becomes more penetrating.

† Barkla & Sadler, Phil. Mag. Sept. 1907, pp. 812-828; also Phil. Mag. May 1909.

‡ Barkla & Sadler, Phil. Mag. Oct. 1908; also present paper.

from cobalt should *not* excite the radiation characteristic of nickel. If, on the other hand, the atomic weight of nickel were less than that of cobalt, then the results would be the reverse of the former.

It has been suggested that the anomalous behaviour of nickel may be due to oxidation of the cobalt or the nickel. The author has found, however, that if iron be used as a tertiary radiator, and the secondary radiations from the group of metals Cr–Ag successively fall upon the iron in the manner described earlier in the paper, the results obtained are exactly similar whether the radiator is coated with a layer of rust or brightly polished, the relative values of k being the same in the two cases.

The result (*b*) would not have been obtained if the nickel experimented with had contained an impurity which emitted a characteristic type of radiation "harder" or "softer" than that characteristic of nickel. For the harder component of the secondary beam (whether due to the nickel or the impurity) would have excited the softer component radiation in the tertiary radiator, and this would have been detected if the impurity had been present to any appreciable extent.

Summary of Results.

(1) The characteristic Röntgen radiation from a substance is found to be identical in character whether it is emitted as a secondary or a tertiary radiation. In each case (*a*) the radiation is homogeneous, (*b*) the absorption coefficients of the radiation by other substances are the same, and independent of the penetrating power of the exciting radiation by which these homogeneous beams are produced.

(2) Using the homogeneous secondary beams emitted by the members of the group of metals Cr–Ag, excited by suitable primary beams, the emission of tertiary radiation by the earlier members of the group Cr–Ag when excited by these secondary beams was found to be governed by the following laws:—(*a*) With a given substance as radiator, its characteristic radiation is only excited by those secondary beams which are more penetrating than the tertiary radiation characteristic of the substance. (*b*) When the secondary beam is only just more penetrating than the tertiary the

intensity of the latter is small, but as the secondary beam becomes more penetrating a very rapid increase in the intensity of the tertiary radiation to a maximum takes place. (*c*) As the secondary beam becomes more penetrating still, the intensity of the tertiary radiation decays as a linear function of the ionization produced in a given volume of air by the secondary beam.

(3) It has been found that when the secondary homogeneous beams from the group of metals Cr–Ag are absorbed by thin sheets of metals from the same group, a big increase in the absorption takes place when the secondary beams become more penetrating than the radiation characteristic of the absorber, this increase in the absorption being intimately connected with the emission of tertiary radiation by the absorber in these circumstances (see paper on "The Absorption of X-Rays," Barkla & Sadler, Phil. Mag. May 1909).

The fraction of this increase in the absorption of the energy of the secondary beam, which is re-emitted as tertiary radiation, is not constant, but decreases as the secondary beam becomes more penetrating, slowly at first, and then more rapidly when a very penetrating secondary beam is used.

In conclusion I wish to thank Dr. Barkla for the interest he has shown throughout this research, and especially for his kindly criticism and advice during the writing of this paper.

George Holt Physics Laboratory,
University of Liverpool,
24th March, 1909.

Discussion.

Prof. Bragg congratulated the Author on his interesting experiments, and said he could not see any satisfactory explanation of them on the pulse theory.

LII. *On a Bifilar Vibration Galvanometer.*

By W. DUDDELL, *Fellow of the Physical Society.*

In a paper read before this Society in May 1907, Mr. Albert Campbell drew attention to the great advantages of vibration galvanometers in the measurement of mutual inductances, and he further described a new type of vibration galvanometer which he had designed.

During the last few years the use of vibration galvanometers for the measurement of capacities, self and mutual inductions has greatly extended, and in the near future they may be expected to supersede the telephone and possibly the secohmmeter, for these purposes. As there are very few published data about the sensibility, etc., of these instruments I think it may be of interest to describe what I believe to be a new type and a series of tests made on it.

Vibration galvanometers like ordinary galvanometers may be broadly divided into two classes,—those in which the moving part consists of a piece of iron or steel and the current to be measured flows round fixed coils as in the case of the Thomson galvanometer,—those in which the current to be measured flows round a moving coil placed in a fixed magnetic field on the siphon recorder principle.

The vibration galvanometers of Max Wien and Rubens belong to the first class, while Mr. Campbell's moving coil vibration galvanometer belongs to the second, and so does the new bifilar instrument described below.

Generally speaking, when one requires to build a sensitive instrument having a short periodic time it is necessary to reduce as far as possible the mass of the moving parts in order to combine high sensibility with short period.

Further, in the case of a vibration galvanometer, it is also necessary to keep the damping forces as small as possible, as the sensibility to alternating currents depends very greatly on the magnification one can obtain by bringing the instrument into tune or resonance with the alternating current to be measured. These considerations have led me to construct a vibration galvanometer in which the mass of the moving

* Read May 14, 1909.

parts is reduced to a minimum, the moving coil being reduced to the two wires forming its two sides, similar to a bifilar oscillograph, but with this difference:—Whereas the bifilar oscillograph is designed so as to make the damping aperiodic, the bifilar vibration galvanometer is designed so as to keep the damping as small as possible.

The design of the instrument* is shown in fig. 1, in which a, b, c, d, is a fine bronze wire passing over a pulley p, and stretched tight by means of a spring, the tension on the spring being capable of variation by a milled head. The wires carry a mirror M in the centre and are placed in a strong magnetic field between the poles N and S of a magnet. The wires pass over two bridge pieces B, B, which limit the length of the wires which is free to vibrate. These two bridge pieces can be moved nearer together or further apart by means of a right and left handed screw as required. The current to be measured passes up one wire and down the other, causing one wire to tend to move forward and the other back in the magnetic field and so tilts the mirror M through a small angle.

Fig. 1.

The periodic time of the wires depends on their mass, length, and tension, as well as upon the moment of inertia of the mirror. In a completed instrument the moment of inertia of the mirror and the mass of the wires are fixed, but their length and tension can be altered in order to adjust the periodic time. Fig. 2 shows the relationship between the free length of the wires and the frequency of the free vibrations of the instrument for different tensions, and fig. 3 gives the relationship between the frequency and the tension for a series of different lengths. It will be seen from the curves that the total range of frequency obtainable with the instrument is very large, namely,

* Messrs. Nalder Bros. are manufacturing the instruments.

from about 90~ per second up to 1900, though the wires are rather too loose below 100~ per second.

Fig. 2.

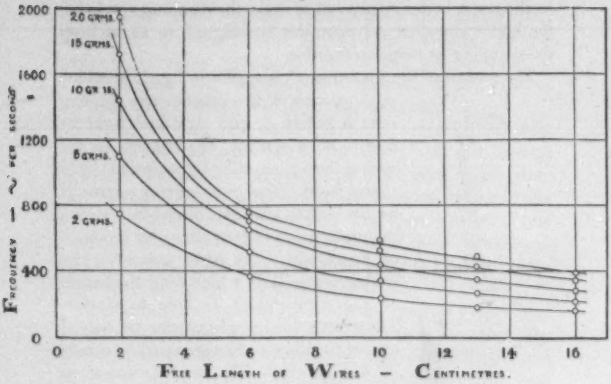
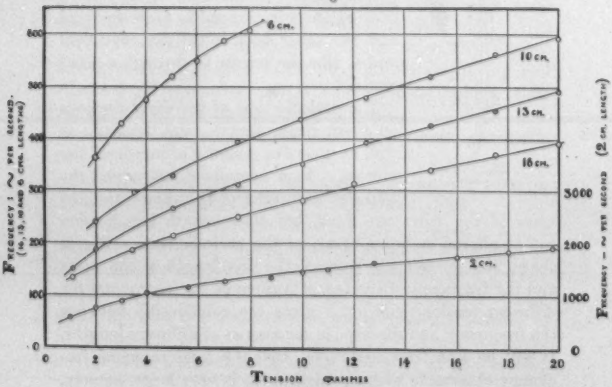


Fig. 3.



As the damping in this instrument is very small the resonance is very sharp. Figs. 4 and 5 show the amplitude

Fig. 4.

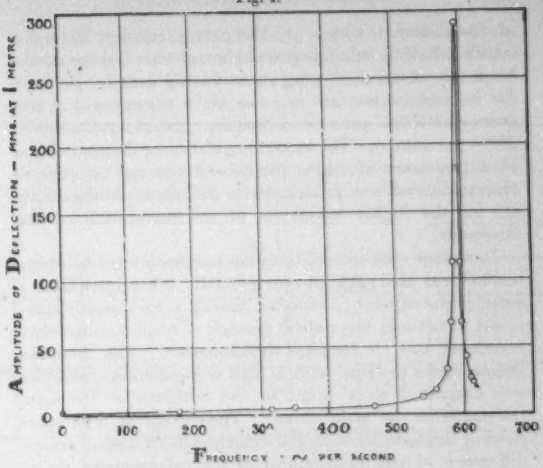
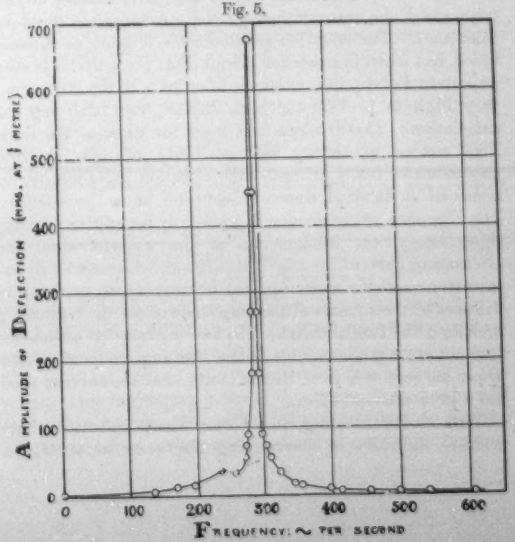

Amplitude of Deflection — mms. at 1 metre
300
250
200
150
100
50
0
0
100
200
300
400
500
600
700
Frequency : ~ per second

Fig. 5.


Amplitude of Deflection (mms. at 1 metre)
700
600
500
400
300
200
100
0
0
100
200
300
400
500
600
Frequency: ~ per second

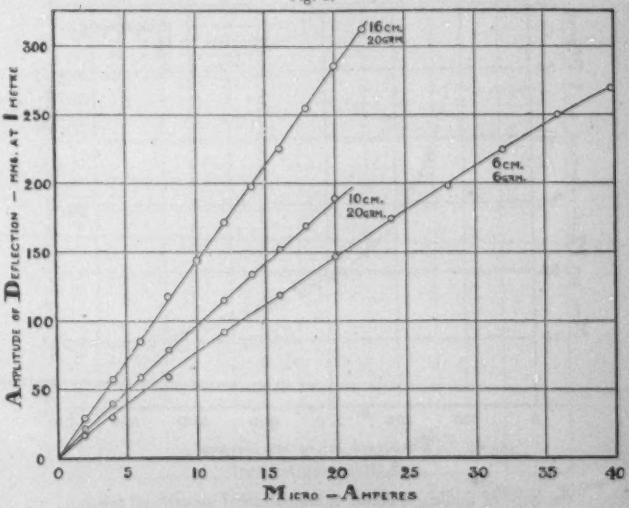
of the deflexion where an alternating current, having a constant R.M.S. value, is passed through the instrument, the frequency of the alternating current being varied. In fig. 4 the instrument was set so as to be in resonance at a frequency of 595~ per second, and in fig. 5 at a frequency of 290~ per second. The small irregularity in the curve fig. 5 at a frequency of 240~ per second was not an error of observation, but was, I think, due to the instrument resonating one of the higher harmonics of the wave form of the alternator.

To measure the sensibility of the instrument the following method was used:—A current generally 0·1 ampere from a small high-frequency alternator, having a very nearly sinusoidal wave-form, was passed through a small non-inductive resistance and an accurate dynamometer. The vibration galvanometer in series with a high non-inductive resistance was connected as a shunt to the terminals of the small resistance in the main circuit. The current of 0·1 ampere flowing through the main resistance produced a small known difference of potential applied to the galvanometer circuit from which the current through the galvanometer could be calculated. The vibration galvanometer it must be remembered, has when in operation a back E.M.F., so that it is very important to keep the resistance in series with the instrument very high to prevent the back E.M.F. from falsifying the calculation; 25,000 ohms was used for most of the tests. This method of testing the sensibility of the vibration galvanometer really calibrates the vibration galvanometer in terms of a standard dynamometer and known resistances. The vibration galvanometer is practically insensitive to anything except the fundamental of the wave-form of the alternating current for which it is tuned, whereas the dynamometer reads the mean squared current which is equal to the sum of the squares of the amplitude of all the harmonics including the fundamental. The two instruments are therefore not strictly comparable unless the amplitude of all the upper harmonics is zero, that is to say, that the current used has a pure sine wave-form.

With an instrument giving a very sharp resonance there is some difficulty in determining the exact value of the

maximum amplitude, a fact which prevents such consistent results being obtained as would otherwise be the case. When the instrument is tuned so as to be in resonance with the alternating current to be measured the amplitude of the vibration is practically proportional to the R.M.S. current, as is shown by the tests recorded in fig. 6, hence it is permissible to quote the sensibility in millimetres of amplitude at a metre per R.M.S. microampere. I have therefore adopted this method.

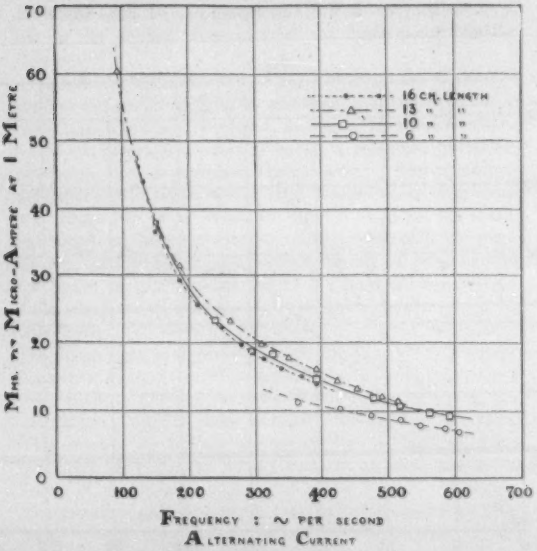
Fig. 6.



In fig. 7 (p. 780) the relationship between the sensibility of the instrument in millimetres of amplitude per R.M.S. microampere, and the frequency of the alternating current is given. The different curves refer to different free lengths of the wires. One interesting point of this figure is that where it is possible to tune this vibration galvanometer to a given frequency, using various combinations of length and tension,

the sensibility so obtained is very nearly the same as long as the wires are longer than the pole pieces. The practical result of this is that if at any given length of wire or tension, one can tune the instrument to suit the frequency of the alternating current, then no further adjustments need be made in

Fig. 7.



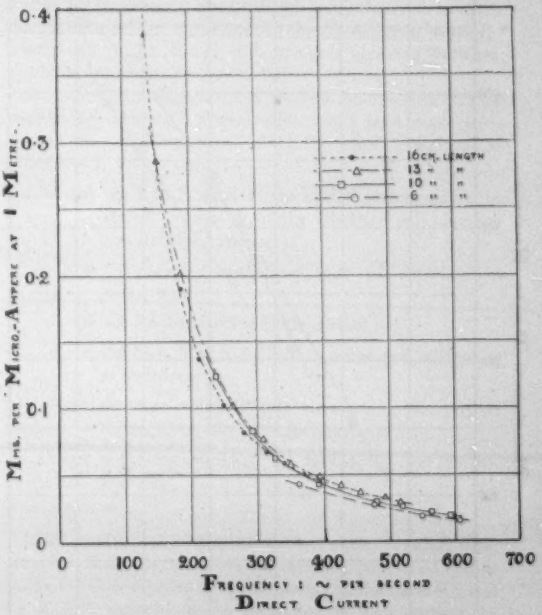
the hope of finding a better combination of length and tension for the purpose.

With each adjustment of the instrument in fig. 7 to suit the various frequencies, the sensibility of the instrument was tested with direct current as an ordinary galvanometer (fig. 8).

It will be noted that the sensibility to alternating current decreases very nearly inversely as the frequency for which

the instrument is adjusted, whereas for direct current the sensibility decreases approximately inversely as the square of the frequency for which the instrument is adjusted, which is what usually takes place with direct current galvanometers. By dividing corresponding ordinates in figs. 7 and 8 a new

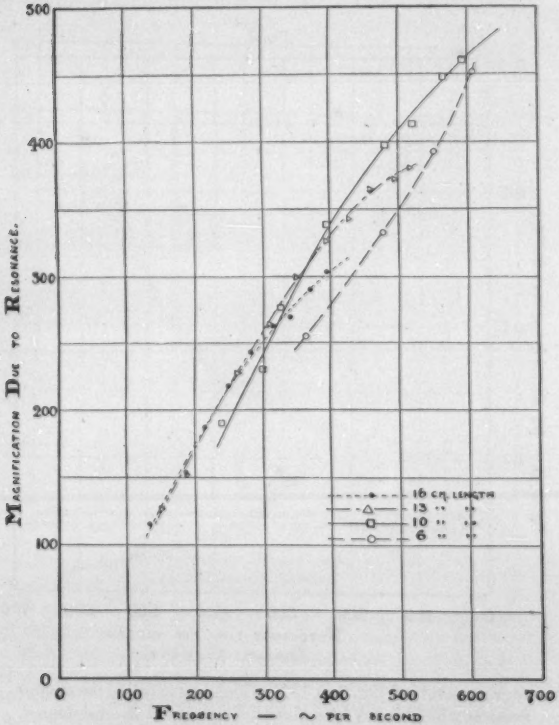
Fig. 8.



curve is obtained (fig. 9, p. 782), which I have called the magnification, which shows for a given adjustment of the instrument, how much more sensitive it is to an alternating current of the proper frequency than to a direct current. The highest value obtained in the curves is about 460. Had the vibration

galvanometer been aperiodic and of sufficiently short period to follow the alternating current then the alternating current sensibility would have been only $2\sqrt{2}=2{\cdot}82$ times the direct current sensibility.

Fig. 9.



The practical applications of the vibration galvanometer nearly all involve using the instrument in some form of bridge or null method for determining when a small difference

of potential vanishes, that is to say the instrument is generally used as a sensitive detector for small alternating voltages. The resistance of the instrument used in these tests is 136 ohms, but owing to the back E.M.F. of the instrument its sensibility as a voltmeter must not be calculated on the basis of this figure. This resistance could be easily reduced by employing a more conducting material than hard phosphor bronze for the wires. There would be no objection to doing this except that the upper limit of the frequency to which the instrument could be tuned, without risk of breaking the wires, would be some somewhat reduced.

In view of the importance of the fact that the back E.M.F. reduces the sensibility of the instrument, I have made some tests to determine its value, using the same connexions as before :—

Let E = the R.M.S. value of the E.M.F. impressed on the circuit consisting of the vibration galvanometer and its added resistance ;

d = the corresponding amplitude of deflexion in millimetres ;

r = the total resistance of the circuit ;

e = the back E.M.F. (R.M.S. value per millimetre of deflexion) ;

c = the true value in microamperes of the current through the vibration galvanometer ;

k = the true sensibility in millimetres per microampere ;

then $d = ck$.

If the mechanical friction of the instrument is small, then, when it is tuned to resonance, the motion of the wires will be nearly 90° out of phase with the force, *i. e.* the current ; and the E.M.F. induced in the wires due to their cutting the magnetic field will also be 90° out of phase with their motion, so that the back E.M.F. will be approximately at 180° to the current. The E.M.F. sending the current through the vibration galvanometer may therefore be approximately

taken as $E-ed$, and the current

$$c = \frac{E-ed}{r}, \text{ whence}$$

$$r + ek = k\frac{E}{d}.$$

If therefore $\frac{E}{d}$ be plotted against r the resistance of the vibration galvanometer circuit, the graphs obtained should be straight lines which should not pass through the zero but should cut the zero line at a point ek which gives the apparent resistance of the instrument due to its back E.M.F.

This test has been carried out and the graphs are given in fig. 10. From these graphs the following results were obtained :—

Free length in cm.	Frequency.	Intercept in ohms.	Back E.M.F. in microvolts per millimetre.
5	538	30	4·8
10	532	130	14·8
13	530	165	18·4
16	228	205	8·4

The very large amount that the apparent resistance of the instrument is increased by its back E.M.F., over 200 ohms in one case, shows how very important it is to try to keep this back E.M F. as low as possible, and how little can be gained by reducing the real resistance of the wires themselves.

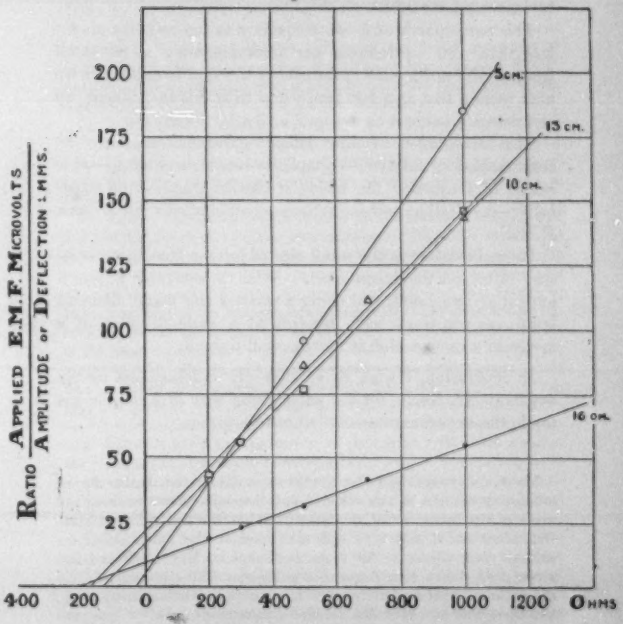
The fact that the back E.M.F. of the instrument is practically proportional to the deflexion and hence to the current through it, and so appears, as far as the outside circuit is concerned, like an addition to the resistance of the instrument, indicates that no arrangement of condensers is likely to improve matters as one might at first expect.

It will be seen that the intercept in the above table decreases for a given frequency with the length of the wires in use, and as the current sensibility does not change at all rapidly with the active length of the wires there is some

advantage in using the wires as short as possible when measuring small P.D.'s in a low resistance circuit.

I have not had an opportunity of testing the back E.M.F.

Fig. 10.



of other vibration galvanometers. In the case of the moving coil vibration galvanometer I should expect that it would be considerably larger than in the bifilar owing to the coil having a number of turns. I trust that the discussion on the paper may bring out some information on this point.

All the sensibilities so far given for the vibration galvanometer correspond to the use of a permanent magnet for the field. The instrument has, however, been tried in an oscillograph electromagnet, and the current sensibility was found to be increased threefold.

This corresponds at a 100 frequency to a sensibility of not less than 160 millimetres per microampere; as a small fraction of a millimetre movement of the spot is noticeable we may reckon that at a 100 frequency with this instrument we can detect a current as low as 2 or 3×10^{-9} ampere.

The advantages of the bifilar galvanometer may be summarized as follows:—Simplicity—ease in tuning—wide range of frequency for which it can be tuned—high sensibility—negligible self-induction—comparatively small back E.M.F.

Its main defect is the small size of mirror that it is necessary to use on the instrument. With a carefully adjusted optical arrangement, and using a small 4 volt metal filament lamp, one can work with comfort at a scale distance of a metre in a room which is not too well lighted.

In conclusion, I wish to express my indebtedness to my assistant Mr. Neale, for the painstaking way in which he has made the experiments recorded in this paper.

Discussion.

Mr. A. Campbell said that thanks were due to the Author for an interesting account of the valuable and thorough investigation he had made of the behaviour of single-loop bifilar vibration galvanometers. Galvanometers of this type had been used at the Reichsanstalt for several years (Zeitschr. für Instrumentenkunde, May 1906) and had given good results with frequencies as high as 2500 ~ per second. He did not know what sensitivity was obtained with the German instruments, but there was no doubt Mr. Duddell's pattern of the same type was much more sensitive than the instrument of the moving coil type which he (Mr. Campbell) described in 1907. Blondel pointed out in 1900 that in an oscillograph there was no advantage in having more than one turn of wire, but this appeared to be only true when the mirror was very small. As a large mirror with plenty of light and good definition greatly improved the ease and accuracy of observation, he gave the preference to a moving coil of more than one turn, even though the single-loop instrument was highly recommended by Dr. Orlich. With a moderately large mirror one could obtain good definition of a dark line with a scale distance of 600 cms. Mr. Campbell mentioned that Wien's vibration galvanometer (Ann. der Physik, 1901) with very small

magnets and minute mirror gave 70 mms. at 1 metre distance for 1 microampere at 100 ~ per second and had a resistance of 200 ohms. Mr. Duddell's curves were most instructive. He had verified experimentally the law which he (Mr. Campbell) had stated from theoretical considerations, namely, that with a given bifilar suspension (tuned to resonance) the sensitivity varies inversely as the frequency. The figures given tended to show that the shorter bifilars gave reduced sensitivity. With much shorter bifilars he had found great loss of sensitivity. In the moving coil type with many turns the back E.M.F. was considerable and might even double the effective impedance of the instrument and lower the volt-sensitivity accordingly.

Dr. J. A. FLEMING asked the Author if he had tried his instrument with intermittent currents of the right frequency. If the galvanometer was sensitive to currents such as are obtained by rectifying trains of oscillations from condensers, it might be useful as an optical call in wireless telegraphic stations.

Dr. RUSSELL congratulated the Author on the notable advance he had made in the design of vibration galvanometers. He much appreciated the clear presentation and the accuracy of the experimental results. He asked whether variations of the barometric pressure had any appreciable effect on the sensitivity of the instrument, and suggested that variations in the humidity of the atmosphere might possibly have some effect. So far as he was aware, the Author was the first to point out the importance of the back electromotive force due to the vibrating wires cutting the magnetic field. He thought that the experimental results given would be a great help in formulating a more exact mathematical theory of this type of apparatus. He showed that to a first approximation the Author's results were in agreement with those deduced from the differential equation ordinarily given for the motion of the mirror. He asked whether the frequency of ordinary alternating-current supply circuits was sufficiently steady to avoid the necessity of constant tuning of the apparatus.

The CHAIRMAN asked the Author if the mirror was always situated in the middle of the vibrating fibres.

The AUTHOR, referring to Mr. Campbell's remarks, said considerations of space prevented him from working with scale distances of 6 metres. With regard to Wien's instrument it was easy to obtain high current sensibility but not high voltage sensibility. He had tried Prof. Fleming's suggestion, but his instrument was not sensitive enough for ordinary signals. With more uniform spark frequency it gave excellent results. In reply to Dr. Russell, he stated that the sensitivity of the galvanometer did vary slightly with the barometric pressure. In the ordinary differential equation the term giving the moment of the applied forces needed amending, and so also did the term for the damping couple. The complete equation was complex and there were difficulties in the way of getting an exact solution. The frequency of ordinary supply circuits was quite constant enough for accurate work. In reply to Dr. Chree, he stated that the mirror was always symmetrically situated on the suspension.

LIII. *On a Method of Testing Photographic Shutters.*

By A. CAMPBELL, *B.A.*, *and* T. SMITH, *B.A.**

(From the National Physical Laboratory.)

[Plate XXIX.]

I. *Principle of the Method.*

THE importance of having accurate methods of testing photographic shutters is well shown by the number and variety of the methods that have from time to time been proposed and used for this purpose. The method described herein is intended to provide a rapid test, while ensuring the maximum of accuracy. The total duration of exposure for low speeds (say longer than ·1 sec.) is determined with an error not exceeding ·005 sec., and for high speeds not exceeding ·0005 sec. For exceptionally fast exposures, measurements can probably be made to within ·0001 sec. Attention has been specially directed to the elimination of all calculations, and a permanent record is obtained of each test.

The essential principle of the method consists in photographing on a moving plate a narrow beam of light reflected from a mirror which is forced to make angular vibrations of known frequency about an axis parallel to the direction of motion of the plate. There is thus obtained on the plate a sine curve, and if the light on its way to the plate passes through the shutter, the length of the exposure can be found by counting the number of vibrations recorded on the plate. The method has been rendered practicable by the fact that a Vibration Galvanometer affords a suitable means of imparting the necessary oscillations to the mirror; with this instrument both the amplitude of the vibration and the frequency are under control, and these are points of great importance. Owing to these conditions not being satisfied, and for other reasons, a tuning fork is not nearly so suitable for imparting vibrations directly to the mirror. When it is desired to find

* Read May 14, 1909.

the total duration of the exposure only, the general arrangement is shown in fig. 1. Light from a point source A is reflected by the mirror M of the vibration galvanometer G, through the shutter S on to the photographic plate P. The mirror makes oscillations in a horizontal plane, and the

Fig. 1.

amplitude is such that the vibrating beam nearly fills the horizontal diameter of the shutter aperture. The plate falls vertically, and in doing so sets off the shutter. In practice, instead of a point source of light a Nernst lamp is used with its filament vertical, and in front of the plate is placed a narrow horizontal slit T to limit the width of the record on the plate. Lenses L_1 and L_2 are employed to regulate the intensity of the light and the amplitude of the vibration recorded aswell as for focussing.

II. *Description of the Special Apparatus employed.*

(a) *Camera.*

A special camera has been constructed to hold the shutter S, the lens L_2, and the slit T (fig. 1). It is provided with brass ways down which the plate-holder and plate slide, and an adjustable mechanism for setting off the shutter. A side elevation is shown in fig. 2, and a view of the back in fig. 3. A is the base carrying the whole apparatus, and into it is fixed a vertical pin B. C is a piece of wood in which is bored a hole through which B passes freely. C slides between two portions of the base D of the camera, and can be held in any position by tightening the nuts E. The front of the camera, with the shutter and lens L_2, is supported by struts from C, and set so that the axis of the pin B, if produced, would pass through the centre of the shutter. By this means the whole vibrating beam of light will be able to pass through

the shutter though the whole camera is rotated through a few degrees about B. The back F of the camera is attached rigidly to the base D. The metal plate S carries a narrow horizontal slit T extending the whole width of the plate, and a window P is provided to facilitate focussing. G, G are

Fig. 2.

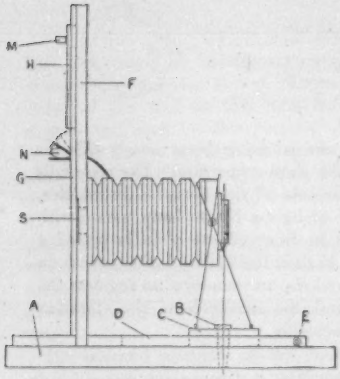
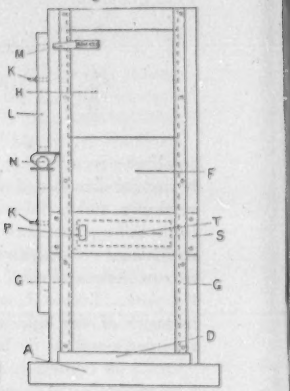


Fig. 3.



the grooved metal plates down which the plate-holder H slides. This holder carries a projection M which engages with the upper plate of a ball compressor N; when the holder falls this upper plate is forced down and the pneumatic bulb of the release is compressed. On the return upward journey of the holder the projection passes the compressor freely. The pneumatic release is carried on the moveable part L and may be fixed at any desired height by the butterfly screws K, K.

A number of records are taken side by side on one plate. This is secured by rotating the camera about B between each exposure. With this apparatus, ten or more records can be made side by side on a 5×4 in. plate. The base A has grooves cut in it to serve as a guide to the amount of rotation to be given to the camera. In making tests it is

possible to make on one plate ten records of the various speeds given by a shutter in one minute.

A view of the front of the camera is shown in fig. 9 (Pl. XXIX.).

(b) *Electrical Apparatus.*

The vibration galvanometer used is of the moving coil type (as already described by one of us*), and can be readily tuned to frequencies of 50 or 500 ~ per second. As its mirror is of fair size (50 to 80 sq. mm.), it gives with a Nernst lamp a sufficient amount of light to avoid "tailing off" in the records for quick exposures with rapid plates. The source of current may be an ordinary lighting circuit, the current through a lamp as resistance being made periodically intermittent by means of a wire interrupter, an electrically maintained tuning fork, or, for the higher frequencies, a microphone hummer†. The wire interrupter, which is merely an electrically maintained monochord, can be set to the desired frequency by the help of a tuning-fork; the maintained fork and the hummer give constant and known frequencies, and do not require setting. The pulsating current is led through a primary coil of 50 or 100 turns, and over this is placed a movable secondary coil connected directly to the vibration galvanometer, which is tuned to resonance with the source of current. The amplitude of its vibration can be brought to the desired amount by changing the position of the secondary coil relatively to the primary.

The distance from mirror to plate is usually about 100 cms.

III. *Determination of the Efficiency.*

For a more complete test the efficiency, in addition to the total duration of exposure, is determined. If τ denote the total duration of the exposure, T the equivalent exposure, a the area of the shutter aperture at the instant t, and A the maximum opening, we have the relation

$$AT = \int_0^\tau a\,dt\ ;\ \text{the efficiency} = \frac{T}{\tau} = \frac{\int_0^\tau a\,dt}{A\tau}.$$

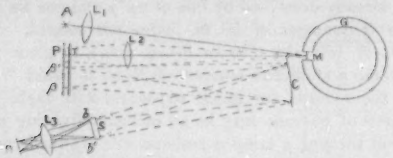
* Proc. Phys. Soc. vol. xx., and Phil. Mag. Oct. 1907.

† Proc. Roy. Soc. A. vol. lxxviii. p. 208 (1906).

Thus it is necessary to obtain a record of the area of the shutter opening at every instant of the exposure.

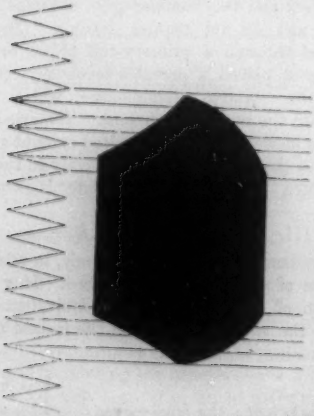
For the determination of the efficiency the method employed is essentially that proposed by Sir William Abney, who uses a siren to measure the time. We proceed as follows:—The

Fig. 4.



shutter is removed from the vibrating beam of light (see fig. 4) so that a continuous sine curve is recorded over the whole length of the plate, and this curve serves now simply

Fig. 5.



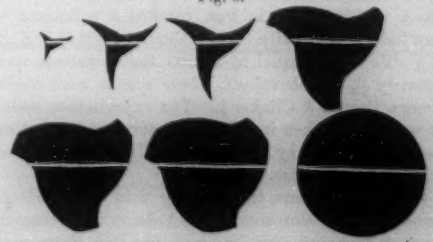
as a time record. A line source of light B is focussed by a lens L_3 on to a narrow horizontal slit $b\,b'$ placed in a diametral plane of the shutter as close as possible to the shutter-leaves.

An image $\beta\beta'$ of this slit is formed by the concave mirror C on the photographic plate by the side of the vibrating beam of light. As the plate falls the shutter is opened as before, and a record is obtained giving at every instant of time the length of slit through which light was passed by the shutter leaves (fig. 5). Measurements are then taken of the area of shutter aperture corresponding to a number of lengths of the slit opening. Fig. 6 shows some stages in the opening of a particular type of shutter, the white line representing the position of the slit. From a pair of records such as shown in figs. 5 and 6 a curve is drawn showing at every instant of the exposure the area of the shutter aperture through which light can pass to the plate. It is now evidently a simple matter to find the amount of light cut off owing to the finite length of time taken by the shutter leaves to open and close, and hence to calculate the efficiency.

IV. *Examples of Records.*

A number of shutters of different types have been tested with the apparatus. Figs. 7 and 8 (Pl. XXIX.) are examples of the records obtained, the former at 50 and the latter at 500~ per second. Figs. 5 and 6 are copies of records obtained in testing the efficiency.

Fig. 6.



For general use the most convenient frequencies to work with are 50 and 500~ per second, with possibly 2000 for special work; or perhaps 100 and 1000~ per second would be suitable. The full advantages that the method offers can

only be secured by the use of round numbers, such as the above, for the frequencies of the oscillations.

In conclusion, we wish to thank Dr. Glazebrook for the kind interest he has taken in the working out of the method.

Discussion.

Mr. H. Beck expressed his interest in the method, which he said was the most satisfactory yet produced. It was difficult to say exactly what to call exposure. In photographing moving objects it was advisable to use as the effective time the time during which the central eight-tenths of the shutter was open.

Mr. Duddell said the method was an ingenious one for determining the efficiencies of shutters. He suggested that by using an arc it might be possible to reduce the size of the mirror and thus work at higher frequencies. Instead of using a slit it might be better to use a short focus cylindrical lens.

Mr. Campbell, referring to Mr. Duddell's remarks, said that with a Nernst lamp it was possible to obtain high frequency curves showing very little tendency to tail off.

LIV. *Effect of Temperature on the Hysteresis Loss in Iron in a Rotating Field.* *By* W. P. Fuller, *B.Eng.*, *and* H. Grace, *B.Eng.**

It was shown by Professor Baily † that the hysteresis losses due to a rotating field in iron reached a maximum value with an induction density (B) of about 16,000 C.G.S. units. With a value of B equal to 20,000 the hysteresis was approximately $\frac{1}{2\frac{1}{0}}$ of the value. These results were confirmed by Messrs. Beattie & Clinker ‡. The experiments described below were undertaken in order to determine to what extent the results attained by Professor Baily were modified by variation in temperature.

In these experiments, the rotating field was produced by means of two phase-currents. Fig. 1 shows the arrangement, the two magnetizing coils carrying the two phase-currents being marked E, D. A is a slab of plaster 2 cms. thick and

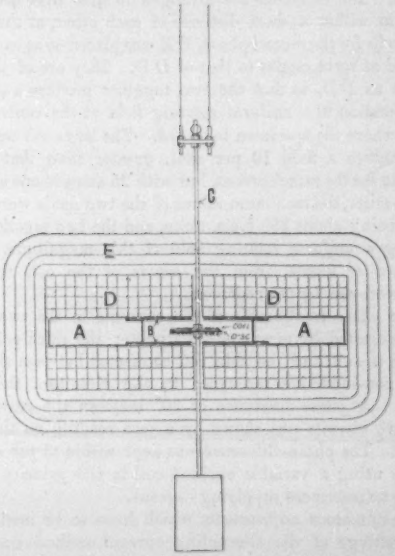
* Read May 14, 1909.

† Phil. Trans. 1896.

‡ 'Electrician,' 1896.

22 cms. square, having a circular hole 8½ cms. diameter in the centre. At the top and bottom of the hole the heaters B are placed, and in the chamber so formed the iron specimen is suspended. Each heater was made by winding No. 40 S.W.G. nickel wire zigzag fashion over the surface of a circular piece of mica. Its resistance cold was 13 ohms, but

Fig. 1.



General arrangement of the magnetizing coils and specimen.

increased rapidly with temperature. With the two in series a current of 1·6 amps. at 180 volts produced a maximum temperature of 500° C. at a point close to the iron disk. The specimen used was a circular disk of iron 4 cms. diameter, ·027 thick: it was attached by nuts to a brass spindle C, which had a weight attached to one end, and a *concave* mirror

at the other for indicating the motion of the specimen. The whole was supported by a bifilar suspension, the sensitiveness of which could be varied by altering the weight or by varying the distance apart of the supporting wires. The weight of the whole moving part was 285 grams.

The two magnetizing-coils D D, belonging to one phase, are each made by winding 14 turns of 7/14 asbestos-covered cable. The two coils are arranged to slide over the plaster slab to within a short distance of each other, at the centre. The coils for the second phase, E E, are placed so as to produce a field at right angles to that of D D. They are of the same shape as D D, so that the two together produce a close approximation to a uniform rotating field at the centre of the coils where the specimen is placed. The large coil was found to produce a field 10 per cent. greater than that of the smaller for the same current, but with 25 amps in one and 27·5 in the other, the maximum values of the two fields were equal, each being about 250 C.G.S. units, and the two together producing a uniform rotating field of this magnitude. At a distance of 2 cms. from the centre of the coil the field produced was 1½ per cent. less.

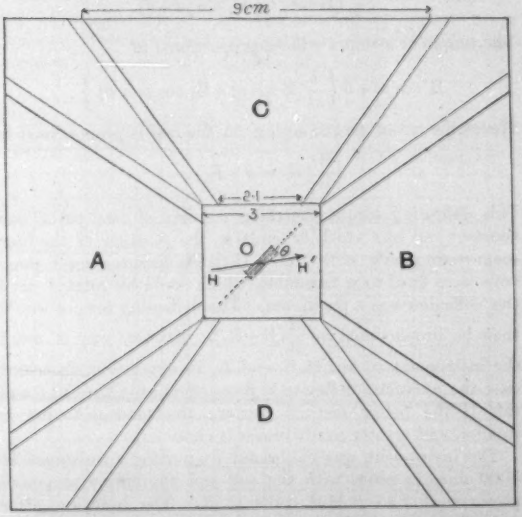
The phase relationship of the currents in the two phases was indicated by means of a wattmeter, the thick coil being in one circuit and the thin coil connected between the ends of a non-inductive resistance in the circuit of the other phase. A phase-difference of 90° between the currents in the two circuits was shown by a zero reading on the wattmeter. The phase-difference was kept within ·3 per cent. of 90° by using a variable choking-coil in the primary of one of the transformers supplying current.

The numerous adjustments which have to be made are a disadvantage of the alternating current method compared with the rotating magnet method used by Baily and others; but the absence of rotating parts is a distinct advantage, and this fact would render the method very suitable for high-frequency experiments.

To measure the flux in the iron, a coil of 8 turns of bare wire was wound round it and insulated therefrom with mica. As the E.M.F. induced in the coil is alternating and of the order of ·04 volt, a suitable galvanometer had to be con-

structed in order to measure it. The instrument constructed for the purpose is shown in fig. 2. A B C D are four coils of conical shape and wound with no. 28 wire. A and B were connected in series through a resistance to one phase, C and D to the other phase, of the machine supplying current to the magnetizing-coils of the test apparatus D E. With a current of ·31 amp. a rotating field of about 20 C.G.S. units

Fig. 2.



was produced at the point O. A small coil, made up with 200 turns of no. 46 wire (140 ohms resistance) 5 cms. long and 1 cm. broad, is suspended in the central space between the coils, and is connected to the search-coil on the iron specimen. In the galvanometer-coil there are two E.M.F.'s, one, E cos pt (say) due to the flux in the iron specimen, and one due to the rotating field of the galvanometer itself. Referring to fig. 2, let H H′ be the direction of the rotating

field of maximum value H when E $\cos pt$ is a maximum. The angle between it and the coil at a time t will be $\theta + pt$, and the component along the coil H $\cos pt + \theta$. The flux normal to the coil is H $\sin pt + \theta$, and the E.M.F. induced is proportional to

$$\mathrm{H}\, p \cos pt + \theta = \mathrm{E}_1 \cos pt + \theta.$$

If R be the resistance of the coil and the self-induction is negligible, the resultant current is equal to

$$\frac{1}{\mathrm{R}}(\mathrm{E} \cos pt + \mathrm{E}_1 \cos pt + \theta).$$

The torque at a time t will be proportional to

$$\mathrm{H} \cos pt + \theta \left\{ \frac{1}{\mathrm{R}}(\mathrm{E} \cos pt + \mathrm{E}_1 \cos pt + \theta) \right\}.$$

Hence the mean torque acting on the coil is proportional to

$$\frac{\mathrm{H}}{\mathrm{R}}\{\mathrm{E} \cos \theta + \mathrm{E}_1\}.$$

This deflecting torque therefore consists of two parts, one constant and one which depends on the position of the four magnetizing-coils A B C D. In the apparatus used these coils were fixed to a turntable, which could be rotated until the deflexion was a maximum. The deflecting torque would then be proportional to $\frac{\mathrm{H}}{\mathrm{R}}(\mathrm{E} + \mathrm{E}_1)$. Another way of using the instrument is to get E and E_1 in opposition, in which case the minimum deflexion is proportional to $(\mathrm{E}_1 - \mathrm{E})$; and this is the better method because the total deflexion is smaller, and greater sensitiveness is obtained.

The instrument was calibrated by putting a resistance of 4000 ohms in series with the coil and applying a measured pressure of 1·95 R.M.S. volts to it. The deflexion, after deducting that obtained by shortcircuiting the coil through the 4000 ohms, equalled 22·4 centimetres, consequently one centimetre corresponds to a voltage of maximum value ·00416 applied to the moving coil.

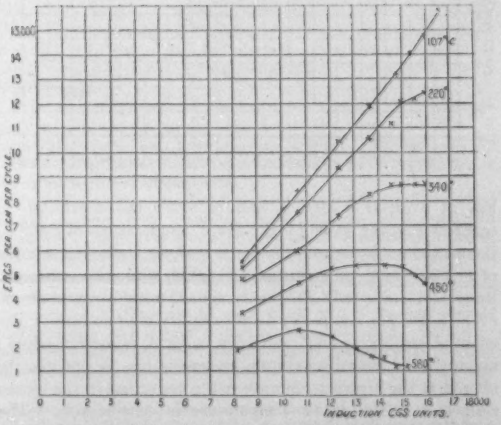
The induction density in the iron disk was readily calculated from the maximum voltage measured as above, the dimensions of the search-coil on the iron being known.

The disadvantage of this type of instrument is that the

constant part of the deflexion depends on the cube of the frequency, so that the speed of the machine must be very exactly regulated for a constant value. The E M.F. induced in the coil on the iron varies approximately as the frequency, and so the torque on the galvanometer due to this current depends on the square of the speed. It was found better in practice to have the two voltages E and E_1 in opposition, as mentioned above, so getting a deflexion proportionate to $E_1 - E$.

One other measurement required to be made, namely, that of temperature. This was effected by means of a Platinum-Platinum-Rhodium junction (placed close to the specimen) connected to a suitable galvanometer.

Fig. 3.

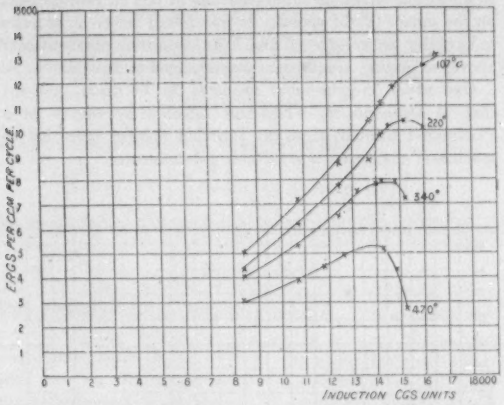


Curves showing relation between Hysteresis Loss and Induction at different Temperatures in a sample of armature iron from Messrs. Sankey.

The results obtained are shown in figs. 3 and 4. The temperature of 107° C. was the lowest at which it was found possible to make any measurements, on account of the heating effect of the magnetizing-coils. The results in fig. 4 were

obtained after the iron had been heated to 580° and cooled slowly. These experiments show that the effect of increasing the temperature of iron is to greatly reduce the hysteresis

Fig. 4.



loss at a given induction, and to cause the maximum to occur at a lower value of B. At a temperature of 580° C. the maximum hysteresis loss occurs at an induction density of 11,000 C.G.S. units instead of 16,000, while the maximum hysteresis loss is only 2600 as compared with a maximum of over 15,000 per c.c. per cycle at ordinary temperatures. The effect of heating to 580°, as shown in fig. 4, is most marked; although the actual maximum hysteresis loss is not greatly altered at the higher temperature, the reduction in the hysteresis loss at the lower temperature is considerable. The shape of the curve between ergs per cycle and B is quite different, the hysteresis loss falling off much more rapidly from the maximum.

The permeability of the iron is not greatly affected by changes in temperature within the limits of the experiment, as was shown by Morris *.

* Phil. Mag. Sept. 1897.

The above experiments were carried out at the Applied Electricity Laboratories of the University of Liverpool, and the authors thank Professor Marchant for his valuable advice and assistance.

Discussion.

Dr. W. H. Eccles remarked that the effects of previous thermal treatment, and of the nature of the material on the properties studied by the Authors, were so enormous that comparison of the present experiments with similar ones with alternating fields was difficult; but the Authors' curves seemed to show that the effect of rise of temperature on hysteresis loss was less for rotating fields than for alternating fields if the fields were below about 12 C.G.S. The curves showed also that the maximum values of hysteresis loss in rotating fields were reached at lower and lower flux densities the higher the temperature, although the permeability varied but little. This was of importance from the point of view of the molecular theory of magnetism: it showed that smaller forces sufficed to rotate the molecules within a magnetically stable group at high than at low temperatures. The Authors' experiments might be regarded as adding strong confirmation to the molecular theory of magnetism.

Prof. Marchant said the iron tested was obtained direct from the manufacturers and its previous thermal treatment was not known. A permeability curve had, however, been obtained. He expressed his interest in Dr. Eccles's remarks upon the bearing of the experiments on the molecular theory of magnetism.

Mr. A. Campbell remarked that the methods and results described by the Authors were interesting, particularly the novel instrument employed. There is a source of considerable uncertainty, however, which does not appear to have been taken into account, namely that the total power spent in the iron disk includes eddy-current losses. For the thickness of sheet used, the eddy-current loss at 50 ~ per sec. and B maximum = 10,000 would be about 20 per cent. of the total amount measured when the induced voltage has sine wave-form. With other wave-forms the eddy-current loss depends on the square of the form factor and might easily rise to 40 per cent. of the whole, even although the source of current was a sine-wave alternator. Thus tests made without simultaneous measurement of the form factor are open to serious error. It is highly probable that the form factor would vary considerably along any one of the curves given in the paper. The eddy-current loss also would be very much reduced at the higher temperatures owing to the increased resistivity of the iron.

LV. *The* Arthur Wright *Electrical Device for evaluating Formulæ and solving Equations.* *By* Alexander Russell, *M.A., D.Sc., and* Arthur Wright, *M.I.E.E.**

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XII. Solution of Transcendental Equations.
XIII. Tracing Curves Electrically.
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I. *Introduction.*

In an ordinary algebraical expression consisting of several terms be computed by the slide-rule or by logarithmic tables, each term must be computed separately and the sum of the negative terms subtracted from the sum of the positive terms. If all these operations were done mechanically it would often save time and lessen the risk of error. In a few cases the form of the expression can be altered with advantage so as to adapt it to logarithmic computation, but there are many cases where the terms of the expression have to be evaluated separately, and in these cases it saves mental labour to do the operations mechanically.

The electrical device described below, which was invented by Mr. Arthur Wright, enables us to evaluate almost all mathematical expressions by simple mechanical and electrical operations. In the present model the accuracy is not high, being of the order of one per cent.; but this accuracy suffices in most engineering calculations. The apparatus, however, can be easily elaborated to give a far higher accuracy. The

* Read June 11, 1909.

main object of the authors in this paper is to explain the principles on which it works, and to point out a few of its many applications in practical calculations.

Although some of the operations described below—as, for example, the finding of the roots of a numerical algebraical equation of the nth degree—can easily be done by the expert mathematician, yet even in his case the apparatus can be a help, as it gives him at once the approximate values of the roots. In general, he can then find them to any required degree of accuracy by the methods given in the Theory of Equations.

II. *Historical.*

The history of calculating machines, and an excellent description of many of them, is given by M. d'Ocagne in his book *Le Calcul Simplifié* (1905). Valuable information is also given in Professor Henrici's article on "Mathematical Instruments" in the *Encyclopædia Britannica* (10th edition). We need only mention, therefore, the statical method * of solving equations due to Professor Vernon Boys, the Torrès logarithmic arithmophore †, which is a very elaborate piece of mechanism, and the extremely ingenious electrical method ‡ due to M. F. Lucas, of finding the real and imaginary roots of an equation of the nth degree. The last method, however, is not rigorously accurate and would be very laborious to apply in practice.

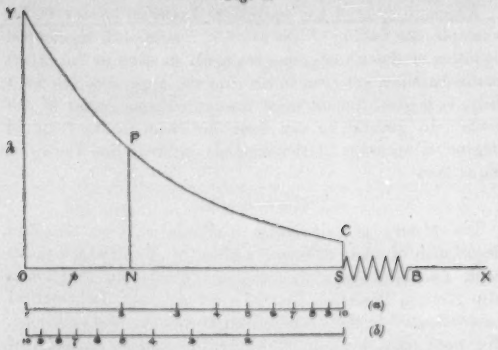
III. *The Slide Resistances.*

In the electrical device described below the principle of the ordinary logarithmic slide-rule is combined with addition and subtraction, by utilising the laws according to which resistances combine in series or parallel. The products found by the slide-rule method are represented either by the resistances or by the reciprocals of the resistances of certain wires. In the latter case the currents through them can easily be added or subtracted by a Wheatstone Bridge Method. Hence the sums or differences of the products can be found.

* Phil. Mag. vol. xxi. p. 241 (1886).
† M. d'Ocagne, *l. c. ante*, p. 123.
‡ *Comptes Rendus*, t. cvi. p. 1072 (1888).

The method of constructing the slide resistances (fig. 1) is as follows:—A template of thin insulating material is made

Fig. 1.



in the shape OSCY, shown in fig. 1. The equation to the curve YPC is

$$y = k\,10^{-x/h},$$

where OY$=k$, OS$=h$, and SC$=k/10$.

This plate is wound uniformly with a hundred turns of No. 36 insulated manganin wire, the temperature coefficient of which is negligible, and the wires are practically parallel to OY. At points along OS the insulation of this wire is removed, and contact with the wires is made by the contact-finger described below. The scale OS is logarithmic. If p, for instance, be the reading at the point N, we have

$$\mathrm{ON} = h \log p,$$

and thus $$y = \mathrm{PN} = k/p.$$

The resistance of the wire between O and N will be approximately proportional to the area ONPY. This area is given by

$$\int_0^x y\,\partial x = k\int_0^x 10^{-x/h}\,\partial x$$

$$= \frac{hk}{\log_e 10}\left(1-\frac{y}{k}\right).$$

If, then, the wire on the frame have a resistance 9R/10, and a coil SB of resistance R/10 be put in series with it, the resistance from N to B will equal (y/k)R, that is, R/p.

When the resistances are fixed on the device, the resistance between N and B is in circuit, and this always equals R/p, where p is the reading on the top scale [(*a*), fig. 1]. If we have n of these slide resistances in parallel, and their readings are $p_1, p_2, \ldots p_n$, the sum of the currents will be

$$(\mathrm{E}/\mathrm{R})\,(p_1+p_2+\ldots+p_n),$$

where E is the applied potential-difference.

Instead of having the logarithmic scale placed as in (*a*), fig. 1, we may reverse it and place it as in (*b*). If p' be the value of ON read on this scale, we obviously have $p'=10/p$; and hence the resistance between N and B is $(\mathrm{R}/10)p'$. If the n resistances be now connected in series, their combined resistance will be

$$(\mathrm{R}/10)\,(p_1'+p_2'+\ldots+p_n')\ ;$$

and if E be the applied potential-difference, the current flowing through them will be

$$(10\,\mathrm{E}/\mathrm{R})\,/\,(p_1'+p_2'+\ldots+p_n').$$

In some problems it is more convenient to use the scale placed as in (*a*), and in others it is more convenient to use it as in (*b*). In what follows, unless otherwise stated, we shall suppose that it is placed as in (*a*).

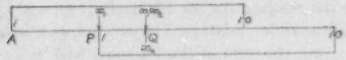
IV. *Multiplication. The Index Line.*

The method of performing multiplication is practically identical with that utilised in the ordinary slide-rule, which was originally designed by Seth Partridge *. In Partridge's

* 'The Description and Use of an Instrument called the Double Scale of Proportion.' London, 1671. Near the beginning of the book we read: "Here might have been expected a print of the rule, but in regard to its sliding it could not well be demonstrated: wherefore I thought good to advertise that this Scale and all other Mathematical Instruments are accurately made by Mr. Walter Hayes at the Cross-Daggers in More-Fields, next door to the Popes-Head-Tavern, London." Slide rules, therefore, were for sale in London 238 years ago.

instrument two logarithmic scales (fig. 2) slide with their edges in contact. If the reading on the top scale at P equals x_1, and on the lower scale at Q equals x_2, the reading on the top

Fig. 2.



scale at Q equals x_1x_2. This follows because by the construction of the scales,

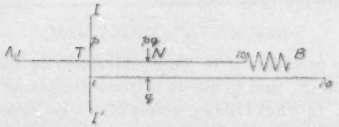
$$AP = h \log x_1, \quad PQ = h \log x_2,$$

and thus

$$AQ = AP + PQ$$
$$= h \log x_1 + h \log x_2 = h \log x_1x_2.$$

In the Arthur Wright device the slide resistances are movable. If we wish to form the product pq, we move the slide so as to make the index-line I I′ (fig. 3) lie over

Fig. 3.



the reading p, and we then move the contact-finger until it makes contact at N, where TN, on the logarithmic scale, is q. The resistance NB will then equal R/pq.

V. *The Contact Fingers.*

The ordinary contact finger is a straight piece of metal wire ONP (fig. 4), which is capable of rotation about a point O fixed on a sliding bar parallel to and vertically over the index-line I I′. This bar can move in the direction of its length. A pointer L attached to it moves along a fixed logarithmic scale KQ. The length of this scale is made equal to h, and so

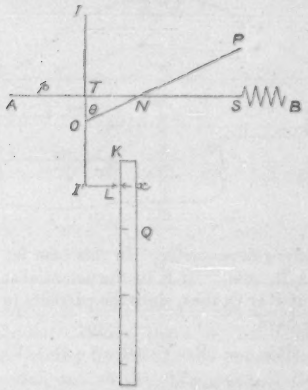
$$KL = h \log x = OT.$$

The contact finger OP can be adjusted so as to make any desired angle θ with the index-line II′. If the contact be made at N, we have

$$TN = OT \tan\theta = h \tan\theta \log x = h \log x^{\tan\theta};$$

and thus, if $AT = p$, the resistance from N to B will be

Fig. 4.



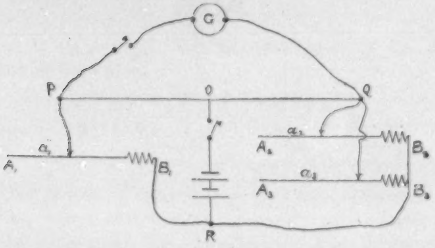
$R/(px^{\tan\theta})$. By giving various values to θ we can, with the help of a table of natural tangents, obtain all positive integral or fractional powers of x. If N lie between A and T, θ is negative, and so we obtain all the negative integral or fractional powers of x.

VI. *Addition and Subtraction.*

In performing addition and subtraction, the action of the apparatus can be understood from the following diagram. P and Q are the terminals of a wire bridge, of which O is the middle point (fig. 5). One pole of a battery of dry cells is connected with O and the other pole is connected with the B ends of the slide resistances, A_1B_1, A_2B_2, and A_3B_3. A galvanometer connects P with Q. Let us suppose that

we have to find the value of a_2+a_3. We connect Q with the contact-fingers of A_2B_2 and A_3B_3 respectively, and set the fingers at the marks a_2 and a_3 on their respective scales. We then join P with the contact-finger of A_1B_1, and closing the keys we adjust the position of this finger until there is no

Fig. 5.



deflexion on the galvanometer. In this case let the reading on the slide A_1B_1 be a_1. If E be the potential at R, and V be the potential at P or Q, then, since the currents in PO and QO are equal, we have

$$(E-V)/(R/a_1) = (E-V)/(R/a_2) + (E-V)/(R/a_3),$$

and thus $$a_1 = a_2 + a_3.$$

To find a_2-a_3, all we have to do is to change the connexion of the finger of A_2B_2 from Q to P. Proceeding as before, a_1 now gives us the value of a_2-a_3.

When subtracting approximate values, it has to be remembered that the percentage error in the result may be large, especially when a_2-a_3 is small compared with a_3. This difficulty, which arises when subtracting the approximate values of nearly equal quantities, affects all approximate methods. In our case it can only be overcome by increasing the accuracy of the device.

VII. *The Use of the Arms of the Bridge.*

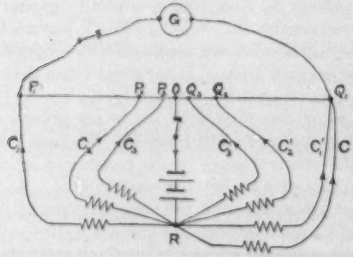
Terminals are connected at points P_1, P_2, P_3, Q_3, Q_2, and Q_1 of the bridge wire (fig. 6). These points are chosen

so that if the resistance P_1Q_1 be $2r$, the resistances of OP_1, OP_2, and OP_3 are equal to r, $r/10$, and $r/100$, and also to the resistances of OQ_1, OQ_2, and OQ_3 respectively.

Let the magnitudes of the currents in RP_1, RP_2, RP_3, RQ_3, and RQ_2 be C_1, C_2, C_3, C_3', and C_2' respectively.

Let us also suppose that the currents in the two branches joining R and Q_1 are C and C_1' respectively.

Fig. 6.



The resistances in these circuits are the slide resistances described above. When there is no deflexion on the galvanometer joining P_1 and Q_1, the potentials of these two points are equal, and so the differences of their potentials from that at O are equal, and thus we find that

$$C_1r + C_2(r/10) + C_3(r/100) = Cr + C_1'r + C_2'(r/10) + C_3'(r/100).$$

Hence

$$C = C_1 + C_2/10 + C_3/100 - (C_1' + C_2'/10 + C_3'/100).$$

Let E be the potential of R, and V_1 the potential of P_1 and Q_1 (fig. 6). Then, if V_2, V_3, V_2', V_3' be the potentials of P_2, P_3, Q_2, and Q_3, and p_1, p_2, p_3, q_1, q_2, q_3, and x be the readings on the scales of the slide resistances in the branches RP_1, RP_2, &c., we have

$$(E-V)x = (E-V_1)p_1 + (E-V_2)(p_2/10) + (E-V_3)(p_3/100) - \{(E-V_1)q_1 + (E-V_2')(q_2/10) + (E-V_3')(q_3/100)\};$$

and therefore

$$x = p_1 + p_2/10 + p_3/100 + \Delta_p - (q_1 + q_2/10 + q_3/100 + \Delta_q),$$

where

$$\Delta_p = \frac{V_1 - V_2}{E - V_1} \cdot \frac{p_2}{10} + \frac{V_1 - V_3}{E - V_1} \cdot \frac{p_3}{100},$$

and

$$\Delta_q = \frac{V_1 - V_2'}{E - V_1} \cdot \frac{q_2}{10} + \frac{V_1 - V_3'}{E - V_1} \cdot \frac{q_3}{100}.$$

The readings p_1 and q_1 on the scales cannot be less than 1, and the readings p_2, p_3, q_2, and q_3 cannot be greater than 10. Hence, remembering that $(V_1 - V_2)/(E - V_1)$ cannot be greater than $(V - V_3)/(E - V_1)$, we see that if $(V_1 - V_3)/(E - V_1)$ be equal to or less than 1/53, Δ_p is not greater than the hundredth part of $p_1 + p_2/10 + p_3/100$. Similarly, if $(V_1 - V_3')/(E - V_1)$ be equal to or less than 1/53, Δ_q is not greater than the hundredth part of $q_1 + q_2/10 + q_3/100$. We have therefore to arrange the relative values of the resistances of the slides and the bridge wire so that this may be true. As we have pointed out above, however, the inaccuracy in the value of x depends on the relative values of x and $p_1 + p_2/10 + p_3/100 + \Delta_p$. When these quantities are nearly equal, approximate methods of computation fail.

By Ohm's law, we see from fig. 6 that

$$\frac{V_1 - V_2}{\frac{9}{10^r}} = \frac{E - V_1}{R_1}, \quad \ldots \ldots \quad (1)$$

and

$$\frac{V_2 - V_3}{\frac{9}{100^r}} = \frac{E - V_1}{R_1} + \frac{E - V_2}{R_2}, \quad \ldots \quad (2)$$

where R_1 and R_2 are the resistances of the circuits RP_1 and RP_2. Hence it readily follows that

$$\frac{V_1 - V_3}{E - V_1} = \frac{9}{10} \cdot \frac{r}{R_1} + \frac{9}{100}\left(\frac{r}{R_1} + \frac{r}{R_2}\right) + \frac{81}{1000} \cdot \frac{r^2}{R_1 R_2}.$$

If R be the resistance of a slide, the minimum possible value of both R_1 and R_2 is R/10. Hence the maximum possible value of $(V_1 - V_3)/(E - V)$ is

$$9\frac{r}{R} + 1{\cdot}8\frac{r}{R} + 8{\cdot}1\left(\frac{r}{R}\right)^2,$$

and this is less than 1/53 when $2r/R$ is not greater than 1/287.

Similarly, if ρ, R_1', and R_2' be the resistances of the slides, the readings on which are x, q_1, and q_2 respectively, we see that

$$\frac{V_1-V_3'}{E-V_1}=\frac{9}{10}\left(\frac{r}{\rho}+\frac{r}{R_1'}\right)+\frac{9}{100}\left(\frac{r}{\rho}+\frac{r}{R_1'}+\frac{r}{R_2'}\right)+\frac{81}{1000}\left(\frac{r}{\rho}+\frac{r}{R_1'}\right)\frac{r}{R_2'}.$$

In the very unlikely case when $\rho=R_1'=\frac{R}{10}=R_2'$, $(V_1-V_3')/(E-V_1)$ has its maximum possible value, and in this case it will be less than 1/53 if r/R be not greater than 1/1100. Hence the desired accuracy can be attained by making the resistance R of each slide 550 times greater than the resistance of the bridge wire.

It is to be noticed that the effect of connecting a slide resistance with P_2 instead of P_1 is to divide its value by 10. Hence, if we were adding 8·8 to 0·75, we would put the contact-finger of one slide on 8·8 and connect it with P_1, and the contact-finger of another slide on 7·5 and connect it with P_2. Suppose also that the value of x was 123. In this case we could only obtain a balance when it was connected with Q_2.

VIII. *Cubic Equation.*

Let us suppose that the cubic equation is

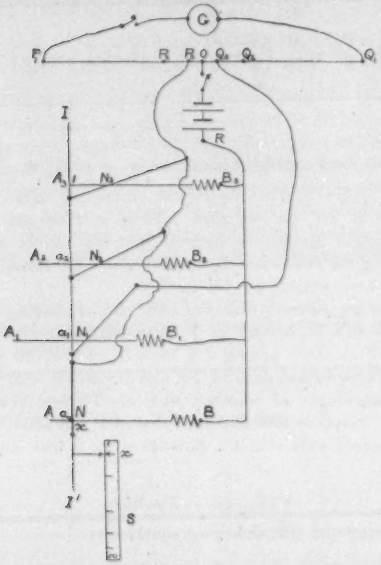
$$x^3+a_2x^2-a_1x+a_0=0,$$

where a_2, a_1, and a_0 are positive numbers.

The arrangement of the apparatus for solving this equation is shown in fig. 7, p. 812. The bridge-wire is the same as in fig. 6. We move the slides AB, A_1B_1, A_2B_2, and A_3B_3 so that the readings 1, a_2, a_1, and a_0 on their logarithmic scales are on the index-line II′. The ends of the contact-fingers touching AB, A_2B_2, and A_3B_3 are connected with P_2 and the finger touching A_1B_1 with Q_2. The fingers through N, N_1, N_2, and N_3 make angles $\tan^{-1}0$, $\tan^{-1}1$, $\tan^{-1}2$, and $\tan^{-1}3$ with the index-line. The battery and galvanometer-keys being

closed, the bar to which the fingers are connected is gradually lowered. When the galvanometer deflexion is zero let us

Fig. 7.



suppose that x is the reading on the logarithmic scale S. In this case we have

$$\frac{E-V_3}{\frac{R}{x^3}}+\frac{E-V_2}{\frac{R}{a_2x^2}}+\frac{E-V_1}{\frac{R}{a_0}}=\frac{E-V_0}{\frac{R}{a_1x}},$$

and thus $x^3+a_2x^2-a_1x+a_0=0$. Hence x is one of the roots of this equation.

When a contact-finger, N_1 for instance, reaches the end of its slide N_1B_1 will be R/10. If we now move the slide A_1B_1 so so that A_1 is on the index-line, unclamp the contact-

finger and move it parallel to itself until N_1 is over A_1 and then reclamp it, the resistance of N_1B_1 will be R. If, then, we disconnect the finger N_1 from Q_3 and connect it with Q_2 the deflexion on the galvanometer will not be altered, and so we can continue to increase x. It will be seen that as x increases from 1 to 10 the contact-finger N_2 traverses A_2B_2 three times. The first time N_2 gets to the end of its scale it is connected with P_2 and moved back to A_2. The second time it reaches the end it is connected with P_1 and again moved back.

The device enables us to find any real root of the equation, greater than 10^n and less than 10^{n+1}. All we have to do is to find the root of the equation

$$x^3+(a_2/10^n).x^2-(a_1/10^{2n})x+a_0/10^{3n}=0,$$

lying between 1 and 10, and multiply the result by 10^n. By solving a similar subsidiary equation we can find the approximate values of the roots of the equation which are less than unity.

We may use Newton's rule to find more accurate values of the roots. If a, for instance, be an approximate value of a real root of $f(x)=0$, $a-f(a)/f'(a)$ gives usually a very much closer approximation x_1. The failing case is when we have two roots very nearly equal to one another. The device always indicates when this occurs. If two roots are each equal to a, or if they are approximately equal to a, the galvanometer deflexion instead of passing to the other side of zero when x becomes greater than a returns to the same side.

In practice if $f'(a)$ is large when $f(a)$ is zero the device is very sensitive, but when $f'(a)$ is small and in general, therefore, when the roots are equal the device is not sensitive.

To find approximate values of the negative roots of the original equation we find by the device the roots of

$$x^3-a_2x^2-a_1x-a_0=0.$$

The imaginary roots* are most readily found by first finding as accurate a value x_1 as possible of the real root. We then divide the cubic expression by $x-x_1$ and equate the dividend to zero. The roots of this quadratic equation give the approximate values required.

* *Cf.* C. P. Steinmetz, 'Transient Electric Phenomena,' p. 136.

IX. *Equations of Degrees higher than the Third.*

The device can be usefully employed to find the values of the roots of equations higher than the third with sufficient accuracy for practical work. When calculating, for example, the requisite resistances for the motor controller of an electric car, the following equation * has to be solved

$$x^{n/(n-1)} + a_1 x - a_0 = 0,$$

where n is the number of steps in the rheostat.

To solve this equation three only of the slides shown in fig. 7 are required. AB is connected with Q_3, and A_1B_1 and A_2B_2 are connected with P_3. The finger making contact with A_2B_2 is inclined at an angle $\tan^{-1} n/(n-1)$ with II′, and the fingers on A_1B_1 and AB are inclined at angles of 45° and 0° respectively. The value of x is then increased until a balance is obtained.

When solving equations containing large numbers of terms it is sometimes convenient to divide the equation by a suitable power of x. In this case the fingers making contact with slide resistances representing terms containing negative powers of x are inclined to the left of II′ (fig. 8).

Let us consider, for example, a sextic equation. As shown above, we alter it so that the required root or roots are multiples or submultiples of the equation

$$x^6 - a_5 x^5 + a_4 x^4 + a_3 x^3 - a_2 x^2 - a_1 x + a_0 = 0,$$

which lie between 1 and 10.

Dividing this equation by x^3 we get

$$x^3 - a_5 x^2 + a_4 x + a_3 - a_2 x^{-1} - a_1 x^{-2} + a_0 x^{-3} = 0.$$

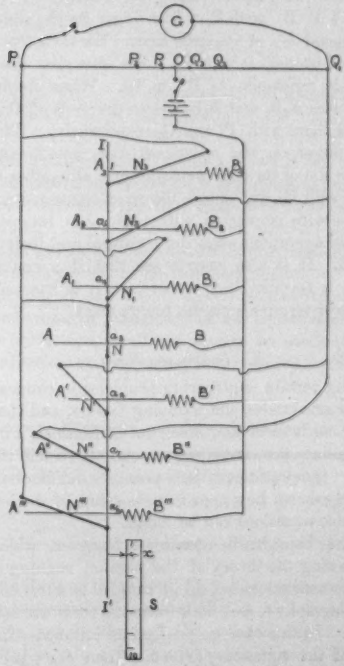
In general, seven slide resistances will be required, but if when x is put equal to unity any term is less than the hundredth part of the sum of the terms of the same sign preceding it, that term can be neglected. The slide resistances are first moved until the readings on the index-line are a_0, a_1, a_2, ... &c., as shown in fig. 8. The contact-fingers are next turned round to the requisite angles $\tan^{-1}3$, $\tan^{-1}2$, ... $\tan^{-1}-3$, respectively. The ends of the contact-fingers are

* E. Wilson, 'Electrical Traction,' vol. i. p. 42.

then connected with suitable points on the arms of the bridge.

At first sight it might be thought that two extra pairs of contact-points P_4, Q_4 and P_5, Q_5 would be required in the

Fig. 8.



bridge arms. If a one per cent. inaccuracy, however, is permissible this is not necessary. To illustrate this point, and also the method of using the device in this case, let us

consider the numerical equation

$$x^3 - 5{\cdot}7x^2 + 0{\cdot}12x + 8{\cdot}6 - 0{\cdot}04x^{-2} + 1{\cdot}3x^{-3} = 0.$$

Putting x equal to unity we see that the term 0·04 can be neglected compared with 5.7, only five slide resistances therefore are required. We connect A_2B_2 with P_2, A_2B_2 with Q_2, A_1B_1 with P_1 and make the reading on it 1·2, AB with P_2 and $A'''B'''$ with P_3. The finger on the slide $A'''B'''$ gets to the end A''' of its scale first. We then disconnect it from P_3 and connect it with P_2. We also alter the reading on the slide resistance $A'''B'''$ to 10. When the fingers on the resistances A_2B_2 and A_3B_3 get to the ends of their scales we connect them with P_1 and Q_1 respectively. When, however, the finger on the resistance A_2B_2 gets for the second time to the end of its scale we disconnect altogether the wires connected with P_3, and move the wire connected with P_2 to P_3 and the wire connected with Q_1 to Q_2. In working the device these operations seem quite natural and little thinking is required. It is also easy to see that if a one per cent. inaccuracy is permissible it is unnecessary to have more than three pairs of terminals on the bridge arms.

X. *Imaginary Roots.*

In solving certain engineering problems in connexion with finding the amplitudes, the damping factors, and the periods of certain mechanical and electrical oscillations, a necessary step is finding the imaginary roots of certain algebraic equations. Quadratic equations present no difficulty, and we have already shown how approximate values of the imaginary roots of cubic equations can be found.

With the biquadratic equations, however, which occur when discussing the theory of the parallel running of alternators*, the oscillations set up in coupled electric circuits in wireless telegraphy†, &c., both pairs of roots are sometimes imaginary. In this case we proceed as follows:—Let $x+yi$ be a root of the equation $f(z)=0$. Then $f(x+yi)=0$, and

* A. Russell, 'Alternating Currents,' vol. ii. p. 184.

† J. A. Fleming, 'Electric Wave Telegraphy,' p. 209.

hence, expanding by Taylor's theorem, we have

$$f(x)-\frac{y^2}{2}f''(x)+\ldots+iy\left\{f'(x)-\frac{y^2}{3}f'''(x)+\ldots\right\}=0,$$

and thus we must have

$$\left.\begin{aligned} f(x)-\frac{y^2}{2}f''(x)+\frac{y^4}{24}f^{iv}(x)=0 \ldots (a)\\ \text{and}\qquad f'(x)-\frac{y^2}{6}f'''(x)=0 \ldots (b)\end{aligned}\right\}.$$

From (*b*) we get y^2 in terms of x, and substituting this value of y^2 in (*a*) we get an equation of the sixth degree to find x. The two real roots of this equation can be found by the machine arranged in the manner shown in fig. 8, and the pairs of corresponding values of y are given at once by (*b*). Approximate values of the four imaginary roots can thus be rapidly found. If a higher degree of accuracy be required we can either use Newton's method of approximation, or better apply Horner's method to the auxiliary sextic.

Theoretically we can always obtain approximate values of all the roots real or imaginary of an equation of the nth degree by the device, as y^2 can always be easily eliminated from the equations corresponding to (*a*) and (*b*), by Sylvester's method *.

XI. *Solution of Equations containing Miscellaneous Functions.*

For this purpose some of the contact-fingers are made from wires bent into the shape of certain curves. Suppose, for example, we desire to find the roots of the equation

$$\frac{a}{x^m}+\frac{b}{f(x)}=cx^n+d\mathrm{F}(x),$$

where the values of $f(x)$ and $\mathrm{F}(x)$ have been computed, or found experimentally, for values of x.

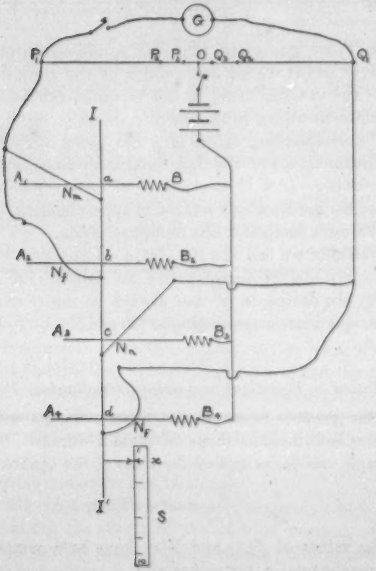
The wire passing through the point N_f (fig. 9, p. 818) is bent so that if y be a horizontal ordinate and x the vertical abscissa on the logarithmic scale, $y=h\log f(x)$, where h is the

* Burnside and Panton, 'Theory of Equations,' p. 296.

length of the scale on the slide resistances. Similarly the equation to the wire through N_F is $y=h \log F(x)$.

The contact-fingers through N_m and N_n are set so that their inclinations to the vertical are fixed at $-\tan^{-1} m$ and $\tan^{-1} n$ respectively. The rod II′ is then gradually lowered and the readings on the logarithmic scale S, when the deflexion on

Fig. 9.



the galvanometer is zero, give the roots of the equation lying between 1 and 10.

The use of a bent contact-finger to take into account the "hysteresis loop" of iron would be of use in certain electrical problems.

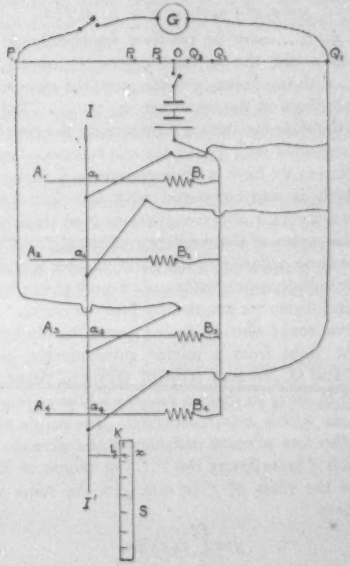
XII. *Solution of Transcendental Equations.*

Let us suppose that the equation we have to solve is

$$a_1 l_1^{b_1 x} + a_2 l_2^{b_2 x} - a_3 l_3^{b_3 x} + a_4 10^x = 0.$$

In this case (fig. 10) the scale S is an ordinary scale, the length KL being equal to kx. The various contact-fingers are adjusted so that the angles they make with the index-line II′ are $\tan^{-1}(b_1 \log l_1)$, $\tan^{-1}(b_2 \log l_2)$, $\tan^{-1}(b_3 \log l_3)$

Fig. 10.



and 45° respectively. We connect the contact-fingers as in fig. 10 and vary x. The values of x for which there is no deflexion on the galvanometer are the roots of the given equation lying between 1 and 10. By altering the given

equation to one whose roots are ten times smaller we find the roots lying between 10 and 100, and proceeding in this way we can get approximate values of all the roots.

By similar settings of the contact-fingers, and by using both logarithmic and ordinary scales, approximate values of the roots of very complicated equations can sometimes be easily found.

XIII. *Tracing Curves Electrically.*

Suppose we desire to find the graph of the curve

$$y = a_p x^p + a_q x^q - a_r x^r + \ldots = f(x),$$

where p, q, r, ... may be positive, fractional, or negative indices. We set the contact-fingers at angles $\tan^{-1} p$, $\tan^{-1} q$, ... with the index-line, and move the slide resistances until the readings on the scales are a_p, a_q, ... The contact-fingers on the slide resistances representing the positive terms are then connected with P, and the other contact-fingers with Q. In addition we have a slide resistance Y with a vertical finger, which is also connected with Q. The fingers are moved down through a given distance x on the logarithmic scale, and the value of the reading y on the slide resistance Y when there is no deflexion of the galvanometer is then found. In this way simultaneous values of x and y can be rapidly obtained, and hence we can readily plot the curve.

The curve could also be traced automatically by making the spot of light from a mirror galvanometer, connected between P and Q, fall on a strip of sensitive paper which is constrained to move so that its velocity is always proportional to the rate at which x is increasing. The points where the trace cuts the line of zero deflexion would give the roots of the equation $f(x)=0$, and the turning points of the trace would give the roots of $f'(x)=0$. In the same way the integral curve

$$y = \int_0^x f(x)\, \partial x,$$

could be drawn automatically.

If the slide resistances were made of large size so that they could carry appreciable currents, recording ammeters and voltmeters could be employed to trace the curves.

XIV. *Conclusion.*

We think that if engineers and physicists recognize that approximate values of roots of very complicated equations can be easily obtained, it may considerably extend the usefulness of theory. Authors are often diffident to publish results which can only be utilized when the roots of equations of degrees higher than the third can be found, or when equations involving miscellaneous functions can be solved. In these cases we hope that a knowledge of the methods of using logarithmic slide resistances described above may be of practical value by showing how the theoretical results can be immediately utilized.

We hope also that this preliminary sketch will induce others to improve the method, and to apply it to other and possibly more practical uses. It seems, for example, particularly suited to harmonic analysis as the integrals representing the coefficients of $\sin nx$ and $\cos nx$ in the expansion of $f(x)$ can be readily found.

In conclusion, we shall quote from the quaint but spirited preface to Seth Partridge's book on the slide rule (1671).

"I am sure here is a good Subject, a good piece of Cloath, if the Garment be not marred in the making; if it be, the fault is in the botching Taylor, not in the stuffe."

Discussion.

Prof. C. H. Lees expressed his interest in the device and referred to the large number of calculations that could be performed with it.

Dr. W. H. Eccles congratulated the Authors, and referring to the fact that the machine could be used to solve a bi-quadratic, asked if it was possible to determine the two quadratic factors by means of it.

LVI. *The Echelon Spectroscope, its Secondary Action, and the Structure of the Green Mercury Line.* By HERBERT STANSFIELD, *D.Sc., Demonstrator in Physics, Manchester University* *.

Thesis approved for the Degree of Doctor of Science in the University of London †.

[Plates XXX.–XXXII.]

Part I.—THE ECHELON SPECTROSCOPE.

The Echelon and its Mounting.

THE echelon spectroscope described in this paper was constructed by Messrs. Adam Hilger for Professor Schuster, a modification, which has proved to be very valuable, being made in the usual design. A front elevation, plan, and two end elevations of the echelon are reproduced on Plate XXX., the plan being drawn with the cover removed.

There are 33 glass plates. The smallest plate is 13 mm. wide, and they increase in width by steps of 1 mm. until the last plate is reached, which is 12 mm. wider than the last but one, the aperture being reduced to 1 mm. by the screen S. The effective aperture of the first plate is similarly reduced in width to 1 mm. by the block B. All the plates are 40 mm. high, and the common thickness is 9·48 mm. The plates are pressed together by the two nickel steel rods marked T, which are intended to have the same coefficient of expansion as the glass. The plates are in very close contact, in most cases the greater part of an interface being taken up by a patch of definite but irregular outline that appears "black" by reflected light. The remaining part of the interface generally shows white of the first order, but here and there the film of air may be thick enough to show the yellow or even the red of the first order.

* Read June 11, 1909.

† The following alterations have been made:—Equations 2, 3, 4, 9, 9A, and the section on the spectrum given by a hot lamp have been added; also the faint lines previously described as doubtful are shown to have their origin in the echelon.

The common thickness of the glass plates is 9·48 mm.; their refractive index, deduced by a Hartmann formula from the values given by the makers of the glass, is 1·5802 for the green line 5461; and the dispersive power, $\frac{d\mu}{d\lambda}$, is −918 per cm. for this wave-length.

Optical Effects due to clamping the Plates.

The echelon produces a slight cylindrical convergence in a beam of light, altering the focus of the observing telescope by 1 mm. (the focal length of the object-glass being 53 cms.). This effect is probably due to the clamping, as Twyman * states that when the clamping pressure is applied a change of focus is produced. He attributes the effect to a uniform increase in the compression of the plates from the largest to the smallest; but there is direct evidence (see p. 830) that the echelon plates become slightly prismatic, and this effect, increasing towards the smaller end, would also produce convergence.

The focus is altered by rotating the echelon about a vertical axis. The convergence is increased, as would be expected, by turning the echelon so as to reduce the width of the emergent beam (see fig. 3 A), and diminished when it is turned so as to make the beam broader (fig. 3 B). Changes in the focus are also produced by covering some of the step-faces at either end of the echelon.

Use of Auxiliary Prism.

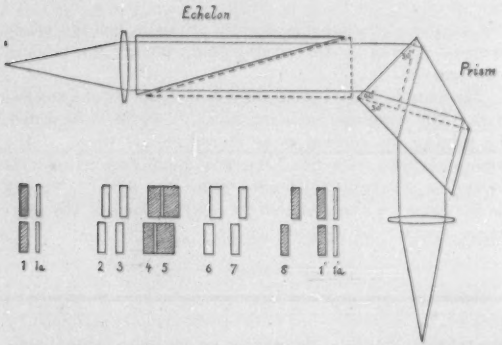
Auxiliary spectroscopes have generally been employed to pick out the particular line in the spectrum to be examined by the echelon, the slit of the echelon spectroscope being illuminated with the selected line; but a modification in the design of this instrument was made for Professor Schuster so that the prism from the auxiliary spectroscope could be mounted next to the echelon, as shown in fig. 1, the prism being made larger than usual in order to take the full width and height of the echelon beam. This arrangement, which appears to have been originally contemplated by Professor Michelson, has been found to have important advantages.

* Twyman, Proceedings of the Optical Convention, 1905, p. 53.

The dispersion produced by the prism, which is 2 per cent. of that given by the echelon, is subtracted from the echelon dispersion when the echelon is in the usual position shown by full lines, and is added to it when the echelon is in the reversed position shown by the dotted lines.

The change of 4 per cent. in the dispersion obtained in this way produces the alteration in the spectrum shown in fig. 1. The distance apart of successive orders of the same wave-length is not altered; but when the dispersion is reduced by the prism, all the lines belonging to the same order approach one another and draw away from the neighbouring orders. It is evident from fig. 1 that all but two of the lines

Fig. 1.



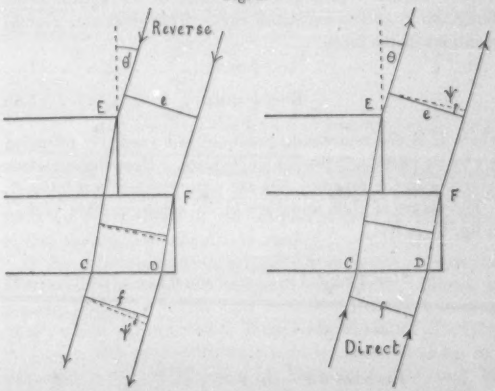
represented belong to the same order, and that these two lines, marked 1′ and 1′*a*, belong to the next higher order. With the prism on an auxiliary spectroscope, no evidence of this kind is obtained; and it has simply been assumed in some cases that the companion lines belonged to the bright central line nearest to them.

Primary Action of the Echelon in the Direct and Reversed Positions.

Equations for the Principal Maxima.

The theory of the echelon given by Michelson * for the case of light falling normally on the larger end, has been extended by Galitzin †, who has investigated, both by calculation and experiment, the motion and changes in the dispersion of the spectra when the echelon is rotated through small angles within the limits of two or three degrees on either side of the normal position. As the theory of the echelon in the reversed position has not, as far as I know, hitherto been considered, a comparison will be made below with the theory of the ordinary action.

Fig. 2.



In fig. 2 the reversed and direct cases are represented diagrammatically. The rays and wave-fronts drawn with full

* A. A. Michelson, Astrophysical Journal, viii. pp. 36–47 (1898). American Journal [4] v. pp. 215–217 (1898). *Journ. de Phys.* [3] viii. pp. 305–314 (1899).

† Fürst B. Galitzin, *Bulletin de l'Académie Impériale des Sciences de St. Pétersbourg*, 5 série, t. xxiii. pp. 67–118 (1905).

lines are those regularly refracted, the dotted wave-fronts are drawn perpendicular to the directions in which a maximum of order n is formed. The angle between this direction and the normally refracted rays is called ψ in the direct case and ψ' in the reversed case. The distance apart in the air of regularly refracted rays passing through corresponding points E and F of neighbouring step-faces is called f at the end of the echelon and e on the step side. The condition for a principal maximum is that the sum of the distances of E from the incident wave-front on one side and the dotted wave-front on the other, shall be n wave-lengths greater than the sum of the distances of the corresponding point F from the same wave-fronts.

When the angles ψ and ψ' are sufficiently small, we may employ Fermat's principle and measure the optical paths along the regularly refracted rays. This gives the general equations in the form

$$R - e\psi = n\lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$R - f\psi' = n\lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1A)$$

where R is the retardation produced in a regularly refracted ray by its passage through a single plate. Here the equations referring to the reversed case are distinguished by a letter A.

The values of R, e, and f are given below, and are plotted in fig. 4 (p. 831).

$$R = t\left(\mu\sqrt{1-\frac{\sin^2\theta}{\mu^2}} - \cos\theta\right), \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$e = s\cos\theta + t\sin\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$f = s\cos\theta + \frac{t}{\mu}\sin\theta\,\frac{\cos\theta}{\sqrt{1-\frac{\sin^2\theta}{\mu^2}}}, \quad . \quad . \quad . \quad (4)$$

Here s is the width of the step-faces, t the thickness of the plates, and θ the angle of incidence of the light on the plates. This angle is in practice generally kept within the narrow limits of $\pm 2°$, and it is sufficient to retain only the lowest

powers of θ. These expressions then become

$$R = R_0\left(1 + \frac{\theta^2}{2\mu}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$e = s + t\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$f = s + t\frac{\theta}{\mu}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

R_0 (the value of R when $\theta = 0$) is $(\mu - 1)t$. The parabolic expression for $R - R_0$ in equation (5) gives a very close approximation to the value given by (2), the difference, which depends on θ^4, being only 1 part in 800 when $\theta = 5°$. The slight deviations of e and f from the linear expressions (6) and (7) may be detected in fig. 4.

Substituting the approximate expressions for R, e, and f in equations (1) and (1 A), we obtain the general equations in the form:

$$R_0\left(1 + \frac{\theta^2}{2\mu}\right) - s\psi\left(1 + \frac{t}{s}\theta\right) = n\lambda, \quad . \quad . \quad . \quad (8)$$

$$R_0\left(1 + \frac{\theta'^2}{2\mu}\right) - s\psi'\left(1 + \frac{t}{s}\frac{\theta'}{\mu}\right) = n\lambda. \quad . \quad . \quad . \quad (8A)$$

Equation (8) is in agreement with Galitzin's calculated values and has the support of his measurements which were made on orders close to the position of greatest brightness so that the angle ψ was always small.

If the principal maxima several places removed from the direction of the regularly refracted rays are considered, it becomes necessary to take into account the terms depending on ψ^2, which were neglected in applying Fermat's principle. When the path of the diffracted light is measured along the diffracted rays, the equation for the direct case may be written

$$R - e\sin\psi + (t\cos\theta - s\sin\theta)(1 - \cos\psi) = n\lambda. \quad . \quad (9)$$

If we now suppose the light to retrace its path, the angle of incidence, θ', on the step-faces is equal to $\theta - \psi$ (see fig. 2), and the angle $\theta' + \psi'$ at which the diffracted rays leave the end plate is equal to θ. Hence the equation for the reversed

case may be found by substituting $\theta'+\psi'$ for θ and ψ' for ψ in equation (9) after substituting for R its value from equation (2). The equation thus obtained may be written in the form:

$$t(\sqrt{\mu^2-\sin^2(\theta'+\psi')}-\cos\theta')-s\{\cos\theta'\sin\psi' - \sin\theta'(1-\cos\psi')\}=n\lambda. \quad . \quad (9A)$$

If the terms whose order in θ and ψ is higher than the second are neglected, equations (9) and (9 A) reduce to

$$R_0\left(1+\frac{\theta^2}{2\mu}\right)-s\psi\left(1+\frac{t}{s}\theta-\frac{1}{2}\frac{t}{s}\psi\right)=n\lambda. \quad . \quad (10)$$

$$R_0\left(1+\frac{\theta'^2}{2\mu}\right)-s\psi'\left(1+\frac{t}{s}\frac{\theta}{\mu}+\frac{1}{2}\frac{t}{s}\frac{\psi'}{\mu}\right)=n\lambda. \quad (10A)$$

These equations only differ from (8) and (8 A) by the addition of small terms depending on ψ^2 and ψ'^2, whose values are given in Table I. below.

Position of the Orders in the Field of View and Dispersion.

According to the first approximation formulæ (1) and (1A)

$$\psi=\frac{R-n\lambda}{e} \text{ and } \psi'=\frac{R-n\lambda}{f}.$$

R, e, and f depend on θ but not on ψ; so the orders are equally spaced and the dispersions, given by

$$\frac{d\psi}{d\lambda}=-\frac{n-t\frac{d\mu}{d\lambda}}{e} \text{ and } \frac{d\psi'}{d\lambda}=-\frac{n-t\frac{d\mu}{d\lambda}}{f},$$

do not vary along the spectrum.

The second approximation formulæ (10) and (10 A) give

$$\psi=\frac{R-n\lambda}{s\left(1+\frac{t}{s}\theta-\frac{1}{2}\frac{t}{s}\psi\right)} \text{ and } \psi'=\frac{R-n\lambda}{s\left(1+\frac{t}{s}\frac{\theta}{\mu}+\frac{1}{2}\frac{t}{s}\frac{\psi'}{\mu}\right)}$$

If the echelon is rotated a little, so that one order, say the mth, is in the position of maximum brightness where

ψ or $\psi'=0$,

$$\psi=\frac{m-n}{1+\frac{t}{s}\theta-\frac{1}{2}\frac{t}{s}\psi}\left(\frac{\lambda}{s}\right) \text{ and } \psi'=\frac{m-n}{1+\frac{t}{s}\frac{\theta}{\mu}+\frac{1}{2}\frac{t}{s}\frac{\psi'}{\mu'}}\left(\frac{\lambda}{s}\right)$$

Table I. gives the values of the small terms in the denominators depending on ψ and ψ' for the first five orders on either side of the central order.

TABLE I.

$m\sim n$.	$\frac{1}{2}\frac{t}{s}\psi$.	$\frac{1}{2}\frac{t}{s}\frac{\psi'}{\mu}$.
1	·003	·002
2	·005	·003
3	·008	·005
4	·010	·006
5	·013	·008

The Wave-Length Intervals between the Orders.

The repetition of a line in a number of orders provides an echelon spectrum with a wave-length scale of nearly equal divisions, and the wave-length value of these divisions, $\Delta\lambda$, is a constant of the echelon which can be calculated for any wave-length from the thickness of the plates and the refractive indices given for the glass. The expressions for $\Delta\lambda$ for the direct and reversed cases may be obtained from the general equations. Neglecting the terms depending on θ^2, which do not affect the values by more than one part in ten thousand, they both reduce to

$$\Delta\lambda=\frac{\lambda}{n-t\frac{d\mu}{d\lambda}}.$$

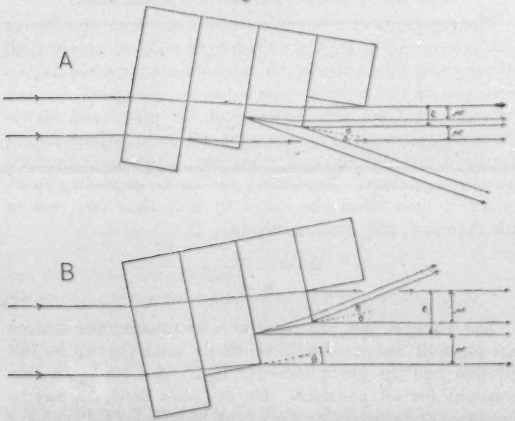
The changes that are made in n by rotating the echelon are so small compared with its whole value (10,070 for this echelon and the green mercury light) that $\Delta\lambda$ is sensibly constant for all positions. On the other hand, $\Delta\lambda$ may be increased or decreased by 2 per cent. by means of the auxiliary prism, as described on p. 824.

Notes on Changes produced by rotating the Echelon about a Vertical Axis.

Changes in brightness, and the position of greatest brightness — As an order is moved across the field of view by rotating the echelon, it crosses the lateral maxima of the distribution of light due to the individual step apertures, disappearing as it crosses the diffraction minima, and having its greatest brightness as it crosses the central diffraction maximum. This position of greatest brightness corresponds to the direction of the regularly refracted rays, and is the origin from which the angle ψ is measured. It does not quite coincide with the position of the image of the slit when the echelon is removed, the position of greatest brightness being displaced towards the side on which the step-faces of the echelon lie. This displacement indicates that the echelon-plates are slightly prismatic.

Changes in dispersion and retardation.—Some of the effects produced by rotating the echelon about a vertical

Fig. 3.

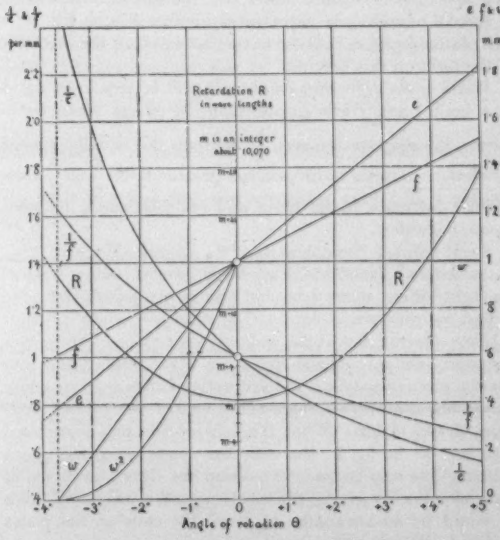


axis are represented in fig. 3. The echelon is shown at A

rotated so as to make θ negative; in this case e is smaller and therefore (see p. 828) the dispersion is larger than in the normal position. B shows the echelon rotated in the opposite direction so that e is increased and the dispersion decreased.

The way in which R, e, and f change with θ is represented in fig. 4. The scale for R is given in wave-lengths, m being the number of wave-lengths in the retardation of the order

Fig. 4.



which is nearest to the position of greatest brightness on the lower retardation side when $\theta=0$. The positions of the row of marks on the $\theta=0$ line represent, by their distances from the apex of the curve, the distances of the orders near the mth from the position of greatest brightness when $\theta=0$, and the abscissæ of the curve passing through the marks which

lie within it, give the angle through which the echelon must be turned in either direction in order to bring orders higher than the mth up to the position of greatest brightness.

The reciprocals of e and f are also plotted to show how the dispersion is changed by the rotation in the direct and reversed cases. These changes in dispersion are simply changes in magnification, as they are not accompanied by any sensible change in the wave-length intervals between the orders.

The way in which the effective width, w, of the individual step apertures is reduced when the echelon is rotated so as to make θ negative, is represented in figs. 3 and 4; w^2 is also plotted in fig. 4, to indicate the falling off in the intensity of the light on this account.

Width of the diffraction-bands.—It will be seen from fig. 4 that both e and f are greater than w, except when $\theta=0$. Hence, the angular interval, $\frac{\lambda}{w}$, between the equally spaced diffraction minima is in general greater than $\frac{\lambda}{e}$ or $\frac{\lambda}{f}$, the angular intervals, in the direct and reversed cases, between successive orders.

Light reflected from the ends of the plates.—The step ends of the echelon plates, which are finely ground, scatter some of the light falling upon them and also give regularly reflected beams, represented in fig. 3. The reflected beams produce echelon spectra; but as the ends are not polished, the spectra are poorly defined and the spreading of the light brings into greater prominence the broad diffraction-bands corresponding to the narrow sources represented by the illuminated end-faces of the plates. When the echelon and prism are used together, as in fig. 1, the reflected spectrum of the green mercury-line may be made to overlap the direct spectrum of the violet line by giving θ a small negative value. I think it would be an advantage to have the ends of the plates polished or blackened.

The position of minimum deviation.—As the echelon is rotated from one side of the normal position to the other, the orders first move across the field of view in the direction of decreasing deviation (measured by ψ) and then turn round and go back again. If the value of R happens to be a whole

number of wave-lengths for normal incidence, say $m\lambda$, then the mth order will come to its position of minimum deviation in the position of greatest brightness when $\theta=0$, but higher and lower orders will not then be quite in their positions of minimum deviation. Consider, for example, a lower order: ψ (or in the reversed case ψ') will be positive, and its value when $\theta=0$ can be reduced a little by increasing θ (in the positive direction), as at first the value of R is almost stationary, while e is increasing, and therefore the dispersion is decreasing.

By differentiating the general equations (1) and (1 A), it will be found that the conditions for the turning-points in the direct and reversed cases are,

$$\frac{d\mathrm{R}}{d\theta}=\psi\frac{de}{d\theta} \quad \text{and} \quad \frac{d\mathrm{R}}{d\theta}=\psi'\frac{df}{d\theta}.$$

Substituting the values of the differential coefficients and calling the minimum values of the deviation in the two cases ψ_{M} and ψ'_{M} and the corresponding angles of incidence θ_{M} and θ'_{M},

$$\theta_{\mathrm{M}}=\frac{\mu}{\mu-1}\psi_{\mathrm{M}} \quad \text{and} \quad \theta'_{\mathrm{M}}=\frac{1}{\mu-1}\psi'_{\mathrm{M}}.$$

Hence as the echelon is rotated, so as to increase θ, the orders higher than the mth come to their minimum deviation positions before the echelon is normal to the light, and the lower orders have their minimum deviation after the normal position is passed. The central orders are very nearly in their positions of minimum deviation when $\theta=0$. If ψ_0 and ψ_0' are written for the values of ψ and ψ' in this case, it may be shown that

$$\frac{\psi_0-\psi_{\mathrm{M}}}{\psi_{\mathrm{M}}}=\frac{1}{2}\frac{t}{s}\frac{\mu}{\mu-1}\psi_{\mathrm{M}} \quad \text{and} \quad \frac{\psi_0'-\psi'_{\mathrm{M}}}{\psi'_{\mathrm{M}}}=\frac{1}{2}\frac{t}{s}\frac{1}{\mu-1}\psi'_{\mathrm{M}},$$

and the values calculated from these formulæ show that, when this echelon is in the direct position, $\psi_0-\psi_{\mathrm{M}}$ is ·7 per cent. of ψ_{M} for an order one place from the position of greatest brightness, 1·4 per cent. of ψ_{M} for an order two places away, and so on. The corresponding values for the reversed position are ·4 per cent. and ·9 per cent.

Effects produced by Temperature Changes.

The position of the various orders in the field of view when the echelon is in a definite position, such as the normal position, depends on the temperature of the echelon and on the refractive index of the surrounding air. Records of the temperature of the echelon and micrometer readings of the position of the orders in the field of view, taken from day to day, show that a rise of temperature of 8·6° C. moves the spectrum through a distance equal to the interval between neighbouring orders, indicating an increase of one wave-length in the retardation, R, for this rise of temperature.

Curving of the Spectrum Lines.

The echelon spectrum lines, like those of a prism spectrum are curved with the concave side toward the violet end. The curving is due to the variation in the angle of incidence, and therefore in the retardation, R, of light from different points along the length of the slit. The theory has been given by Laue *.

Consider the simplest case, in which the slit is vertical and the plates are normal to the axis of the collimator: then light from points above and below the centre of the slit will have a vertical plane of incidence on the plates. Writing i instead of θ for the angle of incidence in equation (3),

$$R - R_0 = \frac{1}{2} t \frac{\mu - 1}{\mu} i^2 ;$$

but from equation (1),

$$R - R_0 = e(\psi - \psi_0) ;$$

and in this case $e = s$, so that

$$\psi - \psi_0 = \frac{1}{2} \frac{t}{s} \left(\frac{\mu - 1}{\mu}\right) i^2. \quad . \quad . \quad . \quad . \quad (8)$$

Hence, for small values of i, the spectrum lines are parabolas. The curvature is very small. If, for example, the extreme values of i are ten times the angular separation of the orders,

* *Physikalische Zeitschrift*, vi. pp. 283-285 (1905).

$\Delta\psi$, then the top and bottom of the curved image of the slit, representing one order of the spectrum, will be 2 per cent. of $\Delta\psi$ to the violet side of the centre of the image.

Effects produced by Rotating the Echelon about a Horizontal Axis parallel to the Plates.

The echelon can readily be tilted in this way by placing a block under the foot at the small end. The block which I employ tilts the echelon through an angle of nearly 3°, and the photographs Nos. 1 to 3 in Pl. XXXI. of the green mercury line were taken with this angle of tilt. The horizontal axis of the parabolic lines, which passes through the normal to the plates, has been raised much above the field of view of these photographs, so that the lines where they cross the field of view are considerably inclined to the vertical. The reproductions in Pl. XXXI. have the centre of the curves below them, as they have been turned round in order to put the shorter wave-lengths on the left.

Part II.—Secondary Action of the Echelon.

Secondary Bands in the Primary Spectrum.

The inclined spectrum lines of photograph No. 1, Pl. XXXI. are broken up and have a ropy or screw-like appearance because they are crossed by a secondary system of bands. This screw-like structure of echelon spectrum lines was observed by Gehrcke *. He does not explain how the new bands are produced, but shows that the appearance can be imitated by tilting an echelon with only two apertures, so that the echelon bands slope across the central vertical diffraction band.

When the echelon is in the ordinary position the secondary bands are parallel to the spectrum lines, and so their effects, though very important, are not so easily recognized as they are in this photograph taken with the echelon tilted.

Character of the Secondary Bands.

The secondary bands are superposed on the echelon lines and resemble them in appearance; they are also affected in the

* *Annalen der Physik*, xviii. p. 1074 (1905).

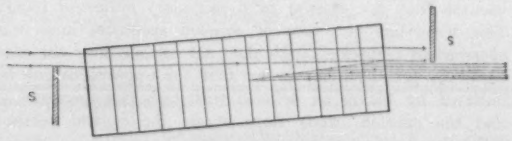
same way, but to a greater extent, by adjustments of the echelon. When the echelon is rotated, for example, the secondary bands move faster than the spectrum lines in the same direction and so move across them. When the echelon plates are vertical, the secondary bands, like the echelon lines, are vertical in the centre of the field; they are also curved in the same direction but more strongly. Their width is about the same as that of the finer spectrum lines, and so, in the ordinary position of the echelon, they are not easily recognized; but when the echelon is tilted they become more inclined than the spectrum lines in the same direction and can then be seen as in photograph No. 1, Plate XXXI., intersecting all the spectrum lines.

The behaviour of the secondary bands suggested the idea that they might be spectrum lines of a higher order, such as might be produced by the reflexion of light in the echelon.

Fabry and Perot Spectrum produced by the Secondary Action.

The secondary light, which has been twice reflected in the echelon, is by no means negligible. The echelon was tilted as shown in fig. 5, screens SS being arranged to cut out the

Fig. 5.



primary light, and it was found that the echelon was bright with secondary light coming out below the second screen, the brightness extending down to the bottom of some of the step apertures.

Suppose that each interface reflects a very small proportion of the light falling upon it and leave out of account for a moment the step structure of the echelon. There will be a secondary beam produced by the combination of the n faint secondary beams which have gone back through one plate, n

being the number of plates coming into action; let the retardation of this beam be $\triangle$. There is another secondary beam whose retardation is $2\triangle$, made up of $n-1$ faint beams which have each gone back through two plates, and so on. Hence the secondary action of the pile of plates in the echelon is similar to that of a Fabry and Perot spectroscope, and, under suitable conditions of illumination, the secondary light by this action would be thrown into a ring spectrum of a high order. The retardation is $2\mu t$ for normal incidence, so it is $\frac{2\mu}{\mu-1}$ times as great, in this case about five and a half times as great, as that of the primary spectra.

The secondary light also undergoes the ordinary echelon treatment as it leaves by the step-faces, and it is therefore confined to the primary spectrum lines.

The photograph No. 2, Pl. XXXI., which shows short portions of the rings of the Fabry and Perot spectrum crossing the echelon spectrum, was obtained by stopping the primary light in the way described above, and making the echelon lines broad by widening the slit. The spectrum lines produced by the primary action of the echelon upon the secondary light are inclined because the echelon is tilted as shown in fig. 5, and the centre of the ring system is above the field of view for the same reason. The echelon has also been rotated (in the positive direction of θ), so the centre of the ring system is displaced laterally as well as vertically.

The Fabry and Perot spectrum lines are not dependent for their definition, like the echelon spectrum lines, on the narrowness of the slit; their want of clearness in this photograph is partly due to the overlapping of lines belonging to different orders, which is produced by the orders overlapping four deep.

The Secondary Point Spectrum.

If the slit, instead of being opened wide to show portions of the rings, is made narrow enough to give good definition to the primary echelon action, the secondary light which has undergone both actions is confined to dots indicating the position of the points of intersection of the spectrum lines in

one system with corresponding lines, that is lines representing the same wave-length, in the other system.

The horizontal dispersion of the dots, given by the ordinary echelon action, now prevents overlapping, as the wave-length interval between the orders in this system is a little greater than the length of the spectrum, and the secondary light gives in this point spectrum, therefore, the advantages of a very high order without the usual overlapping.

Gehrcke and Baeyer * obtained similar spectra which they call interference points, by crossing plane parallel plates, and have pointed out the advantages of combining two independent high dispersions.

A photograph of the secondary point spectrum, obtained with an exposure of one hour, is reproduced in Plate XXXI., No. 3. The echelon had the usual tilt, about 3°, and the echelon table was rotated about 2° in the positive direction of θ from the normal position. The dispersion in the Fabry and Perot system is, in this case, twelve times that in the echelon system, so the wave-length intervals can be best determined by the position of the dots in the former system, while the dots can be recognized and their wave-lengths can be roughly fixed by their position in the latter system.

There was no difficulty in recognizing the dots marked 1, 2, 3, 4, 5, 6, and 7 in the photograph, which represent well-known lines in the green line spectrum.

In order to determine the wave-length intervals, it is necessary to calculate $\triangle\lambda$, the wave-length interval between neighbouring orders of the Fabry and Perot spectrum. It was found from the formula

$$\triangle\lambda = \frac{\lambda}{p - 2t\cos r\dfrac{d\mu}{d\lambda}},$$

where p, the order of the spectra, is given by

$$p = \frac{2\mu t\cos r}{\lambda},$$

r being the angle of refraction of the light in the plates, and t their thickness. The value found in this way for the centre

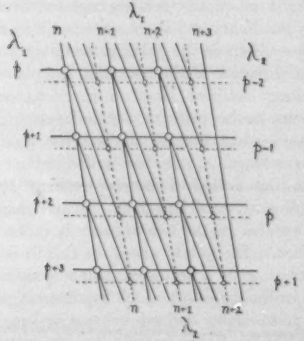
* E. Gehrcke and O. v. Baeyer, *Annalen der Physik*, xx. p. 269 (1906).

of the photograph is 96·5 m.Å. The wave-length intervals between the components deduced from the measurements of the photograph and this value of $\triangle\lambda$ are given in Table III., p. 393; most of them agree closely with the values which have been found by other methods.

Character of the Point Spectra produced by the Secondary Light.

Fig. 6 is a diagram representing the type of spectrum produced by the secondary light when the echelon is in the direct position and tilted about a horizontal axis, that is, the

Fig. 6.



type represented in the photograph of the secondary point spectrum No. 3, Pl. XXXI.; but in order to make the diagram clearer, the ratio of the dispersion in the Fabry and Perot system to the dispersion in the primary echelon system has been made much smaller than it is in the photograph.

In this case all the rays of light from a given point of the slit are parallel to one another during their passage through the plates, as the lateral diffraction does not take place until the light leaves the echelon by the step-faces. The spectra, or lines of constant retardation, in the Fabry and Perot system are therefore drawn in the diagram as horizontal lines.

When the photograph was taken the echelon had been rotated (from the normal position about a vertical axis) as well as tilted, so the plane parallel to the axis of the collimator and the diffracting apertures would not be quite vertical, but the deviation from the vertical is a small angle of the second order (equal to the product of the small angles of tilt and rotation), and it will be seen that lines joining two dots representing the same wave-length and of the same order in the Fabry and Perot system, would be sensibly horizontal, although if the slit had been opened wide so as to show portions of the Fabry and Perot rings, they would have been inclined about 45°, as in photograph No. 2.

When the echelon is reversed, the lateral diffraction takes place as the light enters the echelon and the lines of equal retardation in the Fabry and Perot system will be represented by circles whose centre is the point in the image plane corresponding to the direction of the normal to the plates.

The four long inclined lines in the diagram (fig. 6) represent spectra in the primary echelon system; they are inclined to the vertical because the echelon is tilted. The orders of a wave-length λ_1 are represented in the two systems by full lines, the dotted lines representing similarly the spectra of a second wave-length λ_2 greater than λ_1. The order of the spectra in the two systems is indicated at the ends of the lines. The points where the full lines of the two systems intersect give a series of points (marked by the larger circles in the diagram) which represent λ_1 in the joint system, each point being defined by two orders. The top left-hand point in fig. 6 may be described for example as the np order of wave-length λ_1. In the same way the intersections of the dotted lines give the positions where λ_2 is represented in the joint system, and the shorter inclined lines joining the λ_1 and λ_2 points of the same denomination represent the appearance of the joint spectra corresponding to a spectrum continuous between these limits.

It will be seen that there is no chance of spectra of different denominations overlapping in the joint system as long as there is no overlapping in one of the two systems which are combined.

The spectra in the two systems may be regarded as forming

an oblique system of coordinates; the echelon system giving horizontal dispersion may be called the X system and the Fabry and Perot system the Y system. Then the slope of the spectra in the joint system, $\frac{dy}{dx}$, is the ratio of $\frac{\partial\lambda}{\partial x}$, the dispersion in the X system, keeping y constant, to $\frac{\partial\lambda}{\partial y}$, the dispersion in the Y system, keeping x constant.

One important feature of these point spectra is that they give a system of lines whose definition depends in general on the defining power of the two systems which are combined, but does not depend on the smallness of the range of wave-length in the light examined, so long as that range does not exceed a certain relatively large limit.

If the definition is poor in one of the two systems, a monochromatic radiation would be represented by dots elongated in the direction of the spectrum lines of the other system, and this would in general spoil the sharpness of the lines representing in the joint system a spectrum continuous between narrow limits; but if these lines are nearly parallel to the spectrum lines of one system, the want of definition in the other will not spoil the definition of the lines, as each dot representing a single wave-length will be drawn out in a direction nearly parallel to the length of the joint spectra. This special case is realized when the echelon is in the direct or reversed normal positions.

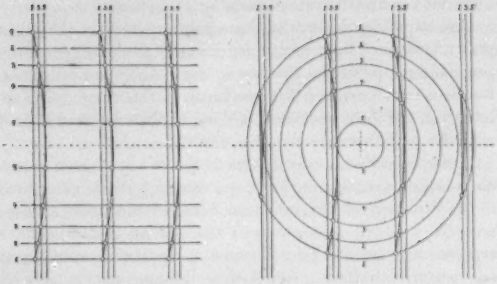
The diagrams A and B in fig. 7 represent the conditions in these cases. They are similar to fig. 6, but spectrum lines have been drawn in the Fabry and Perot system for a series of five wave-lengths whose values increase by equal increments from λ_1 to λ_5. In the primary echelon system the lines representing alternate wave-lengths have been omitted. B represents the case in which the echelon is in the reversed position, the radii of the circles, representing the spectra in the Fabry and Perot system, being chosen so that their squares increase by equal increments. The horizontal lines in diagram A represent the same spectra when the echelon is turned round into the direct normal position.

For convenience in drawing the diagrams, the dispersion in the Fabry and Perot system has been represented as only

about ten times as great as that in the primary echelon system. It should be several hundred times as great even for those parts of the diagram farthest from the centre.

It will be seen that in both diagrams the secondary point spectra, drawn through the points of intersection of lines

Fig. 7.



A B

representing the same wave-length in the two systems which are combined, are curved so as to be concave towards the side of shorter wave-length in the primary echelon system; the curvature has been much exaggerated by taking the ratio of the dispersions so much smaller than it actually is.

Origin of the Secondary Bands.

There is no doubt that the secondary bands which are observed when the echelon is employed in the usual way, that is, with the secondary light superposed on the primary, are very closely connected with the point spectra produced by the secondary light. All the characteristics of the secondary bands described on page 836 may be explained by supposing that they are, like the point spectra of the secondary light, the loci of the points of intersection of lines representing the same wave-length in the primary echelon and Fabry and Perot spectra.

Another characteristic which may be explained in the

same way, with the help of fig. 7, is that when the echelon is in the direct normal position the secondary bands affect each order of the spectrum in the same way, while, when the echelon is reversed, there are marked differences between the different orders.

The secondary bands have other characteristics which may find their explanation in interference taking place between the secondary light and the primary. The dark secondary bands appear to cut through the bright primary spectra. This is shown to some extent in photograph No. 1, Pl. XXXI., but it is much more marked when the secondary light is made relatively stronger by covering half the echelon apertures so as to stop all the light which has passed through less than half the plates in the echelon. The brightness of the primary lines is no doubt increased above and below the dark bands, making them appear darker by contrast, but I think there is little doubt that there has been an actual reduction in the brightness of the primary light in the dark secondary bands.

Another feature of the secondary bands, which may be seen in photograph No. 1, Pl. XXXI., where they cross the line 1, is that they always appear in pairs, fainter and stronger bands alternating with one another. This may be connected with the difference in phase between the primary and the secondary light produced by the two reflexions of the latter. Apart from these phase changes at reflexion, the primary and secondary light would be in phase at the points of intersection of their maxima, as the retardations in both systems are whole numbers of wave-lengths in the direction in which the maxima are formed. There are two cases to be considered for the secondary light, according as it has undergone both reflexions at interfaces or one reflexion at an interface and the other at an external surface. In the former case, as the air-films are very thin, it is possible that a change of phase of $\frac{\pi}{2}$ may be introduced at each reflexion which would give the best phase conditions for interference between the maxima. The latter case may account for the second set of bands shifted relatively to the first.

Part III.—Structure of the Green Mercury Line. (5461.)

Description of Spectrum given by an Arons Lamp, and Comparison with the Results obtained by other Observers.

This spectrum consists of a bright principal line, which is a close double, with six companion lines, three on either side. A photograph of the primary echelon spectrum is reproduced in Plate XXXII. together with a diagram of the spectrum. The photograph shows, in addition to the genuine components, a number of faint lines which have their origin in the echelon. The genuine components are numbered from 1 to 8, and the false lines are marked $1a$, $1b$, $3a$, &c. The lines $1a$, $1b$, $1c$, &c., mark the positions of the secondary diffraction maxima on the longer wave-length side of the principal diffraction maximum 1, and the lines $3a$, $5a$, $6a$, $7a$, and $8a$ represent the first secondary maxima on the longer wave-length side of the lines 3, 5, 6, 7, and 8. When the aperture of the echelon is reduced by covering the first ten step-faces at the smaller end, the faint lines move away from their parents into the new positions of the secondary maxima corresponding to the reduced number of apertures.

The numbers in Table II. and Table III. give the distances of the various components from the component of shortest wave-length. I adopted this method of measuring the positions because the bright central line appeared to be the most variable component, while the component of shortest wave-length is a good reference line. The results of the other observers, given in the Tables, are not convenient for comparison in their original form, because Gehrcke and Baeyer*, Janicki†, and Galitzin and Wilip‡, give the distances from the centre of the principal line, while Fabry and Perot§, Baeyer‖ (in a later paper), and Nagaoka¶, divide

* F. Gehrcke and O. Von Baeyer, *Annalen der Physik*, vol. xx. p. 269 (1906). † L. Janicki, *Annalen der Physik*, vol. xix. p. 36 (1906).

‡ Fürst B. Galitzin und J. Wilip, *Mémoires de l'Académie Impériale des Sciences de St. Pétersbourg*, sér. 8, vol. xxii. no. 1 (1906).

§ Astrophysical Journal, xv. p. 218 (1902).

‖ O. v. Baeyer, *Verhandlungen der Deutschen Physikalischen Gesellschaft*, ix. no. 4, p. 84 (1907). ¶ Nagaoka, 'Nature,' vol. lxxvii. p. 582 (1908).

the principal line into two components and measure the distances from the brighter one.

Another difficulty in the comparison is that the values obtained by Gehrcke and Baeyer, and later by Baeyer, by the use of crossed plane-parallel plates, are systematically higher than the values obtained with echelon spectroscopes, which agree fairly well amongst themselves. I have reduced all Baeyer's intervals 5 per cent., and the earlier values of Gehrcke and Baeyer (the means of three sets of measurements given in their paper) by 3 per cent.; and it will be seen that, apart from these differences in the constants, the two independent methods are in close agreement.

TABLE II.—Measurements of the Green Mercury-line Spectrum.

The distances of the component lines from the component of shortest wave-length are given in milli-Ångström units.

Fabry & Perot.	Crossed Plates.		Echelon Spectroscopes.		Reference Numbers.
	Gehrcke & Baeyer.	Baeyer.	Janicki.	Galitzin & Wilip.	
·01 Å. units	reduced 3 p.c.	reduced 5 p.c.			
0	0	0	0	0	1
15	126	136	133	137	2
17	169	169	166	168	3
...	...	189	...	189	...
22	234	214	232	236	4 Central Band.
23		238			5 Central Band.
31	318	320	320	321	6
36	364	363	365	365	7
...	...	449	...	...	8

The brightest component in each case is underlined.

Fabry and Perot's values* given in the Table are those published by Zeeman† in 1902. They help to confirm

* M. Perot informs me (in a letter dated Oct. 13th, 1908) that they have not published any particulars of this line since then.

† Astrophysical Journal, xv. p. 218 (1902).

TABLE III.—Measurements of the Green Mercury-line Spectrum (*continued*).

Nagaoka.	Author.			Reference Numbers.	
Echelon Spectroscope.	Primary Spectrum.	Secondary Point Spectrum.	Width.		
0	0	0	17	1	
31	...	...	...	...	
72	...	...	...	...	
105	...	...	...	...	
137	135	141	13	2	
163	164	167	13	3	
189	...	...	...	...	
223	232 (217)	217	16	4	Central Band.
247	232 (243)	243	24	5	Central Band.
280	...	...	...	...	
315	319	322	17	6	
356	363	365	12	7	
390	...	...	...	...	
448	448	...	14	8	
477	...	...	...	...	

the accuracy of the constants employed in the echelon calculations.

The results obtained by several other observers are given by Janicki*.

The series of faint lines given by Nagaoka resembles the series of false lines in my primary echelon photographs. I published my measurements for comparison with his before discovering that the faint lines in my photographs were not genuine †.

Width of the Components.

The mean values of the widths of the photographic images of the components are given in Table III. If the echelon acted perfectly, the width of a principal light maximum

* *Loc. cit.* p. 61.

† 'Nature,' vol. lxxviii. p. 8 (1908).

constituting one order of the primary echelon spectrum of a monochromatic radiation, with a narrow slit, would be 2/33 of the interval between the orders, which corresponds in this case to a width of 30 m.Å. The narrowest lines in the photographs which still have a measurable width, are about 10 m.Å. wide. The widths of the brighter lines vary considerably with their exposure, while the width of the companion line 8, which is too faint to be overexposed, is the same on each of the three plates on which its width could be measured.

Measurement of Secondary Spectrum.

The results obtained by measuring the secondary point spectrum are also entered in Table III. The photograph measured was exposed for an hour, and is the one reproduced in Plate XXXI., No. 3. It will be seen that the agreement of these results, obtained by an independent method, with the ordinary echelon values, is fairly close, except for the component 2. The methods agree very closely as to the position of the central line. They both give the centre of the double line at 232 and the dividing dark line at 228. The two components of the central line were not measured separately in the primary spectrum photographs, but the position of the dark dividing line was sometimes recorded. The positions of the centres of the components given in brackets in the second column of Table III., are deduced from the position and width of the whole line and the position of the dark dividing line (neglecting its width).

Spectrum given by a Bastian Lamp.

A striking variation in the spectrum of the green line was observed with a Bastian mercury arc-lamp. The glass tube through which the discharge passes is bent nearly into the form of an S in a horizontal plane, so that when one part of the discharge is parallel to the slit plate, another part may be normal to it. When the image of the "end-on" portion of the discharge is put on the slit, the change in the spectrum is so great that it is difficult at first to recognize the components. On measuring a photograph of the "end-on" spectrum, however, it was found that the companion

lines keep their relative positions, although some become broader and brighter, but the dark space between 4 and 5, the two components of the principal line, is greatly increased (from 6 to 26 m.Å.) and, being no longer brighter or broader than the rest, they look like ordinary companion lines.

The "side-on" spectrum of the Bastian lamp resembles more nearly that given by the Arons lamp.

The separation of the components of the principal line in the "end-on" radiation has been investigated by Galitzin and Wilip*, who observed the phenomenon first in the case of a Geissler tube arranged so that the axis of the discharge was normal to the slit plate.

Spectrum of the Green Mercury line given by a hot Mercury lamp.

Janicki† describes the broadening of the components of the green and yellow mercury lines which takes place when a mercury lamp is allowed to become sufficiently hot, and a peculiar system of five equidistant bands which he observed when the original components of the lines were lost in a continuous spectrum. Galitzin and Wilip‡, who give measurements of the bands, suggest that they may be due to a reversal of the lines, or perhaps to some peculiar property of the echelon spectroscope.

The theory of the secondary echelon spectra (see page 841) indicates that the secondary bands would be well defined in a short continuous spectrum, and it seemed probable that they were the bands which Janicki and Galitzin & Wilip observed.

I have tested this point with a quartz-lamp fitted with an air-manometer, similar to that described by Galitzin and Wilip§.

When an arc is started with the lamp cold, the central line (4 and 5 together) and all the companion lines are at

* Fürst B. Galitzin und J. Wilip, "Ueber die Eigenschaften einiger Emissionslinien des Quecksilberdampfes," *Mémoires de l'Académie Impériale des Sciences de St. Pétersbourg*, sér. 8, vol. xxii. no. 1 (1907).

† *Loc. cit.* pp. 49–55.

‡ *Loc. cit.* pp. 34 & 70.

§ *Loc. cit.* p. 4.

first plainly visible, but the pressure in the lamp increases rapidly and the broadening and coalescing of the lines soon takes place. The position occupied by the bright central line in the various orders is now marked by a dark line, probably due to absorption. As the companion lines become merged in the general brightness the fine secondary bands become clearly visible in all the bright parts of the field, the fact that they are secondary bands being shown by their motion relative to the primary spectrum when the echelon is given a slight rotation. The secondary bands show clearly on the green line when the pressure in the lamp is about one atmosphere.

This research was commenced at the request of Professor Schuster, and I have much pleasure in acknowledging the help I have received from the interest he has taken in its progress.

I wish to thank Mr. E. Marsden for assistance in my first experiments on the secondary effects, and Mr. W. A. Harwood, B.Sc., for measuring several of the photographs. My thanks are also due both to Professor Schuster and to Professor Rutherford, for placing the resources of the Physical Laboratories at my disposal.

Discussion.

Dr. Lees referred to the importance of the secondary action and asked the Author if it was now possible to say definitely whether a line observed in an echelon spectrum is genuine or is produced by the instrument.

The Author said that Gehrcke and Baeyer had hoped to supply the means of settling doubts of this kind when they eliminated the ghosts from their green line spectrum by their method of "interference-points." Since then, however, two faint lines had been added to the list of components. With the possible exception of one faint line agreement had now been arrived at between two independent methods.

LVII. *Inductance and Resistance in Telephone and other Circuits.* By J. W. Nicholson, *M.A., D.Sc.*; *Trinity College, Cambridge* *.

I. Effective Inductance.

A general formula for the effective inductance of a circuit consisting of two long parallel wires has been given by the author †, and is suitable for cases in which the current distribution in either wire is greatly affected by the frequency of alternation. In its general form, although its limitations are clearly defined, the result is not well adapted, in the absence of tables, to rapid calculation. The main object of the present paper is to examine certain important cases in detail, and to obtain formulæ capable of immediate use. A calculation of the effective resistance is also made in each case. A problem to which attention has been mainly directed, which includes several practical cases of great interest, is that of the simple telephone circuit, in which the leads are not twisted round each other in order to annul the inductive effects of the earth and of neighbouring circuits.

In the proof of the general formula, the influence of electrostatic capacity was ignored. This imposes a great limitation upon the types of circuit for which the expression may be used. An estimation of the maximum capacity causing no alteration to a given order of accuracy is given in this paper.

Throughout the investigation, only iron and copper wires, as the two extreme cases, are considered. The large permeability of iron completely changes the character of the effect of frequency on its self-induction, as compared with other metals. To all metals except iron greatly used in practice, the formulæ developed for copper wires may be applied with a nearly identical order of accuracy.

Let a be the radius of either wire, c their distance apart, and (μ, σ) their permeability and resistivity. They are equal in all respects.

* Read June 11, 1909.

† Phil. Mag. Feb. 1909; Proc. Phys. Soc. vol. xxi.

The value of the inductance per unit length is then

$$L=4\log\frac{c}{a}+\frac{4\mu}{x}\cdot\frac{\operatorname{ber}x\operatorname{ber}'x+\operatorname{bei}x\operatorname{bei}'x}{(\operatorname{ber}'x)^2+(\operatorname{bei}'x)^2}+M, \quad . \quad (1)$$

where, if $n/2\pi$ be the frequency,

$$x=2a(\pi\mu n/\sigma)^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and, M and N being real,

$$c^2(M+\iota N)=\frac{4a^2(\log a/c-\mu J_0/ka J_0')(1-\mu J_1/ka J_1')}{(\log ha-\mu J_0/ka J_0')(1+\mu J_1/ka J_1')}. \quad (3)$$

The Bessel functions have an argument $ka=x\iota^{\frac{3}{2}}$, and if V be the velocity of propagation of electromagnetic disturbances in the outer medium,

$$h=n^2a^2c/V^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

M may be described as the correction for closeness of the wires.

The effective resistance per unit length is given by

$$R=\frac{4n\mu}{x}\cdot\frac{\operatorname{ber}x\operatorname{bei}'x-\operatorname{bei}x\operatorname{ber}'x}{(\operatorname{ber}'x)^2+(\operatorname{bei}'x)^2}-Nn. \quad . \quad (5)$$

Now the functions $\operatorname{ber}x$, $\operatorname{bei}x$ are usually defined by

$$J_0(x\iota^{\frac{3}{2}})=\operatorname{ber}x+\iota\operatorname{bei}x,$$

so that

$$J_1(x\iota^{\frac{3}{2}})=-\iota^{-\frac{1}{2}}(\operatorname{ber}'x+\iota\operatorname{bei}'x),$$

$$J_1'(x\iota^{\frac{3}{2}})=-\iota(\operatorname{ber}''x+\iota\operatorname{bei}''x),$$

and

$$\frac{\mu}{ka}\cdot\frac{J_1}{J_1'}=\frac{\mu}{x}\cdot\frac{\operatorname{ber}'x+\iota\operatorname{bei}'x}{\operatorname{ber}''x+\iota\operatorname{bei}''x},$$

whence on reduction

$$(1-\mu J_1/ka J_1')/(1+\mu J_1/ka J_1')=(E+\iota F)/D, \quad . \quad (6)$$

where

$$\left.\begin{aligned}Ex^2&=(x\operatorname{ber}''x)^2+(x\operatorname{bei}''x)^2-(\mu\operatorname{ber}'x)^2-(\mu\operatorname{bei}'x)^2,\\ Fx&=2\mu(\operatorname{ber}'x\operatorname{bei}''x-\operatorname{bei}'x\operatorname{ber}''x),\\ Dx^2&=(x\operatorname{ber}''x)^2+(x\operatorname{bei}''x)^2+(\mu\operatorname{ber}'x)^2+(\mu\operatorname{bei}'x)^2\\ &\quad+2\mu x(\operatorname{ber}'x\operatorname{ber}''x+\operatorname{bei}'x\operatorname{bei}''x).\end{aligned}\right\} \quad . \quad (7)$$

But if accents denote differentiations with respect to x,

$$y=\operatorname{ber}x+\iota\operatorname{ber}x$$

is a solution of

$$xy'' + y' = \iota xy,$$

so that

$$x\,\mathrm{ber}''\,x + \iota x\,\mathrm{bei}''\,x + \mathrm{ber}'\,x + \iota\,\mathrm{bei}'\,x = \iota x(\mathrm{ber}\,x + \iota\,\mathrm{bei}\,x),$$

and therefore

$$\left.\begin{aligned} x\,\mathrm{ber}''\,x &= -\,\mathrm{ber}'\,x - x\,\mathrm{bei}\,x, \\ x\,\mathrm{bei}''\,x &= x\,\mathrm{ber}\,x - \mathrm{bei}'\,x, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (8)$$

from which may be deduced

$$\left.\begin{aligned} &\mathrm{ber}'\,x\,\mathrm{bei}''\,x - \mathrm{bei}'\,x\,\mathrm{ber}''\,x = \mathrm{ber}\,x\,\mathrm{ber}'\,x + \mathrm{bei}\,x\,\mathrm{bei}'\,x, \\ &\mathrm{ber}'\,x\,\mathrm{ber}''\,x + \mathrm{bei}'\,x\,\mathrm{bei}''\,x = \mathrm{ber}\,x\,\mathrm{bei}'\,x - \mathrm{bei}\,x\,\mathrm{ber}'\,x \\ &\qquad\qquad - x^{-1}(\mathrm{ber}'\,x)^2 - x^{-1}(\mathrm{bei}'\,x)^2, \\ &(x\,\mathrm{ber}''\,x)^2 + (x\,\mathrm{bei}''\,x)^2 = (\mathrm{ber}'\,x)^2 + (\mathrm{bei}'\,x)^2 + (x\,\mathrm{ber}\,x)^2 \\ &\qquad + (x\,\mathrm{bei}\,x)^2 - 2x(\mathrm{ber}\,x\,\mathrm{bei}'\,x - \mathrm{bei}\,x\,\mathrm{ber}'\,x). \end{aligned}\right\} \quad . \quad (9)$$

Writing

$$\left.\begin{aligned} \mathrm{P} &= \mathrm{ber}^2\,x + \mathrm{bei}^2\,x, \\ \alpha\mathrm{P} &= (\mathrm{ber}'\,x)^2 + (\mathrm{bei}'\,x)^2, \\ \beta\mathrm{P} &= \mathrm{ber}\,x\,\mathrm{ber}'\,x + \mathrm{bei}\,x\,\mathrm{bei}'\,x, \\ \gamma\mathrm{P} &= \mathrm{ber}\,x\,\mathrm{bei}'\,x - \mathrm{bei}\,x\,\mathrm{ber}'\,x. \end{aligned}\right\} \quad . \quad . \quad . \quad (10)$$

Then

$$\mathrm{E}x^2/\mathrm{P} = \alpha(1-\mu^2) + x^2 - 2\gamma x,$$

$$\mathrm{F}x/\mathrm{P} = 2\mu\beta,$$

$$\mathrm{D}x^2/\mathrm{P} = \alpha(1-\mu)^2 + x^2 - 2x\gamma(1-\mu),$$

where

$$\mathrm{D}\lambda_1 = \mathrm{E} + \iota\mathrm{F} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

For the special case $\mu = 1$; $\mathrm{D} = \mathrm{P}$, so that

$$\lambda_1 = 1 - 2(\gamma - \iota\beta)/x, \quad . \quad . \quad . \quad . \quad . \quad (12)$$

a very simple result, which is true in practice for all but iron wires. Again, on reduction,

$$\log a/c - \mu\mathrm{J}_0/ka\mathrm{J}_0' = \log a/c - \mu(\beta - \iota\gamma)/\alpha x,$$

$$\log ha - \mu\mathrm{J}_0/ha\mathrm{J}_0' = \log ha - \mu(\beta - \iota\gamma)/\alpha x,$$

and their quotient becomes

$$(\mathrm{A} + \iota\mathrm{B})/\mathrm{C},$$

where

$$\left.\begin{aligned} x^2\mathrm{A} &= \alpha x^2 \log ha \,.\, \log a/c - \mu\beta x \log ha^2/c + \mu^2, \\ x\mathrm{B} &= \mu\gamma \log hc, \\ x^2\mathrm{C} &= \alpha x^2 \log^2 ha - 2\mu\beta x \log ha + \mu^2, \end{aligned}\right\} \quad . \quad (13)$$

and it has been noticed that $\beta^2+\gamma^2=\alpha$ identically.

With these values, it is found that, M and N being given by (3),

$$c^2\mathrm{CD}(\mathrm{M}+\iota\mathrm{N}) = 4a^2(\mathrm{E}+\iota\mathrm{F})(\mathrm{A}+\iota\mathrm{B}).$$

Thus

$$\left.\begin{aligned} \mathrm{M} &= 4a^2(\mathrm{AE}-\mathrm{BF})/c^2\mathrm{CD}, \\ \mathrm{N} &= 4a^2(\mathrm{AF}+\mathrm{BE})/c^2\mathrm{CD}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (14)$$

Moreover,

$$\frac{\mathrm{CD}}{\mathrm{P}} = \left(\alpha \log^2 ha - \frac{2\mu\beta}{x}\log ha + \frac{\mu^2}{x^2}\right)\left(\frac{\alpha}{x^2}\sqrt{1-\mu^2} + 1 - \frac{2\gamma}{x}\sqrt{1-\mu}\right). \quad (15)$$

$$\frac{\mathrm{AE}-\mathrm{BF}}{\mathrm{P}} = \left(\alpha \log\frac{a}{c}\log ha - \frac{\mu\beta}{x}\log\frac{ha^2}{c} + \frac{\mu^2}{x^2}\right)\left(\frac{\alpha-\alpha\mu^2}{x^2} + 1 - \frac{2\gamma}{x}\right) - \frac{2\mu^2\beta\gamma}{x^2}\log hc \quad . \quad (16)$$

The succeeding reduction depends upon the particular circumstances of the wires. We now write $\mu=1$, and suppose that the wires are of copper. The only case needing separate treatment is that of iron, where μ is very large.

Inductance when $\mu=1$. High frequency.

Quoting the value of M in (14), the inductance becomes, when $\mu=1$,

$$\mathrm{L} = 4\log_e\frac{c}{a} + \frac{4\beta}{\alpha x} + \frac{4a^2(\mathrm{AE}-\mathrm{BF})}{c^2\mathrm{CD}} \quad . \quad . \quad . \quad (17)$$

where

$$\left.\begin{aligned} \frac{\mathrm{AE}-\mathrm{BF}}{\mathrm{P}} &= \left(\alpha\log\frac{a}{c}\log ha - \frac{\beta}{x}\log\frac{ha^2}{c} + \frac{1}{x^2}\right)\left(1-\frac{2\gamma}{x}\right) - \frac{2\beta\gamma}{x^2}\log hc, \\ \frac{\mathrm{CD}}{\mathrm{P}} &= \alpha\log^2 ha - \frac{2\beta}{x}\log ha + \frac{1}{x^2}. \end{aligned}\right\} \quad . \quad (18)$$

Asymptotic formulæ for the functions (α, β, γ) have been developed by Dr. A. Russell *.

* Proc. Phys. Soc. vol. xxii.

When the argument x is not less than 8, then to four significant figures, if $\lambda = x\sqrt{2}$ *, so that, in terms of the characteristics of the two wires,

$$\lambda = 2a\left(\frac{2\pi\mu}{\sigma}\right)^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

we may write

$$\left.\begin{aligned} \alpha &= 1 - \frac{1}{\lambda} + \frac{1}{2\lambda^2} + \frac{3}{4\lambda^3}, \\ \beta\sqrt{2} &= 1 - \frac{1}{\lambda} - \frac{1}{4\lambda^2}, \\ \gamma\sqrt{2} &= 1 + \frac{1}{4\lambda^2} + \frac{1}{2\lambda^3}, \end{aligned}\right\} \quad . \quad . \quad . \quad (20)$$

and after reduction

$$\left.\begin{aligned} \frac{AE-BF}{P} &= \left(1 - \frac{3}{\lambda} + \frac{5}{2\lambda^2} - \frac{3}{4\lambda^3}\right)\log\frac{a}{c}\log ha \\ &\quad - \frac{2}{\lambda}\left(1 - \frac{3}{\lambda} - \frac{1}{4\lambda^2}\right)\log ha + \frac{1}{\lambda}\left(1 - \frac{5}{\lambda}\right. \\ &\quad \left. + \frac{15}{4\lambda^2}\right)\log hc + \frac{2}{\lambda^2}\left(1 - \frac{2}{\lambda}\right), \\ \frac{CD}{P} &= \left(1 - \frac{1}{\lambda} + \frac{1}{2\lambda^2} + \frac{3}{4\lambda^3}\right)\log^2 ha - \frac{2}{\lambda}\left(1 - \right. \\ &\quad \left. \frac{1}{\lambda} - \frac{1}{4\lambda^2}\right)\log ha + \frac{2}{\lambda^2}. \end{aligned}\right\} \quad . \quad (21)$$

We proceed to examine the limiting frequency for which the four-figure accuracy of these results is preserved. For a copper wire of low resistance as ordinarily used, the resistivity may be taken as 1696 C.G.S. If f be the frequency of alternation, $n = 2\pi f$, leading to $x = \frac{1}{5}af^{\frac{1}{2}}$ approximately.

Thus $x = 8$ leads to the determining relation

$$af^{\frac{1}{2}} = 40. \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

If $af^{\frac{1}{2}}$ is less than 40, the four-figure accuracy, in so far as it depends on x, is lost. When f is only 400 per second, the limiting diameter of a wire becomes as large as 4 centimetres, so that the formulæ are unsuited for practice until

* This modification is more convenient for purposes of printing.

the frequency is really great. The usual range of diameter for wires applied to such purposes as telephonic communication is from 1 to 3 millimetres in the case of cables, and to 5·7 millimetres in that of overhead wires. The frequency in cases of transmission of speech will not be greater than 2000. Taking $f=1600$ as perhaps the real maximum, the limiting radius of a wire becomes 1 centimetre. The results therefore cannot have a great accuracy in such a case, and the formula (43) below must be used. But it will appear that the corresponding formula for iron leads can be so employed.

The present results have a two-figure accuracy even when $x=3$, leading to a radius of 3·75 millimetres, and therefore will apply to many cases even of telephony when thicker wires are used. In such cases, their use is preferable to that of (43), in that the calculation is more rapid.

It is necessary not to overlook the other sources of error. Firstly, in the proof of (1)*, k^2a^2 was neglected in comparison with unity. In the most unfavourable case furnished by the telephone, $f=1600$, $a=3$ millimetres, leading to $k^2a^2=10^{-14}$ approximately.

Thus this source of error needs no further consideration in such a case. But in fact, the neglect of this quantity is always legitimate. For if f be the frequency, $k^2a^2=4f^2\mu 10^{-20}$ for a wire of a centimetre radius. It can only therefore affect the fourth figure if $f=20$ million, and the second if $f=200$ million, and such a frequency is never used in combination with so thick a wire.

Secondly, a^4/c^4 was neglected in comparison with unity. Now even in ordinary telephone construction there is usually about $2\frac{1}{2}$ millimetres of paper and air between the wires, so that in the case of the cable, taking an extreme radius of 1·3 millimetres, $c=4a$, and only the third figure is affected. For overhead wires, the distance apart is about 30 centimetres.

The limiting practical closeness of the wires, when a knowledge of the self-induction is required, is, I believe, attained in Mr. A. Campbell's experiments on variable mutual inductances†. The error may then be so great as one or two per cent.

* Phil. Mag. Feb. 1909.

† Phil. Mag. Jan. 1908.

Effect of a Small Capacity.

Heaviside * has divided circuits into five classes, with the following determining properties:—

(1) Submarine cables proper, in which the capacity is the main factor, and whose treatment must follow the electrostatic theory.

(2) Short lines of low frequency, in which self-induction and resistance determine the effects. This class includes the majority of short telephone circuits.

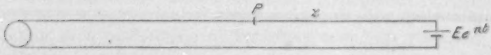
(3) A class in which capacity and self-induction are equally important. These circuits are very difficult in theory.

(4) Circuits of high frequency, but small capacity and resistance, and with the inductance so chosen that signals may travel without distortion. For this class the electrostatic theory cannot be applied however long the circuit.

(5) Circuits in which distortionless propagation is obtained by allowing an electric leakage.

The effect of leakage is small in all but the fifth case, and will be ignored.

Let an impressed force Ee^{nt} act at one end of a circuit consisting of two parallel wires, with terminal apparatus whose inductance, capacity, and resistance are neglected. Let z denote distance of a point P from the end at which the force acts. If R, L, C be the resistance, inductance,



and capacity per unit length, and (ι_1, V) the current and potential at a point P ; then

$$\left.\begin{aligned} C\,\partial V/\partial t &= -\partial \iota_1/\partial z \\ L\,\partial \iota_1/\partial t + R\iota_1 &= -\partial V/\partial z \end{aligned}\right\}, \quad . \quad . \quad (23)$$

where $\partial/\partial t \equiv \iota n$.

The solution of these equations is found to be

$$\iota_1 = (E/I)e^{\iota(nt-\alpha)}, \quad . \quad . \quad . \quad . \quad . \quad (24)$$

* 'Electrical Papers,' vol. ii

where

$$\left.\begin{aligned} I &= LC^{-\frac{1}{2}}(1+R^2/n^2L^2)^{\frac{1}{4}}\,(\cosh 2lQ_1-\cos 2lQ_2)^{\frac{1}{2}} \\ Q_1, Q_2 &= n(\tfrac{1}{2}LC)^{\frac{1}{2}}\{(1+R^2/n^2L^2)^{\frac{1}{2}}\mp 1\}^{\frac{1}{2}}. \end{aligned}\right\} \quad (25)$$

With the value of α we are not concerned. For the effect of leakage, reference may be made to Heaviside's paper *.

$$\text{If } C=0, \qquad I = l(R^2+L^2n^2)^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad (26)$$

which is the ordinary impedance assumed to be valid in the previous paper. In the electrostatic theory, we write $L=0$, which is obviously unjustifiable if R is small or n very great, so that circuits of the first kind lose their character in these cases.

We proceed to estimate the correction to be made in the value of L, to take account of a small capacity, regarding only the first order term. Writing

$$\lambda_1^2,\ \lambda_2^2 = (1+R^2/n^2L^2)^{\frac{1}{2}}\mp 1,$$

then on reduction,

$$\begin{aligned} \cosh 2lQ_1-\cos 2lQ_2 &= LCn^2l^2(\lambda_1^2+\lambda_2^2)\{1+\tfrac{1}{6}LCn^2l^2(\lambda_1^2-\lambda_2^2)\} \\ &= 2Cnl^2(R^2+L^2n^2)^{\frac{1}{2}}(1-\tfrac{1}{3}l^2n^2LC), \end{aligned}$$

so that

$$I = l\sqrt{2}\,.\,(R^2+L^2n^2)^{\frac{1}{4}}(1-\tfrac{1}{6}l^2n^2LC).$$

If L′ be the equivalent induction when this small capacity is taken into account,

$$R^2+L'^2n^2 = (R^2+L^2n^2)(1-\tfrac{1}{3}l^2n^2LC)$$

$$\text{or} \quad (L'-L)/L = -l^2C(R^2+L^2n^2)/6L \quad \text{approximately.} \quad (27)$$

This equation serves to limit the types of circuit to which the uncorrected inductance formulæ may be applied, in so far as error due to capacity is concerned. For example, if a three-figure accuracy is required, it is necessary that

$$l^2C(R^2+L^2n^2)/6L \ngtr 10^{-4}, \quad . \quad . \quad . \quad (28)$$

where l is the length of either wire of the circuit.

* *Loc. cit.*

We have also neglected

$$\frac{1}{90}l^4n^4L^2C^2(\lambda_1^4+\lambda_2^4-\lambda_1^2\lambda_2^2)=\frac{1}{90}l^4n^4L^2C^2(4+R^2/l^2n^2)$$

in comparison with unity. This restricts the capacity for which the correction (27) may be used. Thus, for low resistance,

$$4l^4n^2C^2 \gg 9\,.\,10^{-5}.$$

This equation will ordinarily supply the lower limit for C. Thus for frequencies such that

$$nC \gg (5l^2)^{-1}10^{-2} \quad . \quad . \quad . \quad . \quad (29)$$

the uncorrected (for capacity) formulæ may be used provided that C is also so small as to make

$$Cn^2L \gg 6l^{-2}10^{-4}, \quad . \quad . \quad . \quad . \quad (30)$$

again neglecting R.

For a capacity of one microfarad per kilometre, $C=10^{-20}$. In a case like that of the Atlantic cable the capacity is about a quarter of this amount, and the limiting frequency, with l in centimetres, is about $l^{-2}10^{18}$. This excludes such a cable altogether from the investigation, although a shorter cable of the same radius of wire and capacity gradient can be within its scope. For example, a frequency of so much as 10 million can be treated if l is not greater than 1·2 kilometres, for a suitable range of inductance given by (30). The range of inductance is of course dependent mainly on the distance between the wires.

Obviously all short telephone circuits satisfy both (29) and (30), and their capacity needs no consideration.

Copper Wires with High Frequency.

With the above restrictions as regards capacity and frequency (the latter being of little practical import), writing in (17),

$$\frac{\beta}{\alpha x}=\frac{1}{\lambda}\left(1-\frac{1}{\lambda}-\frac{1}{4\lambda^3}\right)\left(1-\frac{1}{\lambda}+\frac{1}{2\lambda^2}\right)$$

$$=\frac{1}{\lambda}\left(1-\frac{2}{\lambda}+\frac{5}{4\lambda^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Then with the values of (21),

$$L = 4\log_e\frac{c}{a} + \frac{4}{\lambda}\left(1 - \frac{2}{\lambda} + \frac{5}{4\lambda^2}\right) + \frac{4a^2}{c^2 CD}(AE - BF),$$

where $$\lambda = 2a(2\pi\mu n/\sigma)^{\frac{1}{2}} = 4\pi a(8/\sigma)^{\frac{1}{2}},$$

with a four-figure accuracy so far as λ is concerned, if $\lambda > 8\sqrt{2}$. But a further simplification may be introduced. Since ha is a very small magnitude in actual cases, its logarithm is large and negative. Thus if $\rho = \log_e ha$, the functions may be expanded in a descending series of powers of ρ, and

$$\frac{CD}{P\rho^2} = 1 - \frac{1}{\lambda}\left(1 + \frac{2}{\rho}\right) + \frac{1}{\lambda^2}\left(\frac{1}{2} + \frac{2}{\rho} + \frac{2}{\rho^2}\right) + \frac{1}{\lambda^3}\left(\frac{3}{4} + \frac{1}{2\rho}\right)$$

or

$$\frac{P\rho^2}{CD} = 1 + \frac{1}{\lambda}\left(1 + \frac{2}{\rho}\right) + \frac{1}{\lambda^2}\left(\frac{1}{2} + \frac{2}{\rho} + \frac{2}{\rho^2}\right) - \frac{1}{\lambda^3}\left(\frac{3}{4} + \frac{1}{2\rho}\right), \quad (32)$$

whence, after considerable reduction

$$\frac{\rho^2}{CD}(AE - BF) = \rho\log\frac{a}{c} - \frac{1}{\lambda}\left(\rho + 4\rho\log\frac{a}{c} + 3\log\frac{a}{c}\right) + \frac{1}{4\rho^2\lambda^2}\left\{8\rho - 8\rho^2 + 19\rho^3 - (8 - 8\rho + 3\rho^2)\log\frac{a}{c}\right\}.$$

The vanishing of the coefficient of λ^{-2} is curious.

Finally, for a pair of copper wires, the main error being of relative magnitude a^4/c^4 when they are close together, if $\lambda < 8\sqrt{2}$,

$$\lambda = 4\pi a(f/\sigma)^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad (33)$$

if f is the frequency, then

$$L = 4\log_e\frac{c}{a} + \frac{4}{\lambda}\left(1 - \frac{2}{\lambda} + \frac{5}{4\lambda^2}\right) + \frac{4a^2}{c^2}\frac{\log\frac{a}{c}}{\log ha} - \frac{4a^2}{c^2\lambda\rho^2}\left(\rho + 4\rho\log\frac{a}{c} + 3\log\frac{a}{c}\right) + \frac{a^2}{c^2\lambda^2\rho^4}\left\{8\rho - 8\rho^2 + 19\rho^3 - (8 - 8\rho + 3\rho^2)\log\frac{a}{c}\right\}, \quad (34)$$

where $\rho = \log_e ha$, and the limiting capacity is given by (29). For most useful purposes it may be greatly shortened, according to the value of λ.

Iron Wires with High Frequency.

When iron is used, there is no object to be served in obtaining so accurate a formula, on account of the vagaries in the magnetic behaviour of the metal. But the approximate value of μ is known, for Lord Rayleigh * has shown that for feeble magnetizing forces of periodic character, μ usually lies between 90 and 100, and becomes much greater as the forces are increased. It is therefore lawful to take μ as of order 10^2. In the formulæ (20), (α, β, γ) never become large, and therefore whatever the value of x (or λ), the approximation may be conducted by expanding the functions in inverse powers of μ.

Thus if $\rho = \log ha$ as before,

$$\frac{\mathrm{CD}}{\mathrm{P}} = \frac{\alpha\mu^4}{x^4}\left(1-2\beta\rho\frac{x}{\mu}+\alpha\rho^2\frac{x^2}{\mu^2}\right)\left(1-\frac{2}{\mu}\left(1-\frac{\gamma x}{\alpha}\right)+\frac{1}{\mu^2}\left(1-\frac{2\gamma x}{\alpha}+\frac{x^2}{\alpha^2}\right)\right)$$

$$= \frac{\mu^4\alpha}{x^4}\left\{1-\frac{2}{\mu}\left(\beta\rho x+1-\frac{\gamma x}{\alpha}\right)+\frac{1}{\mu^2}\left(1+\alpha\rho^2x^2-\frac{2\gamma x}{\alpha}\right.\right.$$

$$\left.\left.+\frac{x^2}{\alpha}+4\beta\rho x-4\beta\gamma\rho\frac{x^2}{\alpha}\right)\right\}$$

or

$$\frac{\mathrm{P}\mu^4\alpha}{x^4\mathrm{CD}} = 1+\frac{2}{\mu}\left(1+\beta\rho x-\frac{\gamma x}{\alpha}\right)+\frac{1}{\mu^2}\left(3+4\beta^2\rho^2x^2+4\beta\rho x+\frac{4\gamma^2x^2}{\alpha^2}\right.$$

$$\left.-4\beta\gamma\rho\frac{x^2}{\alpha}-\frac{x^2}{\alpha}-\frac{6\gamma x}{\alpha}-\alpha\rho^2x^2\right). \quad (35)$$

This neglects the ratio x^3/μ^3. Taking a wire of gauge $\frac{1}{2}$ cm., an approximate practical limit for overhead wires, and writing, as an ordinary value for iron, $\sigma = 10{,}000$ C.G.S.,

$$x = \frac{3}{20}f^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

if $\mu = 100$. Thus even when $f = 10{,}000$ per second, neglect of x^3/μ^3 only affects the third figure of CD, and therefore at most the fourth of L, when the ratio a^2/c^2 is noticed. This process is justifiable whether the ascending or descending series are used for (α, β, γ).

* Scientific Papers, iii.

In a similar manner,

$$-\frac{(AE-BF)}{P} = +\frac{\alpha\mu^4}{x^4}\left(1-\frac{\beta x}{\mu}\log\frac{ha^2}{c}+\frac{\alpha x^2}{\mu^2}\log ha\log\frac{a}{c}\right)\left(1-\frac{\alpha+x^2-2\gamma x}{\alpha\mu^2}\right)$$
$$+2\mu^2\frac{\beta\gamma}{x^2}\log hc\,;$$

and after some reduction,

$$-\frac{AE-BF}{CD} = 1+\frac{1}{\mu}\left(2+\beta\rho x-\frac{2\gamma x}{\alpha}-\beta x\log\frac{a}{c}\right)$$
$$+\frac{1}{\mu^2}\left(2+2\beta\rho x-\alpha\rho^2x^2+2\beta^2\rho^2x^2+\frac{4\gamma^2x^2}{\alpha^2}-\frac{4\gamma x}{\alpha}-\frac{2x^2}{\alpha^2}\right)$$
$$+\frac{1}{\mu^2}\log\frac{a}{c}\,.\,(\alpha x^2\rho-2\beta^2x^2\rho-2\beta x).$$

In the extreme case above, $\frac{x}{\mu^2} \not> \frac{1}{600}$, and the final error consequent upon its neglect is of order $(6000)^{-1}$ relative to unity even when the wires have limiting practical closeness.

Accordingly,

$$-\left(\frac{AE-BF}{CD}\right) = 1+\frac{1}{\mu}\left(2+\beta\rho x-\frac{2\gamma x}{\alpha}-\beta x\log\frac{a}{c}\right)$$
$$+\frac{1}{\mu^2}\left(2\beta^2\rho^2x^2-\alpha\rho^2x^2+\frac{4\gamma^2}{\alpha^2}x^2-\frac{2x^2}{\alpha}+x^2\rho(\alpha-2\beta^2)\log\frac{a}{c}\right).$$

Consider now the case in which $x>8$. In the coefficient of μ^{-2}, the first order values of $(\alpha\beta\gamma)$ may be used, with the second order in that of μ^{-1}.

Under these conditions, the coefficient of μ^{-2} is found to vanish, leading to the very simple result, to order μ^{-3},

$$-\left(\frac{AE-BF}{CD}\right) = 1+\frac{1}{\mu}\left\{2+\frac{1}{2}\lambda\left(1-\frac{1}{\lambda}\right)\log hc-\lambda\left(1+\frac{1}{\lambda}\right)\right\}$$
$$= 1+(\lambda-1)(\log hc-2)/2\mu,$$

and thus for iron wires with $\lambda>8$,

$$L = 4\log_e\frac{c}{a}+\frac{4\mu}{\lambda}\left(1-\frac{2}{\lambda}+\frac{5}{4\lambda^2}\right)-\frac{4a^2}{c^2}-\frac{2a^2}{c^2\mu}(\lambda-1)(\log hc-2), \quad (37)$$

where

$$\lambda = 4\pi a\left(\frac{f\mu}{\sigma}\right)^{\frac{1}{2}}, \quad \text{and} \quad \frac{a^2}{c^2}\frac{\lambda}{\mu^2} \text{ is neglected.}$$

On account of μ, this formula may be applied, for any given frequency, to a wire of only about a quarter the corresponding limiting radius in the case of copper. For example, for a frequency 800, and radius 2 millimetres, the value of x becomes 5, with $\mu = 100$. Thus a three-figure accuracy is obtained for the most unfavourable case of the telephone circuit. Accordingly, for a telephone circuit of iron wires not twisted together, this result is of universal application.

Iron Wires with Small Frequency.

This case is not very important, but the result may be given at once. Thus if x is not greater than 2, we may neglect the term involving μ^{-2}, and write, to three significant figures,

$$\mathrm{L} = 4\log\frac{c}{a} + \frac{4\mu S}{x\alpha} - \frac{4a^2}{c^2} - \frac{4a^2}{c^2\mu}\left(\beta x \log hc + 2 - \frac{2\gamma x}{\alpha}\right). \quad (38)$$

The values of (α, β, γ) suited to this case have been given by Russell *.

If

$$z = \frac{x}{2} = a\pi\left(\frac{2\mu f}{\sigma}\right)^{\frac{1}{2}},$$

$$\left.\begin{aligned} \alpha &= z^2\left(1 - \frac{5}{12}z^4 + \frac{143}{720}z^8 - \frac{7661}{4^2.7!}z^{12}\right) \\ \beta &= \frac{1}{2}z^3\left(1 - \frac{11}{24}z^4 + \frac{473}{3.6!}z^8 - \frac{304107}{4.12^2.7!}z^{12}\right) \\ \gamma &= z\left(1 - \frac{1}{3}z^4 + \frac{19}{5!}z^8 - \frac{687}{7.6^4}z^{12}\right) \end{aligned}\right\}. \quad (39)$$

Moreover,

$$\left.\begin{aligned} \frac{\beta}{\alpha} &= \frac{1}{2}z\left(1 - \frac{1}{24}z^4 + \frac{13}{4320}z^8 - \frac{647}{12^2.360.56}z^{12}\right) \\ \frac{\gamma}{\alpha} &= z^{-1}\left(1 + \frac{1}{12}z^4 - \frac{1}{180}z^8 + \frac{11}{12.28.30}z^{12}\right) \end{aligned}\right\} \quad (40)$$

* *Loc. cit.*

Heaviside* has also given the value of β/α, and Lord Rayleigh† that of γ/α, by different modes of proof. Thus finally, when

$$z = a\pi(2\mu f/\sigma)^{\frac{1}{2}} \not> 1, \quad . \quad . \quad . \quad . \quad . \quad (41)$$

$$\mathrm{L} = 4\log\frac{c}{a} + \mu\left(1 - \frac{1}{24}z^4 + \frac{13}{4320}z^8 - \frac{647}{12^2 . 360 . 56}z^{12}\right) - \frac{4a^2}{c^2}$$

$$- \frac{4z^4a^2}{c^2\mu}\log_e hc\left\{1 - \frac{11}{24}z^4 + \frac{473}{3 . 6!}z^8 - \frac{304107}{4 . 12^2 . 7!}z^{12}\right\}$$

$$+ \frac{8a^2}{c^2\mu}\left(1 + \frac{1}{6}z^4 + \frac{1}{90}z^8 - \frac{11}{28 . 180}z^{12}\right). \quad . \quad . \quad . \quad . \quad (42)$$

Small Frequency, Copper Wires.

In the notation of (18), with $\rho = \log_e ha$,

$$x^2\frac{\mathrm{CD}}{\mathrm{P}} = 1 - 2\beta x\rho + \alpha x^2\rho^2.$$

The frequencies for which this formula will ordinarily be required lie between 200 and 1600, and the radii between 1 and 3 millimetres. These values cause ρ to vary between about 13 and 16 only, and z between $\frac{1}{4}$ and unity, so that P/CD cannot be expanded in ascending or descending powers of ρ or z in general. The result may be most simply written in the form

$$\mathrm{L} = 4\log\frac{c}{a} + \frac{2}{z}\frac{\beta}{\alpha} + \frac{4a^2}{c^2}\left(1 - 2\beta\rho z - \frac{\gamma}{z}\right)\Big/(1 - 4\beta\rho z + 4\alpha\rho^2z^2)$$

$$+ \frac{8a^2}{c^2}\log\frac{a}{c} \cdot (2\alpha\rho z^2 - 2\alpha\rho\gamma z - \beta z + 2\beta\gamma)/(1 - 4\beta\rho z + 4\alpha\rho^2z^2), \quad (43)$$

where $(\alpha\,\beta\,\gamma\,z)$ are given by (39), and $\rho = \log_e ha$.

The two final terms in the last numerator are only about $\frac{1}{50}$ of the other two, and may be ignored.

* Elec. Papers, ii. p. 64.

† Phil. Mag. xxii. p. 381 (1886); Scientific Papers, vol. ii.

II. Effective Resistance.

Copper Wires with High Frequency.

The limitations of all the formulæ below, except when otherwise stated, are the same as in the corresponding inductance formulæ. With the previous notation, the effective resistance per unit length of the two wires is

$$R = \frac{4n\mu}{x}\frac{\operatorname{ber} x \operatorname{bei}' x - \operatorname{bei} x \operatorname{ber}' x}{(\operatorname{ber}' x)^2 + (\operatorname{bei}' x)^2} - \mu N$$

$$= \frac{4n\mu}{x}\cdot\frac{\gamma}{\alpha} - \frac{4a^2 n}{c^2 \mathrm{CD}}(\mathrm{AF}+\mathrm{BE}). \quad . \quad . \quad . \quad . \quad (44)$$

where CD/P has the value in (15), and on reduction

$$\frac{\mathrm{AF}+\mathrm{BE}}{\mathrm{P}} = \frac{2\mu\beta}{x}\left(\alpha \log ha \log\frac{a}{c} - \frac{\mu\beta}{x}\log\frac{ha^2}{c} + \frac{\mu^2}{x^2}\right)$$

$$+ \frac{\mu\gamma}{x}\log hc\left(1 - \frac{2\gamma}{x} + \frac{\alpha}{x^2}\overline{1-\mu^2}\right). \quad . \quad (45)$$

When $\mu = 1$, and the frequency is high, so that, if $x\sqrt{2} = \lambda$, the formulæ (20) may be employed, then after further reduction,

$$\frac{\mathrm{AF}+\mathrm{BE}}{\mathrm{P}} = \frac{2}{\lambda}\log\frac{a}{c}\log ha\left(1 - \frac{2}{\lambda} + \frac{5}{4\lambda^2}\right) - \frac{4}{\lambda^2}\log ha\left(1 - \frac{2}{\lambda}\right)$$

$$+ \frac{1}{\lambda}\log hc\left(1 - \frac{15}{4\lambda^2}\right) + \frac{4}{\lambda^3},$$

and with the help of (32), we ultimately obtain

$$\frac{\lambda\rho^2}{\mathrm{CD}}(\mathrm{AF}+\mathrm{BE}) = 2\rho\log\frac{a}{c} + \rho - \log\frac{a}{c} - \frac{1}{\lambda\rho}\left\{\rho^2 + \log\frac{a}{c}(2 - 3\rho + 2\rho^2)\right\}$$

$$+ \frac{1}{4\lambda^2\rho^2}\left\{8\rho - 8\rho^2 + 3\rho^3 - \log\frac{a}{c}(8 - 8\rho + 3\rho^2 + 2\rho^3)\right\}$$

where $\rho = \log_e ha$.

Finally, for a pair of copper wires, with high frequency, and with the limitations of (34),

$$R = \frac{4n}{\lambda}\left(1 + \frac{1}{\lambda} + \frac{3}{4\lambda^2}\right) - \frac{4a^2 n}{\lambda c^2\rho^2}\left\{\rho + (2\rho - 1)\log\frac{a}{c}\right\}$$

$$+ \frac{4a^2 n}{\lambda^2 c^2\rho^3}\left\{\rho^2 + (2 - 3\rho + 2\rho^2)\log\frac{a}{c}\right\}$$

$$- \frac{a^2 n}{\lambda^3 c^2\rho^4}\left\{8\rho - 8\rho^2 + 3\rho^3 - (8 - 8\rho + 3\rho^2 + 2\rho^3)\log\frac{a}{c}\right\}. \quad (46)$$

Iron Wires with High Frequency.

When μ is large, we may write from (45),

$$\frac{AF+BE}{P}=\frac{\mu^3}{x^3}\left(2\beta-\alpha\gamma\log hc-2\beta^2\frac{x}{\mu}\log\frac{ha^2}{c}\right), \quad . \quad (47)$$

and by (35),

$$\frac{P\mu^4\alpha}{x^4CD}=1+\frac{2}{\mu}\left(1+\beta\rho x-\gamma\frac{x}{\alpha}\right).$$

so that

$$\frac{AF+BE}{CD}=\left(\frac{2\beta}{\alpha}-\gamma\log hc\right)$$
$$-\frac{2x}{\alpha\mu^2}\left\{\left(1+\beta\rho x-\gamma\frac{x}{\alpha}\right)(2\beta-\alpha\gamma\log hc)+\beta^2x\log\frac{ha^2}{c}\right\}.$$

Neglecting x/μ^2 as before, and writing the first approximations to $(\alpha\beta\gamma)$ in the second term, and the second in the first, we finally obtain

$$\frac{AF+BE}{CD}=\frac{\lambda}{2\mu}\left(2-\rho+\log\frac{a}{c}\right)-\frac{\lambda^2}{2\mu^2}\left(4\rho-2-\rho^2+\rho\log\frac{a}{c}\right),$$

the term in $\lambda^0\mu^{-1}$ vanishing identically.

Thus the effective resistance becomes

$$R=\frac{4n\mu}{\lambda}\left(1+\frac{1}{\lambda}+\frac{3}{4\lambda^2}\right)-\frac{2a^2n\lambda}{c^2\mu}\left(2-\rho+\log\frac{a}{c}\right)$$
$$+\frac{2a^2n\lambda^2}{c^2\mu^2}\left(4\rho-2-\rho^2+\rho\log\frac{a}{c}\right). \quad . \quad (48)$$

This result is much more accurate than the corresponding formula (37) for inductance, owing to the presence of μ in the numerator of the first term. Moreover, λ here denotes the previous λ multiplied by $\mu^{\frac{1}{2}}$. It is applicable to all iron circuits of importance in practice, for a frequency greater than about a hundred with a radius of 1 millimetre.

Iron Wires with Low Frequency.

This result has little interest, but may be obtained at once. For $(\alpha\beta\gamma)$ are all comparable with unity in this case, and x is not greater than 2. We therefore ignore x^2/μ^2, and obtain

$$\frac{AF+BE}{CD}=\frac{x}{\mu}\left(\frac{2\beta}{\alpha}-\gamma\log hc\right). \quad . \quad . \quad . \quad (49)$$

This term also may be ignored unless a very high order of accuracy be desired, which can rarely happen owing to the uncertainty as to the value of μ. But if it be retained, the resistance becomes

$$R = \frac{2n\mu}{z^2}\left(1-\frac{1}{12}z^4-\frac{1}{180}z^8\right)-\frac{8n\alpha^2z^2}{\mu c^2}\left(1-\frac{1}{24}z^4+\frac{13}{4320}z^8\right)$$
$$+\frac{8n\alpha^2z^2}{\mu c^2}\left(\rho-\log\frac{a}{c}\right)\left(1-\frac{1}{3}z^4+\frac{19}{120}z^8\right). \quad . \quad (50)$$

where z has the value in (39), and the brackets may be shortened except when z is nearly unity.

When the frequency is smaller, this makes $R = 2\sigma/\pi a^2$, the value appropriate to steady currents.

Copper Wires with Low Frequency.

The formula suited to this case is, writing $\beta^2+\gamma^2=\alpha$, if $(z\,\alpha\,\beta\,\gamma)$ are defined as in (39), and under the same limitations,

$$R = \frac{2n}{z}\cdot\frac{\gamma}{\alpha}-\frac{4n\alpha^2}{zc^2}(\beta-2\alpha\rho z+2\gamma\rho z^2)/(1-4\beta\rho z+4\alpha\rho^2z^2)$$
$$-\frac{8n\alpha^2}{c^2}\log\frac{a}{c}\left(\gamma^2-\beta^2-\gamma z+2\alpha\beta\rho z)/(1-4\beta\rho z+4\alpha\rho^2z^2\right), \quad (51)$$

again reducing to $2\sigma/\pi a^2$ for steady currents.

Trinity College, Cambridge,
April 21, 1909.

LVIII. *The Proposed International Unit of Candle Power.*

By CLIFFORD C. PATERSON *.

(From the National Physical Laboratory.)

THE intercomparison of light units between the National or other standardizing Laboratories of America, France Germany, and Great Britain has been proceeding at intervals since 1905.

The values which have been obtained for the ratios between the different units are now found to be in sufficiently close accord to warrant the establishment of a working basis of agreement between this country, America, and France in the matter of a common unit of Candle Power.

The writer has been conducting the photometric measurements connected with the work, in this country, and it is the intention in this paper to give the results of the comparisons which have been made, and to explain briefly those facts connected with the different standards concerned which have a bearing on the agreement which has been reached.

The possibility of agreement between the British and French units was demonstrated by Dr. Glazebrook in a paper on light standards read before the B.A. at Dublin in 1908 †. The chief factor in the present movement has been the desire of the authorities in the United States to establish one unit for both Gas and Electrical industries in that country, and the possibility of their adopting a unit which should be identical with those existing in Great Britain and France led them to take the initiative in an attempt to obtain international cooperation. The agreement which has resulted has the approval of the Metropolitan Gas Referees and now forms the subject of an announcement which is reproduced on page 877.

It is not necessary for the purpose of this paper to enter into a detailed description of the various standard lamps and

* Read June 11, 1909.

† B.A. Report, 1908, Dublin, "The Photometric Standard of the National Physical Laboratory"; also 'Journal of Gas Lighting,' vol. 103, 1908, p. 713.

units referred to in the memorandum. Dr. J. A. Fleming discussed the question of light standards very fully in his paper before the Inst. of Elect. Engineers, vol. 32, to which reference should be made *. Some facts, however, connected with the units in question have a more especial bearing on the experimental results, and should be borne in mind in connexion with the table of ratios given on pages 874, 875.

The British Unit.—As defined in the above-mentioned recommendation, the unit of Candle Power in this country is the Harcourt 10 C.P. Pentane Lamp burning in an atmosphere at normal barometric pressure and containing 8 litres of water-vapour per cubic metre as measured by a ventilated hygrometer.

A difficulty arises in the use of all flame standards in connexion with the method to be employed for measuring the humidity. When flame lamps are burning in a closed room it is well known that their candle power diminishes, due probably to the vitiation of the air in the immediate neighbourhood of the flame †. Two standards will not necessarily diminish in Candle Power at the same rate, and it is therefore necessary to take readings after the air of the room has been changed and before the C.P. of the lamps has had time to be affected. The method of measuring humidity must therefore be a rapid one, and it is now generally agreed that from considerations of accuracy and quickness of reading the ventilated hygrometer is the best instrument to use ‡. In the German and French comparisons this has been used, but in the English comparisons (as reported to the Photometric Commission meeting in Zurich in 1907) the unventilated hygrometer was employed§, and in the tables, published in the proceedings of the Commission the author's

* See also paper by the author, Journ. Inst. Elect. Eng. vol. xxxviii. p. 271.

† Report Amer. Gas Inst., "Methods of taking C.P. of Gas," Illum. Eng. 1909, p. 203.

‡ Proc. Roy. Soc. Edinburgh, vol. xliii., 1905; also "Zur kenntniss des Ventilierten Psychrometers," Akademische Abhandlung der Fakultat der Universitat zu Upsala, by Aron Svensson, 1898.

§ See B.A. Report, Dublin 1908.

results are given in terms of humidity measured by this instrument. The ratios tabulated in the present paper are corrected so as to be in terms of the ventilated hygrometer. In each case values are taken for the humidity at which each lamp, in the country where it forms the standard, is considered to give its nominal Candle Power.

The Unit of the United States of America.—In the initial adoption of a unit of Candle Power the United States of America endeavoured to make its value as nearly as possible the same as that accepted at the time in this country *. This was before Prof. Vernon Harcourt and the Metropolitan Gas Re'erees (London) had established the 10 Candle Pentane Lamp on the present definite basis.

The American Inst. of Electrical Engineers recommended the derivation of their unit from the Hefner Lamp by increasing its value in the ratio of 0·88 to 1. This seemed at the time, from different observers' work, to be the most probable ratio between the Hefner and British units. The gas industry in America, however, did not follow this course but developed their unit along the lines of the 10 Candle Pentane Lamp †. The result is that there has been, up to now, an appreciable difference between the units adopted in the two industries in that country. The Illuminating Engineering Society and other bodies took the matter up energetically, and the Bureau of Standards, Washington, now has the support of the leading institutions in America, in defining the value of a common standard, to be accepted throughout the States. This Institution has ascertained by means of electric intercomparisons the ratio of their present unit to those of Germany, France, and Great Britain respectively ‡, and has arranged to adjust the value of the American unit as already indicated.

German Unit.—The unit accepted in Germany is the light given by the Hefner Lamp burning in an atmosphere at

* Bulletin of the Bureau of Standards, vol. iii., no. 1, p. 65; Report to the American Gas Institute on "A Unit of Light," Journal of Gas Lighting, vol. 104, 1908, p. 426.

† "The Working Standards of Light and their Use in the Photometry of Gas," Ch. O. Bond, Franklin Inst. 1908.

‡ Ref. cit.

normal barometric pressure and containing 8·8 litres of water vapour per cubic metre. The researches of Liebenthal * at the Reichsanstalt on the Hefner Lamp and the variation of its C.P. with atmospheric change were the earliest systematic experiments undertaken of this nature and are too well known to require more than passing mention.

French Unit.—The Candle Power Unit adopted by the Electrical Industry in France is the Bougie Decimale. This is the 20th part of the light given out by a sq. cm. of platinum at the temperature of solidification. The unit was suggested by M. Violle and adopted by the Congrès International des Electriciens in 1881.

This standard has been found very difficult to reproduce and the French authorities still use the Carcel lamp, burning colza oil, as the standard for all photometric work.

A determination of the value of the Carcel lamp in terms of the Violle platinum standard has only been made once. This was by M. Violle himself in 1884 †. Measurements were made by him using two or three photometric methods, and all his values except one showed the bougie decimal to be a little less than 4 per cent. greater than the Carcel unit.

A multiplying factor of 1·04 for the Carcel unit was therefore given by him, and has been adopted ever since for reducing the values in terms of one standard to those of the other. As no account was taken by M. Violle of the pressure and humidity of the atmosphere in which the Carcel lamp was burning, the accepted figure of 1·04 must be regarded as liable to a certain inaccuracy due to this cause. It should be remarked, also ‡, that no correcting factor has as yet been determined for the variations in the C.P. of the Carcel lamp due to atmospheric changes. Hence, in the table given later on in the paper a correcting factor has had to be assumed in cases where the Carcel lamp is corrected for a difference of humidity.

Accuracy of Comparisons.—It is well to explain in giving the results of experiments that different limits of accuracy

* "Zeitschrift für Instrumentenkunde," vol. xv. 1895, p. 157.

† Séances of the French Physical Soc., May to July, 1884.

‡ "Rapport de Trois Lamps," Laporte & Jouaust, Bull. Soc. Inst. des Elect. 2nd série, tome vi. no. 58.

must be attributed to photometric measurements of different types of standards.

It is usual in giving photometric results to write down the fourth figure, but even in the *most favourable* circumstances this must be written small and the value considered liable to an error of + or − 0·1 per cent. In the case of the best comparisons of electric sub-standards this inaccuracy should not be occasioned by imperfection in the bench or photometer head nor yet to the electrical measurements, but must be attributed, in the author's opinion, chiefly to want of constancy in the individual who is making the photometric observations.

In some of the comparisons which are tabulated the electrical measurements are probably not so accurate as in others. The fuller appreciation, however, of the exact values of the national and international electrical units which has recently resulted from the labours of the International Conference, makes it possible now to attain an accuracy which leaves nothing to be desired from this point of view.

As matters stand now, undoubtedly the photometric comparisons in which the highest precision is attainable are those between properly seasoned electric glow-lamps of the same coloured light. With a potentiometer which is accurate to one part in 10,000 and a substitution method of photometric comparison * on a bench which can be read to 0·5 mm., an accuracy is attainable with a set of good sub-standards in which the fourth figure is almost definite.

When, on the other hand, comparisons are made against or between flame standards, the probable inaccuracy is greater.

How much the inaccuracy is must depend largely on the flame adjustments and the consistent behaviour of the standard in question. It also depends upon the accuracy of measurement of atmospheric conditions and the precise knowledge which we have of their effect on the light of the standard lamps.

It follows from this that a relatively large number of observations must be made, when using a flame standard,

* See "Photometry of Electric Lamps," by Dr. J. A. Fleming, M.A., F.R.S., Journ. Inst. Elect. Eng. vol. xxxii. p. 144.

if the same order of accuracy is to be attained that is possible with a much smaller number of electric comparisons. It must further be remembered when considering the question of photometric measurements to two or three parts in a thousand that the estimation of the height of the flame in some lamps, or the exact reproduction of the standard conditions, may not be identical when carried out by different observers. Hence it is conceivable that owing to this cause the observations in one laboratory on some flame standard may differ consistently by a small amount from those in another on the same standard. This, however, is not the case when electric sub-standard comparisons are made if the electrical measuring apparatus is accurate. To a certain extent, therefore (in some cases more than others), a flame standard needs to be "interpreted" when its absolute value is desired to a high accuracy.

The value of electric sub-standards comparisons thus becomes apparent. If (as is generally the case) a Laboratory has sets of electric sub-standards which have been compared at intervals for years with the primary flame standard whose value they represent, an opportunity is given for realising the absolute value of this unit to an accuracy which could hardly be attained with certainty by others who might endeavour to reproduce it in a single set of observations, however carefully made. When these electric sub-standards are intercompared through the medium of a travelling set of lamps, there is no reason why we should not obtain accurate knowledge of the relative values of the different units as each is interpreted in the country where it is the recognized standard.

The ratios between the four units of light given in the Table are the results of measurements which have been made at the specified laboratories in the four countries concerned. Other determinations were made previous to these *, but the standards used for obtaining the British Unit were of several different forms and the atmospheric conditions have not always been taken into consideration. I have deemed it

* For a discussion of these, see J. A. Fleming, "The Photometry of Electric Lamps," ref. cit.

desirable, therefore, to insert only the more recent determinations, in all of which the 10 Candle Harcourt Pentane Lamp has been used and atmospheric changes have been allowed for.

The Table is divided into two portions. Columns 1 to 9 give the various ratios obtained through the medium of electric lamps which have been measured at some or all of the laboratories. These have been chiefly initiated by the Americans, who have from time to time sent sets of lamps to Europe to have values assigned in London, Paris, and Berlin. It is not suggested that all the results given in the Table should receive equal weight. In some of the electric comparisons the conditions allowed of a greater accuracy than in others, when fewer lamps were employed and time only allowed a single set of measurements to be made.

Columns 10, 11, and 12 contain the values which resulted from the intercomparison of flame-standards undertaken at each of the laboratories. These were initiated by the International Commission on Photometry and gave a set of ratios which brought the knowledge of the relative values of the candle power units to within an accuracy of about + or −1 per cent. As in the case of the electric comparisons, the conditions in some cases probably allowed of a higher accuracy than in others—but the results of all the measurements have been tabulated in order that the bearing may be seen of each on the agreement which has been established.

The first series of ratios (columns 1 to 9) may therefore be regarded as representing the ratios of the standards as they are interpreted in the countries to which they belong, whilst in the second series we have the interpretation of the values of the standard lamps by experimenters who are not so accustomed to their manipulation.

Lines marked A B, C, give the values of the standards in terms of the Pentane unit. D, E, F, give them in terms of the Hefner and similarly other sets are in terms of the Bougie Decimale and the Bureau of Standards Candle.

Without going into detailed comments upon the experiments from which each ratio in the table is derived it will suffice to say that as far as the electric lamp comparisons are

TABLE *giving determinations of the Ratios of*

Pentane = British. Bougie Decimale = French.

		ELECTRIC					
COLUMN		1.	2.	3.	4.	5.	6.
Tests conducted by		Sharp, 1903.	Paterson, 1905.	Fleming, 1905.	Hyde, 1906.	Laporte & Jouaust, 1907.	Laporte, 1907.
Number of Lamps.....		...	6	3	12	6	2
A	Hefner / Pentane Unit	....	$0{\cdot}89_0$	$0{\cdot}88_5$	$0{\cdot}89_3$	$0{\cdot}90_8$	....
B	Bougie Decimale / Pentane Unit	....	....	....	$0{\cdot}99_8$	$1{\cdot}01_3$	$1{\cdot}00_3$
C	Bur. of Standards / Pentane Unit	....	....	....	$1{\cdot}01_6$	....	....
D	Pentane Unit / Hefner	....	$1{\cdot}12_3$	$1{\cdot}13_0$	$1{\cdot}12_0$	$1{\cdot}10_1$	....
E	Bougie Decimale / Hefner	$1{\cdot}11_8$	....	....	$1{\cdot}11_6$	$1{\cdot}11_3$	....
F	Bur. of Standards / Hefner	....	....	....	$1{\cdot}13_8$	....	....
G	Pentane Unit / Bougie Decimale	....	....	....	$1{\cdot}00_2$	$0{\cdot}98_7$	$0{\cdot}99_7$
H	Hefner / Bougie Decimale	$0{\cdot}89_4$	....	....	$0{\cdot}89_6$	$0{\cdot}89_8$	....
I	Bur. of Standards / Bougie Decimale	....	....	....	$1{\cdot}01_8$	....	....
J	Pentane Unit / Bur. of Standards	....	....	....	$0{\cdot}98_4$	....	....
K	Hefner / Bur. of Standards	....	....	....	$0{\cdot}87_9$	....	....
L	Bougie Decimale / Bur. of Standards	....	....	....	$0{\cdot}98_2$	....	....

* Pentane values corrected to a humidity of

National Candle Power Units from 1903 *to* 1908.

Hefner = German. Bureau of Standards = U.S.A.

Comparisons.			Direct Comparisons.			
7.	8.	9.	10.	11.	12.	13.
Rosa, 1908. Spring.	Rosa, 1908. Autumn.	Laporte, 1908.	Paterson.	Liebenthal.	Perot & Laporte.	Photometric Commission, Zurich,* 1907.
12	6	11				
$0{\cdot}89_6$	....	....	$0{\cdot}90_2$	$0{\cdot}90_4$	$0{\cdot}92_1$	$0{\cdot}90_2$
$1{\cdot}00_8$	....	$1{\cdot}00_8$	$1{\cdot}00_9$	$1{\cdot}00_9$	$1{\cdot}01_9$	$1{\cdot}00_8$
$1{\cdot}01_5$	$1{\cdot}01_6$	....	....	....	....	....
$1{\cdot}11_6$	....	....	$1{\cdot}10_9$	$1{\cdot}10_6$	$1{\cdot}08_6$	$1{\cdot}10_9$
$1{\cdot}12_4$	....	....	$1{\cdot}11_8$	$1{\cdot}11_6$	$1{\cdot}11_7$	$1{\cdot}11_8$
$1{\cdot}13_1$	....	....	....	....	....	....
$0{\cdot}99_2$	....	$0{\cdot}99_2$	$0{\cdot}99_1$	$0{\cdot}99_1$	$0{\cdot}98_1$	$0{\cdot}99_2$
$0{\cdot}89_0$	....	....	$0{\cdot}89_4$	$0{\cdot}89_5$	$0{\cdot}89_6$	$0{\cdot}89_4$
$1{\cdot}00_6$	....	....	....	....	....	....
$0{\cdot}98_5$	$0{\cdot}98_4$	....	....	....	....	....
$0{\cdot}88_4$	....	....	....	....	....	....
$0{\cdot}99_4$	....	....	....	....	....	....

8 litres of water vapour per cubic metre of air.

concerned, greatest stress should be laid on the results in columns 4 and 7. This is partly on account of the large number of lamps employed, and also because of the more prolonged measurements made. Line J illustrates the high accuracy it is possible to secure in comparisons of this nature.

In the certification of glow-lamps in terms of the Hefner Unit the Reichsanstalt only give candle-power values to the nearest one per cent. If the average of 10 or 12 lamps is taken the error thus introduced is probably not great, but when the number in small, appreciable inaccuracies may be introduced into the mean, and the rather low value obtained by Fleming in 1905 may be attributed to the fact that only 3 lamps were tested *.

As regards the flame-lamp comparisons, it will be noticed that Perot and Laporte (Column 12) found a value for their Pentane lamp which was appreciably lower than that obtained by other observers. Except for this difference the agreement between the ratios is fairly close. The exceptionally close agreement shown in columns 10, 11, and 12 for the ratio Bougie Dec./Hefner can only be attributed to a coincidence, since in the experiments from which two of the three ratios were determined the observations varied between 3 and 4 per cent. from the mean.

The chief point of interest in these comparisons is the near coincidence of the value of the Bougie Decimale with that of the Pentane unit as indicated by lines B and G. An inspection of these values indicates that the Bougie Decimale, as interpreted by the Laboratoire Central, may be slightly larger than the Pentane unit—but the amount is less than 1 per cent. When we remember that, at present, the value of the Bougie Decimale depends on the interpretation of the Carcel lamp and the ratio between it and the platinum unit, determined by Violle in 1884, it must be admitted that this small apparent difference is well within the limits of the errors of experiment.

The second point to notice is the difference of 1·6 per cent.

* See Journ. Inst. Elect. Eng. vol. xxxviii. p. 311: Discussion by Dr. Fleming of the Author's paper on "Investigation of Light Standards etc."

between the units of the Bureau of Standards and the National Physical Laboratory. It is generally recognized * that the unit at present adopted by the Gas interests in the States is about 4 per cent. smaller than the Bureau of Standards unit. It will then be seen that by lowering the value of their unit by 1·6 per cent. the Bureau comes into exact agreement with this country and approximately halves the difference between the units employed by the Gas and Electrical industries in the States.

A further point of interest and importance which results from the comparisons (see line A in the table) is that the Hefner unit is in the ratio 9/10 to the new candle. The French authorities have for some time taken the ratio Hefner/Bougie Dec. as 0·895. It is of interest, therefore, to see from lines A and H how nearly the value for the Hefner unit in terms of both Pentane and Bougie Dec. units approaches the figure 0·90. The mean of all the ratios Hefner/Pentane comes to $0{\cdot}90_0$ and those of Hefner/Bougie Dec. to $0{\cdot}89_5$, so that although the comparisons between the Pentane and Bougie Decimale units indicate a difference of 0·8 per cent., the same units compared through the Hefner Standard only appear to differ by 0·5 per cent.

The author's acknowledgments are due to Dr. R. T. Glazebrook, F.R.S., Director of the National Physical Laboratory.

Appendix.

Copy of Announcement made in France, America, and Great Britain, relative to the proposed International Unit of Light.

In order to determine as accurately as possible the relations between the photometric units of America, France, Germany, and Great Britain, comparisons have been made at different times during the past few years at the Bureau of Standards, Washington; at the Laboratoire Central d'Electricite, Paris; at the Physikalisch-Technische Reichsanstalt, Berlin, and at the National Physical Laboratory, London.

* Report of Committee on Nomenclature and Standards, Annual Conv. Illum. Eng. Soc. Oct. 5, 1908, Dr. A. C. Humphreys.

The unit of light at the Bureau of Standards has been maintained through the medium of a series of incandescent electric lamps, the values of which were originally intended to be in agreement with the British unit, being made 100/88 times the Hefner unit.

The unit of light at the Laboratoire Central is the bougie decimale, which is the twentieth part of the standard defined by the International Conference on Units of 1884, and which is taken, in accordance with the experiments of Violle, as 0·104 of the Carcel lamp.

The unit of light at the Physikalisch-Technische Reichsanstalt is that given by the Hefner lamp burning at normal barometric pressure (76 cm.) in an atmosphere containing 8·8 litres of water-vapour per cubic metre.

The unit of light at the National Physical Laboratory is that given by the 10 candle power Harcourt Pentane lamp burning at normal barometric pressure (76 cm.) in an atmosphere containing 8 litres of water-vapour per cubic metre

In addition to the direct intercomparison of flame standards carried out recently by the national laboratories in Europe, one comparison was made in 1906 and two in 1908 between the American and European units by means of carefully seasoned carbon filament electric standards, and as a result of all the comparisons, the following relationships are established between the above units.

The Pentane unit has the same value within the errors of experiment as the bougie decimale. It is 1·6 per cent. less than the standard candle of the United States of America, and 11 per cent. greater than the Hefner unit.

In order to come into agreement with Great Britain and France, the Bureau of Standards of America proposed to reduce its standard candle by 1·6 per cent. provided that France and Great Britain would unite with America in maintaining the common value constant, and with the approval of other countries would call it the International Candle. The National Physical Laboratory, London, and the Laboratoire Central d'Electricité, Paris, have agreed to adopt this proposal in respect to the photometric standardization which they undertake, and the date agreed upon for the

adoption of the common unit and the change of unit in America is April 1, 1909.

The following simple relations will therefore hold after that date:

Proposed New Unit = 1 Pentane Candle.
= 1 Bougie Decimale.
= 1 American Candle.
= 1·11 Hefner Unit.
= 0·104 Carcel Unit.

Therefore 1 Hefner Unit = 0·90 of the proposed unit.

The Pentane and other photometric standards in use in America will hereafter be standardized by the Bureau of Standards in terms of the new unit. This, within the limits of experimental error, will bring the photometric units for both gas and electrical industries in America and Great Britain and for the electrical industry in France to a single value, and the Hefner unit will be in the simple ratio of 9/10 to this international unit.

The proposal to call the common unit of light to be maintained jointly by the national standardizing laboratories of America, France, and Great Britain, the "International Candle" has been submitted to the International Electrotechnical Commission, and through it to all the countries of the world which are represented on that Commission.

It is to hoped that such general approval will be secured, and that in the near future the term "International Candle" for the new unit will have official international sanction.

Discussion.

Dr. Fleming said that it was interesting to hear that the chief Powers had come to an understanding with each other as to the Unit of Light. It must be remembered, however, that this proposed International Unit had no objective existence, and no greater value as a unit of comparison than the Hefner or Pentane units, to which it was related by an arbitrary definition. Dr. Fleming said that he greatly regretted that the National Physical Laboratory authorities had acquiesced in the adoption of a flame standard of light with all its difficulties and variabilities. Influenced as they are by atmospheric pressure, moisture, CO_2, height of flame, composition of fuel, and number of persons in the photometric room, these flame standards could not possibly be considered as a final solution of the problem of obtaining a primary standard of light. What was really

required was the concrete realization of a permanent primary standard, which would be the standard of reference for secondary standards like the Fleming Ediswan Large-bulb glow-lamp standards, which he (Dr. Fleming) had introduced seven years ago. Dr. Fleming remarked that Mr. Paterson made only a very brief reference to M. Violle's work on the platinum standard and ignored altogether the careful work of Prof. Petavel carried out in 1899 in the Davy-Faraday Laboratory. Prof. Petavel's conclusions were that with suitable precautions the unit of illumination could be reproduced within 1 per cent. by means of the molten platinum standard. He asked Mr. Paterson if any attempt had been made at the National Physical Laboratory to repeat or extend Prof. Petavel's work, and if not why not? Investigations of this kind, which were difficult to carry beyond a certain point in private laboratories, were peculiarly the province of a State-aided institution like the National Physical Laboratory.

Dr. Fleming remarked that he was pleased to see that Mr. Paterson endorsed the conclusions which he (Dr. Fleming) had stated seven years previously in a paper read before the Institution of Electrical Engineers, viz., that properly prepared (large bulb) glow lamps constitute the best secondary standards. He (Dr. Fleming) had now employed for fourteen years secondary standards of this type, and had not found anything to surpass them in convenience and accuracy. The flame standards were unequally affected by changes in atmospheric pressure and moisture. Hence any figures for ratios such as are given in Mr. Paterson's paper are true only under certain accurately defined conditions of surrounding atmosphere which are very difficult to reproduce. Accordingly elaborate experiments to ascertain how many Hefners are equal to 1 Pentane is not a matter of nearly such importance as the construction of some final constant primary standard of light, and in his (Dr. Fleming's) opinion the most satisfactory form for this primary standard of light is to derive it from the light emitted normally by a defined area of some substance in a state of incandescence at a known fixed temperature. He was sure that many practical photometrists, especially those connected with the electric lighting industry, were not at all convinced that the best primary standard was a flame standard, or that the Pentane or Hefner units were a completely satisfactory solution of the problem of obtaining a Primary Standard of Light.

Dr. Russell complimented the Author on his experimental results. The *bougie décimale* was the unit adopted by the International Congress of Electricians in 1889, and was defined to be the twentieth part of the Violle standard. He was not prepared to accept that it was equal to 1·11 Hefner Unit. Lummer's and Petavel's results rather discounted the importance to be attached to Violle's number. In connexion with Dr. Fleming's remarks he stated that the unit suggested by Waidner and Burgess had many advantages. They proposed to adopt as the unit of intensity the radiation from a square centimetre of a black body maintained at the temperature of the fusion of platinum. He referred also to the unit suggested by Steinmetz, and as Mr. Dyott was present he asked if he could give any information about this unit.

Mr. Dyott said that his experiments had been made exclusively in connexion with Professor Steinmetz's magnetic arc. He had made no experiments on his photometric unit.

Mr. J. S. Dow remarked that the discussion had turned on the subject of standards rather than units. He wished, however, to make it clear that he did not condemn the Pentane standard as Dr. Fleming's quotation might, perhaps, suggest. Considered as a flame standard, he thought it ranked high, and gave very constant results when used in a scientific manner. Certainly flame standards had very serious drawbacks. But one must be cautious as yet in basing an incandescent standard upon results on a black body or other variety of radiation which might in the future be subject to modification. For instance, Féry had shown that some surfaces hitherto regarded as black exercised marked selective action; this seemed to suggest that the results of Lummer and others might need revision.

Whatever standard we adopted, the decision regarding the international unit was a very welcome one, and great credit was due to the Illuminating Engineering Society in the United States for taking the initiative. The co-operation between gas and electric authorities in both countries had rendered international action feasible, and proved that those connected with both illuminants could work together for the common good. The present step forward might appear to some to be but a small one, but it formed an important precedent. The ultimate goal was a single international unit, and it might be hoped that Germany would soon fall into line, for, as Mr. Gaster had recently pointed out, it was to her advantage, as a large exporter of glowlamps, to adopt the same unit as other countries. Meantime the adoption of the convenient round number 0·9 for the ratio between the Hefner and the proposed international candle was a very satisfactory compromise.

In this connexion we would like to enquire, however, whether Mr. Paterson felt sure that there was no physiological obstacle to comparing lights so different in colour as the Pentane and the Hefner with an exactitude of one per cent?

He had a recollection of some experiments carried out by M. Laporte in France which seemed, at that time, to suggest this possibility. It was found that when the ratio of the Hefner to the Pentane was obtained direct a value was obtained which differed consistently from that obtained by using the Carcel as an intermediate standard. This difference was then thought to be of physiological origin.

He understood, too, that Mr. Paterson had been experimenting with a series of incandescent lamps of graded efficiency (such lamps being used "in cascade" in order to avoid such an inconvenient colour difference as existed between an ordinary carbon filament and the Pentane lamp), but that the result of a "cascade" comparison was not always identical with a direct one. In view of his own experiences on this point (Proc. Phys. Soc. London, 1906, vol. xx.), it seemed conceivable that a small difference, physiological in origin, might be found to exist.

Dr. Drysdale thought that Mr. Paterson was to be congratulated on

his summary, and the international agreement arrived at was most welcome. As he understood the paper however, it was simply an attempt to obtain agreement between present existing units rather than standards, and left the matter of the best form of standard perfectly open. He thought that everyone having experience with flame standards would thoroughly agree with Prof. Fleming's condemnation of them, and there could be no doubt that the primary standard should be an incandescence one. He, however, did not agree with Dr. Fleming's suggestion of reviving the Violle standard. What was wanted in an incandescence standard was a definite area of a definite surface at a definite temperature. When the Violle standard was suggested we had little knowledge of the radiating properties of surfaces, or high temperature measurement, and therefore the only suitable thing was to take a very pure substance using its melting-point as a bench-mark of definite though unknown temperature. But everyone who had studied the history of the Violle standard was aware of the great difficulties of setting it up, and it had the disadvantage, according to Petavel, that the surface was dependent on the gas mixture used, besides an extremely short period of constancy and high expense. In the meantime we have realised that a perfectly black body is easily obtainable, and that it has perfectly definite radiating properties, we have the laws of Stefan and Wien, and optical pyrometry has advanced to a high degree of accuracy, and it therefore seemed decidedly preferable to suggest a unit area of a black body at a definite temperature. Mr. Jolley and he had come to the conclusion that a square cm. of a black body at a temperature of 2000° absolute would perhaps be a good unit and would be probably of the order of 100 candle-power. This temperature was probably pretty close to that of the ordinary carbon filament glow-lamp, so that there should be no colour difficulty and it should not be exceptionally difficult to maintain constant. If the temperature were measured by an optical pyrometer of say the Fery form based on the Stefan law, the deflection would be proportional to the 4th power of the absolute temperature, while the light according to Lummer and to integration from Wien's law was proportional to T^{12}. Hence the light would be proportional to the cube of the deflection only, and the probable error would not be large. Finally, a point in favour of the black body was the perfectly definite character of its spectrum, which made it a standard of colour as well as intensity and suitable for spectro-photometric comparison. As the surface would be that of a solid, it would be unnecessary to maintain it in a horizontal position, as with the Violle standard, and the amount of light could be easily varied by a diaphragm.

Dr. Drysdale said that he thought Mr. Dow had slightly misunderstood the nature of Prof. Fery's results, and it would be unfortunate if this should militate against the idea of the black body as a standard. There was no difficulty in obtaining a perfectly black body either by an enclosure or reflector. What Prof. Fery's recent experiments had shown was not that Kurlbaum's black radiators were at fault, but that he had been in error in assuming the perfect absorption of platinum-black with which his receiving bolometer was coated. This had necessitated an

increase of the constant in the Stefan formula from Kurlbaum's value of 5·32 to 6·32, but this was a point which could easily be settled and did not indicate any real difficulty in the use of the black body or the determination of its temperature, which could be simply extrapolated from known temperatures by the aid of the Stefan law.

Prof. C. H. Lees said that Prof. Petavel's recent work on the radiations from heated platinum strips suggested that he was not altogether satisfied with the Violle standard.

Mr. Paterson expressed his interest in Dr. Fleming's remarks, but was not sure that in his criticism he appreciated the object of the paper. This was not to discuss the general question of light standards, but rather to deal with light units as they are at present. The unit which is officially or legally recognized in this country has been in existence for many years, and the National Physical Laboratory had no power to establish a new unit. A more constant and reliable standard of light than a flame was a great desideratum, but its value would have to be adjusted to be in agreement with the existing legal unit. Dr. Fleming had dwelt at some length upon the various disturbing factors for which allowance had to be made in using the Pentane lamp, but given long enough, the Author could and had reproduced the value of a set of electric sub-standards in terms of the Pentane lamp to an accuracy of + or − 0·1 to 0·2 per cent. He felt that years of work would be necessary on such a standard as Violle's before an accuracy of that order could be obtained; and if the laboratory had started work of this kind initially, they might still be without a definite unit of light.

On the general question of light standards raised by Dr. Fleming, he was inclined to agree with other speakers that the Violle platinum standard was not an ideal one. A serious objection he saw, even though the standard could be easily reproduced, was the colour of the light emitted. Molten metal at 1700° C. would give a considerably redder light that the low efficiency carbon filament, and now that temperatures of light sources were becoming so much higher, it seemed to him that the standard should not be retrograde in the matter of colour.

He suggested, in comment on Dr. Drysdale's desire for a black body standard, that the same colour difficulty would come in unless the furnace could stand a temperature of 1900° C. or 2000° C. Considerable difficulty would be experienced in keeping a clear reducing atmosphere in the furnace, and the fixing of the temperature in order to secure constancy of illumination to 0·1 per cent. would demand extremely sensitive and accurate temperature measurement. The proposal certainly seemed more promising than in the case of any other incandescent standard, and the possibility of obtaining direct horizontal radiation had very great advantages from the practical standpoint.

Dr. Russell criticised the ratio of 1·11 between the Bougie Decimale and the Hefner unit. The Author was not prepared to express an opinion on the point. By the Bougie Decimale in the paper the Author implied the unit as interpreted in France, which was the only country as yet which had nominally accepted it and professed to interpret it.

LIX. *On the Form of the Pulses constituting Full Radiation or White Light.* By ALBERT EAGLE, *B.Sc., A.R.C.S., Imperial College of Science and Technology* *.

ACCORDING to the modern theory of White Light founded by Gouy, and subsequently developed by Lord Rayleigh, Schuster, and others, White Light does not consist of periodic wave-trains of all wave-lengths, which are simply separated or "dispersed" when it is drawn out into its spectrum, but consists essentially of a succession of non-periodic pulses emitted independently of one another. It is out of such pulses that the spectroscope builds up the periodic wave-trains we observe in the spectrum.

If these pulses are all similar and follow one another quite at random, it follows that the distribution of energy in each pulse must be the same as the distribution of energy in the total succession of pulses. The distribution of energy in the spectrum obviously depends on the shape or form of the pulses making it. Lord Rayleigh has shown how † the distribution of energy in a pulse of any given form could be calculated, and calculates the distribution of energy for a pulse of the form $f(t) = e^{-\alpha t^2}$. Other suggestions as to the form of the pulse have been made by other writers, and the distribution of energy which would be obtained from them has been calculated. In no case, however, has a form been hit upon which gave a distribution in accordance with fact.

The inverse problem—viz., from the distribution of energy, to find the form of pulse which would give rise to it—has not, as far as I know, been published, and its solution is the object of the present paper. We ought, however, to state that the problem is not one which admits of a definite solution, as the distribution of energy in the spectrum is independent of the relative phases of the infinitesimal harmonic components out of which the pulse may be considered to be built up; whereas its form must obviously

* Read June 11, 1909.

† Phil. Mag. vol. xxvii. p. 465 (1889), or Collected Works, vol. iii. p. 268.

depend as much on the relative phases of the components as upon their relative amplitudes.

Let $y = f(x)$ denote the form of the pulse in space. Lord Rayleigh has given * the now well-known relation

$$\int_{-\infty}^{\infty} f(x)^2\, dx = \frac{1}{\pi}\int_0^{\infty} (A^2 + B^2)\, d\alpha, \quad . \quad . \quad . \quad . \quad (1)$$

where

$$A = \int_{-\infty}^{\infty} f(\mu) \cos \alpha\mu\, d\mu,$$

and

$$B = \int_{-\infty}^{\infty} f(\mu) \sin \alpha\mu\, d\mu.$$

The left-hand side of (1) is clearly proportional to the whole energy of the pulse, and the equation is to be interpreted as implying that the energy belonging to the waves comprised between α and $\alpha + d\alpha$ is proportional to $(A^2 + B^2)\, d\alpha$. The wave-length λ is of course $\frac{2\pi}{\alpha}$. Hence, if we are given that the energy between α and $\alpha + d\alpha$ is proportional to $F(\alpha)\, d\alpha$, we have

$$F(\alpha) = \left[\int_{-\infty}^{\infty} f(\mu) \cos \alpha\mu\, d\mu\right]^2 + \left[\int_{-\infty}^{\infty} f(\mu) \sin \alpha\mu\, d\mu\right]^2. \quad . \quad (2)$$

This equation solved for f will be the solution of the problem.

Two particular solutions may very easily be obtained, first when the pulse is an even function, and second when it is an odd function. In the first case (2) reduces to

$$\int_0^{\infty} f(\mu) \cos \alpha\mu\, d\mu = F(\alpha)^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad (3)$$

and in the second case to

$$\int_0^{\infty} f(\mu) \sin \alpha\mu\, d\mu = F(\alpha)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad (4)$$

Now, in Fourier's Theorem

$$\phi(x) = \frac{1}{\pi}\int_0^{\infty} d\alpha \int_{-\infty}^{\infty} \phi(\lambda) \cos \alpha(\lambda - x)\, d\lambda,$$

* *Op. cit.*

let $\phi(x)$ be equal to zero if $x<0$, and be equal to $f(x)$ if $x>0$. Then we have

$$\frac{1}{\pi}\int_0^\infty d\alpha \int_0^\infty f(\lambda)\cos\alpha(\lambda-x)\,d\lambda \quad . \quad . \quad . \quad . \quad (5)$$

is equal to $f(x)$ for positive values of x and equal to zero for negative values of x. Hence

$$\frac{1}{\pi}\int_0^\infty d\alpha \int_0^\infty f(\lambda)\cos\alpha(\lambda+x)\,d\lambda \quad . \quad . \quad . \quad . \quad (6)$$

is equal to zero for positive values of x.

Adding and subtracting (5) and (6), we obtain

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\alpha x\,d\alpha \int_0^\infty f(\lambda)\cos\alpha\lambda\,d\lambda \quad . \quad . \quad . \quad (7)$$

$$= \frac{2}{\pi}\int_0^\infty \sin\alpha x\,d\alpha \int_0^\infty f(\lambda)\sin\alpha\lambda\,d\lambda \quad . \quad . \quad . \quad (8)$$

for positive values of x.

In much of what follows we shall frequently have an equation involving a cosine with a similar one involving a sine. In order to prevent having to duplicate such equations, we will write them with $\begin{matrix}\cos\\ \sin\end{matrix}$, in which either the upper or the lower may be taken; but in one equation upper must be taken with upper and lower with lower.

Equations (7) and (8) may very conveniently be expressed in the form of the theorem:

$$\left.\begin{aligned} &\text{If} && \int_0^\infty f(x)\begin{matrix}\cos\\ \sin\end{matrix} mx\,dx = \phi(m),\\ &\text{then} && \int_0^\infty \phi(x)\begin{matrix}\cos\\ \sin\end{matrix} mx\,dx = \frac{\pi}{2}f(m). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (9)$$

In this form the equations are useful for finding certain definite integrals; for instance, from the result

$$\int_0^\infty e^{-ax}\cos mx\,dx = \frac{a}{m^2+a^2},$$

we can at once deduce

$$\int_0^\infty \frac{\cos mx}{x^2+a^2}dx = \frac{\pi}{2a}e^{-am}.$$

Further examples will not be given here, as the author hopes to discuss some results which may be obtained from equations (9) in another paper.

Equations (7) and (8) enable us to solve at once (3) and (4). Multiplying each side of (3) by $\cos \alpha x\, d\alpha$, and integrating from 0 to ∞, we obtain by (7)

$$f(x) = \frac{2}{\pi}\int_0^\infty \mathrm{F}(\alpha)^{\frac{1}{2}} \cos \alpha x\, d\alpha, \quad . \quad . \quad . \quad . \quad (10)$$

which gives the form of the even pulse; similarly, the odd one is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty \mathrm{F}(\alpha)^{\frac{1}{2}} \sin \alpha x\, d\alpha. \quad . \quad . \quad . \quad . \quad (11)$$

We will now determine the form of the pulses for some of the different formulæ that have been proposed for representing the distribution of energy in full radiation.

Both Lord Rayleigh's and Wien's formulæ are included under

$$\mathrm{E}_\lambda\, d\lambda = \mathrm{C}\theta^n \lambda^{n-5} e^{-\frac{c}{\lambda\theta}} d\lambda,$$

the former being obtained by putting $n=1$ and the latter by putting $n=0$. Transforming this so as to obtain the energy between α and $\alpha + d\alpha$ where $\alpha = \frac{2\pi}{\lambda}$, we get

$$\mathrm{F}(\alpha)\, d\alpha = \mathrm{A}\theta^n \alpha^{3-n} e^{-2b\alpha} d\alpha,$$

where $2b = \frac{c}{2\pi\theta}$.

Hence, dropping factors outside the integral sign, the forms of the pulses are given by

$$f(x) = \int_0^\infty \alpha^{\frac{3-n}{2}} e^{-b\alpha} \begin{matrix}\cos \\ \sin\end{matrix} \alpha x\, d\alpha \quad . \quad . \quad . \quad . \quad (12)$$

$$= \frac{\Gamma\left(\frac{5-n}{2}\right) \begin{matrix}\cos \\ \sin\end{matrix} \left\{\frac{5-n}{2} \tan^{-1}\frac{x}{b}\right\}}{(b^2+x^2)^{\frac{5-n}{4}}}.$$

Putting $n=1$ for Lord Rayleigh's formula, the expressions reduce to

$$\frac{b^2-x^2}{(b^2+x^2)^2} \quad \text{and} \quad \frac{2bx}{(b^2+x^2)^2}.$$

Taking Planck's formula for the distribution of energy,

$$E_\lambda d\lambda = \frac{C\lambda^{-5}d\lambda}{e^{\frac{c}{\lambda\theta}}-1},$$

this transforms into

$$F(\alpha)\,d\alpha = \frac{A\,\alpha^2\,d\alpha}{e^{2b\alpha}-1}, \text{ where } 2b = \frac{c}{2\pi\theta} \text{ as before.}$$

Extracting the square root of this expression by expanding the denominator in ascending powers of $e^{-b\alpha}$, substituting the result in (10) and (11), and integrating term by term, we get for the form of the pulses

$$f(x) = \frac{\cos\left\{\frac{5}{2}\tan^{-1}\frac{x}{b}\right\}}{\sin(b^2+x^2)^{\frac{5}{2}}} + \frac{1}{2}\frac{\cos\left\{\frac{5}{2}\tan^{-1}\frac{x}{3b}\right\}}{\sin(3^2b^2+x^2)^{\frac{5}{2}}} + \frac{3}{8}\frac{\cos\left\{\frac{5}{2}\tan^{-1}\frac{x}{5b}\right\}}{\sin(5^2b^2+x^2)^{\frac{5}{2}}} + \ldots. \quad (13)$$

To the eye these pulses have much the same form as the simple ones obtained from Lord Rayleigh's formula.

It is interesting to observe that all the pulses we have found satisfy the condition $\int_{-\infty}^{\infty} f(x)dx=0$. This is obvious for the odd pulses; for the even ones, we have only to show that $\int_0^{\infty} f(x)dx=0$.

Now $f(x)$ consists of one or more terms of the form

$$\int_0^{\infty} \alpha^p e^{-q\alpha}\cos\alpha x\,d\alpha.$$

This integrated with respect to x gives

$$\int_0^{\infty} \alpha^{p-1} e^{-q\alpha}\sin\alpha x\,d\alpha = \frac{\Gamma(p)\sin\left\{p\tan^{-1}\frac{x}{q}\right\}}{(q^2+x^2)^{\frac{p}{2}}} \quad . \quad (14)$$

which vanishes when taken between 0 and ∞ if p be positive.

On the electron theory these pulses are due to the radiation from moving electrons, and the function giving the form of the pulse as a function of t is the same as the function giving the acceration of the electron at time t. Hence, to find the displacement at any time, we must integrate the function giving the form of the pulse twice. By integrating the left-hand side of (14) twice with respect to x, we observe that we merely change p into $p-2$. Hence, to integrate the right-hand side twice we have only to change p into $p-2$. Applying this to equations (12) and (13), we get the motion of an electron which will give rise to these pulses.

For the pulses $\frac{b^2-x^2}{(b^2+x^2)^2}$ and $\frac{2bx}{(b^2+x^2)^2}$, obtained from Lord Rayleigh's formula, the motion of the electron is given by

$$y = \mathrm{A} \log (a^2+t^2) \quad \text{and} \quad y = \mathrm{B} \tan^{-1} t/a,$$

where $a=\frac{b}{\mathrm{V}}$.

Equations (10) and (11) show that the pulses may be regarded as resulting from a superposition of a series of sine curves of all wave-lengths, each with a suitable amplitude so as to give the required energy distribution. This being so, we may expect that we shall be able to obtain a more general form of pulse by assigning an arbitrary phase to the component sine curves, without thereby altering the distribution of energy in any manner. That is, we may expect

$$f(x) = \frac{2}{\pi}\int_0^\infty \mathrm{F}(\alpha)^{\frac{1}{2}} \cos \{\alpha x - \phi(\alpha)\}^{\frac{1}{2}} d\alpha \, . \quad . \quad . \quad (15)$$

to be a more general solution of (2) than (10) or (11). This in fact is so, as may be verified by splitting up (2) into the two equations

$$\mathrm{F}(\alpha) \cos \phi(\alpha) = \int_{-\infty}^{\infty} f(\mu) \cos \alpha\mu \, d\mu$$

and

$$\mathrm{F}(\alpha)^{\frac{1}{2}} \sin \phi(\alpha) = \int_{-\infty}^{\infty} f(\mu) \sin \alpha\mu \, d\mu,$$

and trying to find a solution of these—in the same manner in which (3) and (4) were solved—which at the same time satisfies both of them. Equation (15) will be the result.

It is no longer, of course, believed that white light consists of a succession of pulses all of exactly the same form. It consists rather of a succession of pulses capable of being represented by an equation with one or more arbitrary parameters, the number of pulses emitted per second which have their parameters lying within definite limits being given by some law analogous to the Maxwellian distribution law. But the form we have obtained may probably be looked upon as some mean or average form of the pulses, and may perhaps be of some value in determining, to a first approximation at least, by what intermolecular forces the free electrons in a substance must be acted upon in order to give the observed distribution of energy in the spectrum. Although, as we have seen, the problem does not admit of a definite solution, yet it is not impossible that physically all the pulses in white light may be (say) even ones. For instance, if an electron moves in a straight line against an opposing force which is a function of the distance, and which is insensible at the beginning of the motion but becomes sufficiently powerful to bring it to rest and reverse its motion, the pulse produced will obviously be an even one, under which conditions the form of the pulse for a given distribution of energy is unique.

LX. *Note on Terrestrial Magnetism.*

By G. W. Walker, *M.A.**

In several papers dealing with an explanation of terrestrial magnetism by extraneous magnetic force, I have been surprised to find it assumed as obvious that if the earth had a large magnetic permeability, the effect of a given extraneous

* Read June 11, 1900.

force would be largely magnified, and that thus small forces might produce effects as large as those actually observed *.

The following considerations show that this view is quite erroneous.

Suppose we have a sphere of radius a and magnetic permeability μ, and let an extraneous magnetic field be represented by a potential $F P_n \frac{r^n}{a^{n-1}}$, P_n being a zonal harmonic of order n. The effect of the sphere is represented at outside points by the additional potential

$$\frac{Fn(1-\mu)}{\mu n+n+1} P_n \frac{a^{n+2}}{r^{n+1}}.$$

We thus find that the normal force at the surface is altered in the proportion $\frac{\mu(2n+1)}{\mu n+n+1}$ and that the tangential component of force at the surface is altered in the proportion $\frac{2n+1}{\mu n+n+1}$. The most likely case in the application to terrestrial magnetism is $n = 1$, and so for a high value of μ the normal force is increased in proportion 3 to 1, while the tangential component becomes nil.

For higher values of n the normal force is increased in a less proportion than 3 : 1. Only in the case $n = 0$ is the normal force increased in proportion $\mu : 1$. But an extraneous field which should be entirely radial in the vicinity of the earth does not appear to me possible.

It seems to me therefore desirable to point out that the assumption of large magnetic permeability of the earth is no real help in the explanation of terrestrial magnetic effects by extraneous magnetic forces.

* *Cf.* Arrhenius, *Kosm. Phys.* p. 984; Pflüger, *Phys. Zeit.* 1905, p. 415; and Humphreys, 'Terrestrial Magnetism,' Dec. 1908, p. 151.

LXI. *A Method of producing an intense Cadmium Spectrum, with a proposal for the use of Mercury and Cadmium as Standards in Refractometry.* By T. MARTIN LOWRY, *D.Sc., F.C.S.* *

Of the different line spectra that are available for spectroscopic standards—hydrogen, mercury, cadmium, &c.—the simplest and purest is undoubtedly the cadmium spectrum. The visible spectrum is made up of four strong lines (red, green, blue, and dark blue), which are so narrow and of such a high degree of purity in respect of the absence of satellites that they have been used by Michelson to produce interference-bands of an order of retardation that has apparently never been reached in the case of any other lines. Michelson's measurements of the wave-lengths of the three chief Cadmium lines :—

Cd red.....	6438·4722	10^{-10} metre.
Cd green.....	5085·8240	,, ,,
Cd blue	4799·9107	,, ,,

have indeed formed the standards from which all other wave-lengths have been deduced. It is therefore evident that the cadmium spectrum is destined to play an extremely important part in optical determinations of all kinds. Unfortunately, the difficulty of producing a cadmium lamp which shall burn steadily and give out light of high intensity has been so great that the four cadmium lines have been used only very occasionally in optical experiments.

Sodium.

The standard monochromatic light employed almost universally for refractometric and polarimetric measurements has been the yellow flame-spectrum of sodium, which has the advantage of being produced with very great readiness, but with all the drawbacks inseparable from the use of a doublet, instead of a single line, as a standard. Thus in determining the refractive index n_D of a liquid, the Pulfrich refractometer

* Read June 25, 1909.

gives readings for the less refrangible constituent, whilst a hollow prism mounted on a spectroscope gives an average value for the two constituents, unless indeed the resolution be sufficient to read them separately. In polarimetric work the double character of the sodium line renders it impossible to secure a proper extinction for large values of α_D, since one wave-length is transmitted with considerable intensity when the other is extinguished, and in addition there is always some uncertainty as to the "optical mass-centre" of the doublet, which may indeed vary in different types of sodium-lamp, on account of changes in the relative intensity of the two constituents *. It should also be noted that the sodium flame emits a considerable amount of light of other colours, which in accurate work, or in reading large rotations, must be removed by filtering through a coloured screen, or, better, by means of a spectroscopic eyepiece (Perkin).

Hydrogen.

The hydrogen lines,

H_α (red), w.-l. = 6560·04,

H_β (blue), w.-l. = 4861·49,

H_γ (violet), w.-l. = 4340·66,

have been employed universally with the sodium doublet in refractometric work when dispersion-values were required. The choice has been wholly one of convenience and has no other merit to recommend it. The vacuum-tube, though easily fitted up, can hardly be considered seriously as a source of light. The red line is by far the strongest, and has been used with advantage to produce interference-fringes in measurements of length †, but would be utterly useless for polarimetric work in which the source of light must be viewed through a Nicol's prism set within 2° or 3° of the extinction position. The violet line is unpleasantly weak even for refractometric measurements, and demands the use of the

* Compare Landolt, *Optische Drehungsvermögen*, 1898, pp. 362 *et seq.*

† See, for instance, Tutton's measurements of the coefficients of expansion and of elasticity of crystal-plates (Phil. Trans. 1903, A. 202 p. 143). Compare also Tutton, Proc. R. S., June 10, 1909.

full power of a six-inch coil, with an efficient optical condenser, before readings can be made with any degree of comfort. The hydrogen spectrum has the further disadvantage of showing, at least in an ordinary vacuum-tube, an almost continuous back-ground of weak lines. Although, therefore, refractometers are regularly sent out with tables for the sodium and the three hydrogen lines—and no data whatever for light of any other wave-length—it is evident that this position is radically unsound and cannot be maintained permanently.

Choice of Standards.

The essential properties for a standard source of light are, (1) that it should be of sufficient intensity to be used for *all* the various types of optical measurements, so that, for instance, refractive indices and optical and magnetic rotatory powers may be determined for the same wave-lengths, (2) that it should be strictly monochromatic and as far as possible free from satellites, and (3) that it should be produced with sufficient readiness to render it generally available. These requirements, as has been shown, are only partially fulfilled by sodium light and fail completely in the case of the hydrogen spectrum. The purpose of the present communication is to suggest that the spectra of mercury and of cadmium fulfil most of the essential conditions outlined above, and to describe a method by which the cadmium spectrum may be rendered more generally available for spectroscopic and other optical work.

The suggestion—which is made on the basis of practical experience in the actual measurement of optical and magnetic rotations and of refractive indices for a large range of wave-lengths (see for instance Proc. Roy. Soc. 1908, 81. p. 472) —that sodium should give place to mercury and cadmium as a chief standard source of light, is fully supported by the theoretical considerations recently advanced by Bates ("Spectrum Lines as Light Sources in Polariscopic Measurements," Bureau of Standards, Bulletin, 1906, ii. p. 239) and by Nutting ("Polarimetric Sensibility and Accuracy," *ibid.* p. 249, "Purity and intensity of Monochromatic Light

Sources," *ibid.* p. 439). The former author has worked out a formula showing the errors due to the use of a doublet in polarimetry, and has redetermined the ratio of the sodium-yellow and mercury-green rotations for quartz; the latter has developed formulæ in reference to polarimetric sensibility, and spectral purity. The two points in these papers that bear directly on the practical problem now under consideration are, (1) the confirmation by Bates of the purity of the mercury green line, which gave very sharp readings in the case of a quartz plate of about 5½ mm. thickness; this point is, however, seriously discounted by the fact that he professes to read the sodium doublet to 0·0001°, and gives the ratio of sodium to mercury to six significant figures (0·850944 : 1), (2) the statement by Nutting that on one basis of reckoning the "spectral purities" of cadmium green, mercury green, and sodium yellow, are represented by the ratios $\frac{1}{100,000}$, $\frac{1}{10,000}$, and $\frac{1}{700}$, whilst on another basis the "specific impurities" of the mercury green and sodium yellow lines are given by the ratios $\frac{1}{1,000,000}$, $\frac{1}{2000}$; these figures serve to show that the change of principal standard now proposed on the basis of practical polarimetric work is fully justified by minute spectroscopic tests on the lines themselves.

Mercury.

The use of the enclosed mercury arc as a source of light in spectroscopy dates back to 1860 (J. H. Gladstone, "On the Electric Light of Mercury," Phil. Mag. [4] xx. pp. 249–253), but its use in polarimetric measurements was apparently introduced by Disch (Ann. Phys. (4) xii. p. 1155) in 1903, who made use of the Arons lamp. The Bastian mercury lamp, which has been in use in my own laboratory since 1906, and at the Central Technical College since 1907, has the advantage of being a commercial article of much lower cost; it is constructed with a suitable resistance in the holder, so that it can be plugged into the ordinary lighting circuit without using a resistance-frame or any special leads. This lamp is unfortunately no longer on the market, though it is still constructed to order by the Brush Electrical Engineering Company; but

silica lamps of moderate price are promised which may prove to be as economical in working as, and even more efficient in illumination than, the earlier glass lamps.

Of the six chief mercury lines,

5790·49
5769·45 } a yellow doublet,

5460·97 a splendid green line,

4358·58 a strong violet line,

4078·03
4046·78 } at the extreme limit of the visible spectrum,

two, the green and the violet, have already proved to be of the utmost value in polarimetry, and are likely in the future to prove of equal value in the measurement of refraction and dispersion.

Their use in polarimetry has been due to the following considerations. For accurate measurements of the specific rotatory power of a substance, and to any even larger extent for tracing the course of chemical changes (isomeric change, sugar-hydrolysis, &c.) by polarimetric observations, it is essential to use an intense source of light in association with a very small half-shadow angle, since only thus can a maximum of sensitiveness be secured: it is also desirable to use a light of relatively short wave-length, in order that the actual readings may be large *, without incurring the loss of optical intensity and the fatigue which result from the use of blue light. The intense green mercury line, which can be read with a considerably smaller half-shadow angle, and gives readings about 15 per cent. larger than the sodium doublet, is therefore much superior to the traditional standard, apart altogether from the question of spectral purity. In the latter respect the contrast is extreme; in the case of quartz I have been able, without any noticeable loss of accuracy, to secure readings showing a total rotation of 50 right angles for the mercury green line, two independent series of determinations

* As a rule the specific rotation is doubled on passing from yellow to violet.

giving average values 4487·78° and 4487·79°; sodium under similar conditions gave no extinction at all. The green mercury line promises, indeed, wholly to replace the sodium doublet as a chief standard in polarimetric work, and it is highly desirable that it should acquire as quickly as possible a like predominance in the measurement of refractive indices, and in all other optical determinations.

The violet mercury line has proved indispensable in the measurement of rotatory dispersion on account of its extraordinary brilliancy. In spite of the low sensitiveness of the eye for light of such small wave-length it has been found possible to read this line with a half-shadow angle of only 6°, and to secure series of readings (each an average of 10 settings) which only differed from one another by a hundredth of a degree. The violet line is less pure than the green, as it is accompanied by two satellites of smaller wave-length, but these are so weak that they cannot be seen at all in the polarimeter, and cannot, therefore, produce any large disturbance in the readings. The yellow doublet is made up of two lines separated by about three times as great an interval as in the case of sodium; for small rotations they may be read as one line, but I have also been able, by using a narrow slit, to read them separately; they are, however, altogether unsuited for general use.

For refractometer work the mercury lines are at least as easily available as those of hydrogen; a warmed vacuum-tube containing a drop of mercury gives the lines with greater brilliance than those of hydrogen, and it is therefore not unreasonable to suggest that—as a minimum concession to the correlation of optical measurements of various kinds—the use of $H\beta$ 4861 and $H\gamma$ 4341 shall be abandoned in favour of Hg 5461 and Hg 4359 in future refractometric work, and that tables for these wave-lengths shall be supplied as a matter of course with instruments of the Pulfrich pattern. It may be noted that the violet mercury and hydrogen lines differ by only 18 Ångström units, the mercury line having the longer wave-length: in a Pulfrich instrument the two lines are indistinguishable, but the edge that is read with a

hydrogen-mercury vacuum-tube (such as is sometimes sent out with the instrument) is due to mercury and not hydrogen. The adaptation of a polarimeter for use with mercury light costs about £2, with a further £3 for the lamp.

Cadmium.

The cadmium spectrum is much less easy to produce than that of mercury. Michelson made use of a strongly heated vacuum-tube with aluminium electrodes connected to platinum wires passing through the glass. This was improved upon by Hamy (*Comptes Rendus*, 1897, cxxiv. p. 749) who used a copper heating-jacket and external electrodes, thus avoiding the risk of cracking the hot glass by wires passing through it. I have had no personal experience of such lamps, but am doubtful whether they would give a sufficiently intense light for use in polarimetry. The amalgam lamp with an arc enclosed in silica can be made to give a splendid series of lines for use in spectroscopy, but I have found that it is useless for polarimetric work, since even the green cadmium-line can only be read with a half-shadow angle of nearly 20°. Apparently the current is carried mainly by the mercury, and the other metals show only weakly in the spectrum. An enclosed cadmium arc has been described by Stark & Küch (*Phys. Zeitschr.* 1905, vi. pp. 438–443), but does not appear to have come into general use.

The method set out below is not put forward as the ideal way of producing an intense cadmium spectrum, but rather as an intermediate stage in the development of the perfect cadmium lamp of the future. It was found that brilliant spectral lines could be sent into the polarimeter by using an arc burning between metallic poles rotating in opposite directions. Copper, for instance, gave a valuable series of lines, and brass electrodes were found to be very efficient for developing a zinc spectrum, the red line Zn 6364, and the three blue lines Zn 4811, Zn 4722, and Zn 4680, standing out very distinctly from the copper lines. A brilliant cadmium spectrum could be produced by melting the metal onto copper electrodes, but it soon burned off, and in any case it

was difficult to avoid a displacement of the readings by the appearance of the copper line Cu 5106, as a ghostly partner of the green cadmium line Cd 5086. After many unsuccessful attempts a workable method of producing the cadmium spectrum was found in the use of an alloy of silver and cadmium. It is perhaps not very widely known that these metals are isomorphous, and form an excellent series of alloys. These have the advantage that no eutectic is formed, the melting-points throughout the series lying above that of cadmium and over a considerable range approximating somewhat closely to the melting-point of silver. Thus whilst the addition of 28 per cent. of copper (mp. 1082°) lowers the melting-point of silver from 960° to 780°, the addition of 28 per cent. of cadmium (mp. 322°) only lowers the melting-point to 860°. Nearly 50 per cent. of cadmium must be added to lower the melting-point to 780°, and even a 60 per cent. alloy melts as high as 700°. These alloys, which can be turned up like pure silver, were supplied by Messrs. Johnson and Matthey in the form of rods $\frac{1}{4}$ inch in diameter and $1\frac{1}{4}$ inch long.

For spectroscopic work a tiny arc can be burnt quite steadily between the points of the rods, in great contrast to the behaviour of pure cadmium, which splutters very badly and gets choked up with oxide, even when the current is kept so small as not to melt the metal.

For polarimetric work a greater intensity of light is desirable, and this is obtained by using a heavier current and rotating the electrodes in opposite directions (compare Baly, Spectroscopy, p. 370) in order to maintain the arc in a central position. The rods of alloy were screwed for half their length into copper cylinders $\frac{3}{8}$ inch in diameter, which served the double purpose of cooling the electrodes—a point of some importance—and connecting them with the iron spindles by means of which the rotation was produced. When run at the highest intensity both rods become red hot, and one of the copper cylinders is usually luminous, but it is not desirable to over-run the arc since even if the electrodes do not melt the cadmium distils out irregularly and causes a certain amount of spluttering.

The electrodes are filed up before the arc is started, and are

carefully adjusted so as to run true to centre; alternatively they may be allowed to burn until the ends are flat and then used without further attention except to adjust the length of the arc from time to time. The cadmium spectrum thus produced is of great brilliance—the green line is even brighter than that of mercury, and can be read with a half-shadow angle of 3° or less. From some points of view it would be a better chief standard than mercury green, as it is considerably brighter and gives readings about 15 per cent. higher, but in view of the greater trouble involved in producing the light, and persuading it to burn steadily, it is better to use it as a secondary line for the study of rotatory dispersion. The red and blue lines are also very bright and can be read quite easily. The dark blue line is of much less intensity, and is not likely to be widely used, as it is difficult to read, and does not differ sufficiently in wave-length from the light blue line to justify the extra trouble involved; this observation applies, however, only to the existing arrangements, as it is quite possible that when a more powerful source of steady light is available the dark blue line may prove to be of considerable value in shortening the gap between Cd 4800 and Hg 4359.

Unlike copper the silver spectrum does not clash at all with that of cadmium; the brilliant silver green lines

$$\left.\begin{matrix}5471{\cdot}72\\5465{\cdot}66\\3209{\cdot}25\end{matrix}\right\}\ \text{doublet (compare sodium)}$$

are separated from the cadmium green, Cd 5086, by an interval nearly as great as that which separates the two cadmium blues, and the only other line that shows at all strongly in the spectrum is a line in the far-violet, perhaps Ag 4055.

In conclusion: It is suggested that the mercury line Hg 5461 should be used as chief standard in optical work of all kinds, and that dispersion should be measured from this line to Hg 4359 instead of from H_α 6561 or H_β 4861 to H_γ 4341. As secondary standards are suggested the flame spectra Li 6708 and Na 5893, purified spectroscopically,

together with the three cadmium lines Cd 6438, Cd 5086, Cd 4800, giving a well-distributed series of seven wave-lengths.

6708,	6438,	5893,	5461,	5086,	4800,	4359.
Red	red	yellow	green	green	blue	violet.

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Note added July 1909.—I am glad to find that the desirability of a change in the standard wave-lengths for use in refractometry is apparently recognized by other workers: in particular, Dorn & Lohmann in their measurements of the refractive indices of liquid crystals (*Ann. Physik*, June 10th, 1909, [4] xxix. pp. 535–565) have used the series Li 6708, Na 5893, Hg 5461, Hg 4359, which is identical with a series that I am using for the measurement of the refractive dispersions of the alcohols and acids of the aliphatic series, and differs from the series of seven lines used in the measurement of rotatory dispersion only in the omission of the three cadmium lines.

Discussion.

Mr. Twyman remarked that during the last few weeks he had seen a cadmium tube, similar to a mercury lamp, working with satisfactory results. Such tubes had been used for some time by Paschen & Röntgen. He agreed with the Author with regard to the greater use of mercury light for spectroscopic and similar purposes.

Dr. A. E. H. Tutton said that he had been working lately with Cadmium tubes and found that they worked well for a considerable time. With regard to the measurement of refractive indices he was astonished that more use was not made in this country of the monochromatic illuminator, which he described some years ago. The instrument was used in Germany and gave satisfactory results.

The Author agreed with Dr. Tutton that the monochromatic illuminator had certain well-recognized advantages, but hoped that those who used it would make a point of setting it to the wave-lengths suggested in the paper. Uniformity in the measurement of optical properties could only be achieved by adopting some series of standard wave-lengths such as the one whose advantages he had indicated.

LXII. *On the Measurement of Wave Length for High Frequency Electrical Oscillations.* *By* ALBERT CAMPBELL, *B.A.**

(From the National Physical Laboratory.)

[Plate XXXIII.]

§ 1. *Introduction.*

In all work with high frequency electrical oscillations, such as for example in wave telegraphy, it is of the utmost importance to be able to determine with accuracy the wave-length actually employed, and for this purpose several types of wavemeter are now in common use. In order to ensure accuracy in such measurements, it was suggested some time ago by the Post Office Authorities that arrangements should be made for the calibration of wavemeters at the National Physical Laboratory. As the results of our first series of investigations in the matter appeared to be of general interest, we publish them here by the kind permission of Major O'Meara, C.M.G., Engineer-in-Chief to the Post Office. Our experiments comprised the construction and testing of a standard wavemeter, and the verification of an ordinary commercial wavemeter sent to us by the Post Office. I shall designate this instrument (A) and our standard wavemeter (B) respectively. While it is unnecessary here to go into the history of the subject, I should like to give one or two references to earlier work by other observers which gave us much assistance, namely the experiments of Pierce † and those of Gehrcke‡.

While our work was in progress an important paper by Diesselhorst § appeared, and the results there published were in ample confirmation of those we were obtaining, as also were the earlier results of Pierce.

* Read June 25, 1909.

† Phys. Review, vol. xxiv. p. 152, 1907.

‡ *Elektrotech. Zeitschr.* (26), p. 697, 1905

§ *Jahrbuch der drahtlosen Telegr. u. Teleph.* vol. i. (1908).

§ 2. *Description of Wavemeter* (A).

Wavemeter (A) is of the Dönitz type and consists of a variable air condenser, with a range from about 100 to 1070 mmfd. (micromicrofarads), a thermo-junction and galvanometer, and a series of coils (A 1, A 2, ... to A 10) of self inductances ranging from 0·76 to 2313 microhenries. The combination is capable of measuring oscillation frequencies over a range extending from about 20,000,000 down to 100,000 ~ per sec., or wave-lengths from $\lambda = 15$ metres up to $\lambda = 3000$ metres. It must be kept in mind, however, that the accuracy obtainable depends on the part of the condenser scale at which the reading is taken. For example, at a reading of 20 the frequency can barely be read to 0·5 per cent.

As the coils (A 1, A 2, ...) of (A) are of solid (not stranded) wire of diameters from 0·32 up to 3·2 mm., the values of their self inductances at the high frequencies are, for most of the coils, considerably lower than those obtained at ordinary frequencies of 0 to 1000 ~ per sec.

With a view to checking the results of the direct experiments by calculation from the measured inductance and capacity of a wavemeter in which the inductances would be much less affected by frequency, and thus to obtain a standard instrument for future use, we constructed a wavemeter (B) in which the coils are all of highly stranded wire.

§ 3. *Description of Standard Wavemeter* (B).

The general arrangement of (B) was similar to that of (A). The variable condenser (from 100 to 900 mmfd.) was of special design, with amberite washers to give very high insulation. It was kindly presented to the Laboratory by Dr. Alexander Muirhead, F.R.S. We added to it a direct reading scale, which can be read to 1 in 2000 at the upper end and to 1 in 200 at the lower. The scale was constructed by a very careful series of tests by Maxwell's Commutator Method and was found to be very uniform.

For the inductances, three coils (Q *b*, Q 1, and Q 2) were used. They were all wound on ebonite tubes with stranded wire (7/36"), *i. e.* containing seven insulated strands, each of

diameter 0·19 mm. In each coil the terminals were brought to a considerable distance (18·5 cm.) from the centre of the coil by fixed leads run parallel to one another 1 cm. apart. In this way too close proximity between the coil and the condenser plates was avoided. A Duddell thermo-ammeter of 1·5 ohms resistance was used to complete the circuit of the coil and the condenser, and by observing its maximum deflexion the point of resonance was obtained. The sensitivity of course varies with the coil used and with the nature of the oscillatory circuit; it was found to be sufficient for the purpose in the experiments described below. Fig. 1 (Pl. XXXIII.) shows a photograph of the wavemeter and of one of the coils separately.

§ 4. *Measurement of the Self Inductances of the Coils.*

The self inductances of all the coils were measured at ordinary frequencies (0 to 1000 ~ per sec.) by a method specially designed for the accurate measurement of such low values (A. Campbell, Phil. Mag. Jan. 1908). The comparisons were made against a standard variable mutual inductance with ranges of 0 to 200 and 0 to 2000 microhenries, the lower range being readable to 0·01 microhenry. The absolute value of this was measured in terms of the new Laboratory Standard of mutual inductance, whose value has been calculated to a very high degree of accuracy (see Proc. Roy. Soc. A. vol. 79, 1907). The subdivision of the scale of the variable mutual inductance was effected by the help of Maxwell's method of comparing two mutual inductances, Its accuracy was verified by an independent method as follows. A mutual inductance coil was constructed with the primary and secondary circuits both of well stranded wire (10 and 20 strands respectively), and, with all the strands in each circuit in series, the value was adjusted to be equal to the 10 microhenry reading on the scale. Then, by taking p strands of the primary and q strands of the secondary, the value would be $\frac{pq}{200} \times 10$, and thus inductances of $\frac{1}{200}$, $\frac{2}{200}$, $\frac{3}{200}$, ... of the full scale reading were obtained. On testing these against the actual scale, the subdivision was found to be perfectly satisfactory.

The details of winding and values of the self inductances of the coils of Wavemeter (B) are given in Table I. In actual working these values had to be increased by the

TABLE I.

Coil.	Coil Diameter. cm.	Axial Length. cm.	Turns.	Self Inductance. Microhenries.
QB.....	13·5	3·6	40	360·3
Q1	7·4	3·9	48	170·7
Q2	7·6	1·7	23	57·8,

addition of the measured self inductance of the rest of the circuit, consisting of the leads, ammeter, and condenser ; the total addition was about 0·5 microhenry, of which 0·1 microhenry was due to the condenser.

§ 5. *Tests of Standard Wavemeter* (B) *by Photographing Sparks.*

In order to obtain an absolute calibration of the standard wavemeter, the frequencies of the oscillations with which it was tested were determined by including a spark gap in the main oscillation circuit to which the wavemeter was loosely coupled and photographing the spark trains by help of a rotating mirror.

The apparatus was arranged as follows :—

The source of current was a small alternator whose frequency could be varied from 50 to 200 ~ per second ; this was connected through a small high voltage transformer to a spark gap with cadmium electrodes shunted by a glass plate condenser and a bare wire inductance in oil. A large variable air condenser consisting of six aluminium disks, each 100 cms. in diameter, could be put in parallel with the glass plate condenser, and, by suitable variation of the capacity and inductance in circuit, oscillations of frequencies from 300,000 to 1,200,000 ~ per sec. could be obtained. A

special camera (fig. 2) was constructed and mounted for rigidity on a long slab of sandstone. At one end of the camera was the rotating mirror, which was concave and of 200 cms. radius of curvature. It turned on a vertical axis and was driven by a small motor, the speed being kept constant by an arrangement described below. The other end of the camera had two branches which carried respectively the spark electrodes and a pair of guides allowing the plate carrier to be smoothly raised or lowered while the rotating mirror threw the images of the spark trains as streaks upon the plate. The distance from the mirror to the plate was 200 cms. The speed of rotation of the mirror was kept constant (usually at about 60 revs. per second) as follows. On the axis was mounted a commutator and this was connected with a condenser, a bridge, battery, and galvanometer, for Maxwell's Commutator method of measuring capacity. By applying a slight variable brake to the rotating axis, the galvanometer light-spot could be kept at zero. When this was the case the speed was steady, and could be measured with a counter or deduced from the value of the condenser and the resistance in the bridge. From measurements of a number of the spark train photographs on each plate, the average displacement from spark to spark was found; and the frequency was calculated from this displacement, the distance from the mirror to the plate, and the speed of rotation of the mirror.

While each photograph was being taken the reading of the standard wavemeter was also observed, care being taken to keep the coupling to the spark circuit very loose.

The value (n) of the frequency obtained from the spark photographs was in each case compared with the values (n_1) calculated from the measured values K and L of the capacity and inductance in the wavemeter circuit. Since the wavemeter coils have a certain amount of distributed capacity, it was necessary to take account of this. The required correction was made by Glazebrook and Lodge's formula (Cambridge Phil. Trans. p. 171, vol. xviii. 1899),

$$p^2\mathrm{LK} \doteqdot 1 - \frac{\mathrm{N}-2}{\mathrm{N}^2} \cdot \frac{k}{\mathrm{K}},$$

where $p=2\pi n$, N = number of turns in coil, k = capacity from turn to turn, and the capacities and inductances are in absolute measure. The value of k was found by testing the capacities of coils with bifilar windings of wire similar to that in the coils used. The correction is small, being in no case more than 1 part in 1000. The results are given in Table II.

TABLE II.

Plates Nos.	n By Sparks ~ per sec.	n_1 From K & L of Standard Wavemeter. ~ per sec.
61 & 62	290,300	290,500
47	516,800	516,800
57 & 58	818,300	821,200
55	1,042,000	1,039,000

We may remark that the small differences between n and n_1 are quite within the limits of probable experimental error. On the same plate the value of n deduced from the various spark trains sometimes showed an extreme variation of 1·2 per cent. from the mean; in the best experiments the variation was about 0·5 per cent. from the mean. In general the mean of 5 to 10 spark trains was used and the average variation from the mean was from 0·2 to 0·6 per cent. in the value of n.

§ 6. *Comparison of Wavemeter* (A) *with Standard* (B).

The condenser of (A) was tested throughout its range by Maxwell's Commutator Method and gave the results shown in Table III. which shows the scale readings to be very nearly proportional to the capacity. At the higher readings the accuracy of measurement of K is here of the order of 2 or 3 parts in 1000.

TABLE III.

Reading. Degrees.	K mmfd.	K/Reading.
20	126	6·1
40	245	$6{\cdot}0_8$
60	367	$6{\cdot}1_2$
80	486	$6{\cdot}0_8$
100	607	$6{\cdot}0_7$
120	724	$6{\cdot}0_3$
140	851	$6{\cdot}0_8$
160	966	$6{\cdot}0_4$
180	1075	$5{\cdot}9_7$

The coils (A4, A5, ... to A9) were tested for self inductance (L_0) at ordinary frequencies. In Table IV. are given the values found (including the working circuit in each case). Approximate dimensions of the winding are also

TABLE IV.

Coil.	$2a$ Coil Diameter. cm.	l Axial Length. cm.	d Wire Diameter. cm.	N. Turns.	Self inductance. microhenries. L_0.	L_∞.
A4	5·2	10·7	0·32	32	$21{\cdot}6_2$	$19{\cdot}5_5$
A5	8·0	11·3	0·32	33	$44{\cdot}9_6$	42·0
A6	5.1	5·6	$0{\cdot}09_1$	55	100·5	97·5
A7	5·1	9·6	$0{\cdot}09_1$	95	195·0	189·8
A8	7·6	5·3	$0{\cdot}05_6$	90	531·8	524
A9	7·6	8·9	$0{\cdot}05_6$	150	1039	1025

given. In the last column are given the values of L_∞ (*i. e.* the value of the self inductance for infinite frequency, assuming that the current in that case is practically confined to the skin of the wire). These values have been calculated

by the approximate formula (due to Heaviside):*

$$L_0 - L_\infty \doteq \frac{26N^2ad}{1000\,l} \text{ in microhenries,}$$

where N = number of turns,
a = radius of coil,
l = axial length of coil,
d = diameter of wire,

all the dimensions being in centimetres.

The application of the formula to such short coils is not quite appropriate, however, as it has been deduced on the assumption that the solenoid is long.

The two wavemeters were then loosely coupled to the same oscillation circuit and simultaneous readings taken at various frequencies. From the calibration of (B) already established and the results in Table III. the effective values of the self inductances of the coils of (A) were deduced; they are given in Table V.

TABLE V.

Coil.	Comparison Frequency ~ per sec.	Effective L (deduced). Microhenries.	L. (Mean).
A4	1,125,000	20·1	19·9
	1,338,000	19·8	
A5	837,400	42·6	42·5$_5$
	975,500	42·5	
A6	665,100	99·1	99·1
A7	366,700	190·5	
	394,800	190·5	
	466,800	191·1	191·2
	473,400	192·0	
	665,100	191·8	
A8	284,500	530	
	290,300	530	53$_1$
	322,800	532·5	
A9	290,300	104$_8$	104$_8$

* Collected Papers, vol. i., p. 356.

From Table VI. it will be seen that the observed effective values of L (to be used in working the instrument) lie between L_0 and L_∞ in all cases but the last. The discrepancy here may be due to an error of 0·5 per cent. in determining n which might occur in consequence of the reduced sensitivity when using a coil of such high resistance as A 9.

The column headed L_n gives the values of L calculated from L_∞ for the actual values of n by means of Cohen's formula *. It will be noticed that this brings the observed and the calculated values of the inductances (at the high frequencies) considerably closer.

TABLE VI.

Coil.	Effective L.	L_n.	L_0.	L_∞.
A4	19·9	19·6	21·6$_2$	19·5$_6$
A5	42·5$_5$	42·1	44·9$_6$	42·0
A6	99·1	98·1	100·5	97·5
A7	191·2	190·8	195·0	189·8
A8	531	526	531·8	524
A9	104$_6$	1029	1039	1025

§ 7. *Conclusion.*

Thus it appears that within the limits of wave-length used, the wavemeter with coils of well stranded wire gave results in close agreement with theory, while in the case of the instrument with coils of solid wire the agreement was as close as could be expected, as the correcting formulas are only strictly applicable to long solenoids.

In conclusion I would express my best thanks for kind assistance to Major O'Meara and his staff; to Prof. R. Ll. Jones, Messrs. H. C. Booth and T. L. Eckersley, who skilfully aided in the experiments; and to Dr. Glazebrook for valued help and advice throughout the work.

* Bulletin, Bureau of Standards, vol. iv., no. 1, p. 177 (1907).

Discussion.

Mr. W. Duddell said the paper was a valuable one because accurate experiments on self-induction and capacity at high frequencies were required. With reference to the photographs he asked if the Author had used his method to photograph arcs and, if so, with what results. It would be interesting to know how low it is possible to get the apparent resistance of a coil with high frequency currents by stranding the wire.

Dr. Erskine-Murray, referring to Mr. Duddell's remarks, said that in actual practice the resistance could not be reduced more than about 10 per cent. by stranding.

Mr. Taylor congratulated the Author, and pointed out that the wavemeters described could only be used at the transmitting station. Wavemeters were required which could be used at receiving stations. Referring to the question of stranding he pointed out that it was possible to overdo the stranding and obtain less satisfactory results than could be obtained by stranding with a fewer number of wires.

Mr. G. B. Dyke called attention to experiments similar to those described which were being carried out by Dr. Fleming at University College. With reference to the spark photographs he was surprised that an accuracy of 1 in 1000 could be obtained.

The Author in reply stated that as the whole distributed capacity of the inductance coils had very little effect on the frequency of resonance, the dielectric hysteresis of the ebonite would be negligible except as regards damping. Sparks appeared to give sharper and more accurately measurable photographs than vacuum tube discharges or arcs, except mercury arcs, which gave the clearest and best pictures.

LXIII. *An Electromagnetic Method of Studying the Theory of and Solving Algebraical Equations of any Degree.* *By* Alexander Russell, *M.A., D.Sc., and* J. N. Alty, *A.I.E.E., Faraday House, London* *.

Contents.

1. Introduction.
2. The Electromagnetic Method.
3. Quadratic Equations.
4. The Equation to Curves passing through the Neutral Points.
5. Cubic Equations.
6. Finding the Roots of an Equation.
7. Description of Apparatus.

1. *Introduction.*

The electrical device recently invented by Mr. Arthur Wright enables us to find approximate values of the real

* Read June 25, 1909.

roots of an equation at once by simple mechanical and electrical operations. In order, however, to find approximate values of the imaginary roots it is necessary to perform certain analytical operations, and then apply the device to find the roots of an equation of a higher degree. The method described in this paper has the great merit of giving approximate values of all the imaginary roots as well as all the real roots. It is not capable of such high accuracy as the Arthur Wright device, and it cannot be directly applied when the indices of the powers of the unknown quantity are fractional. On the other hand, it is exceedingly instructive, as it shows how the numerical values of both the real and imaginary roots vary as the coefficient of any power of the unknown quantity in the equation is varied. The apparatus required is exceedingly simple, and is to be found in practically every physical laboratory. We have found it quite a suitable experiment to include in a laboratory course for first year students.

The method suggested itself to one of the authors when studying a series of papers by Mr. F. Lucas which are published in the *Comptes Rendus*, t. 106 (1888). The final electrical method (p. 1072) devised by Mr. Lucas is a practical one. A sheet of tinfoil is spread over a large flat plate of glass or other insulating material. If the equation is of the nth degree $n+2$ sources and sinks for electrical current are provided. These are arranged on a line at equal distances apart. The currents in $n+2$ wires touching at the sources and sinks are adjusted so that they have certain definite values. The method of calculating these values is practically the same as in the method described below. The equipotential lines are then traced out either by an electrochemical method or by the Kirchhoff and Carey Foster method. If the line of sources and sinks be taken as the axis of x, and the origin be chosen midway between the outer wires, the coordinates of the nodal points, that is, the points where an equipotential line intersects itself, enable us to write down all the roots, real or imaginary, of the given equation at once. If (x_1, y_1) be the coordinates of a nodal point, we can see from the symmetry of the arrangement that $(x_1, -y_1)$ will be the coordinates of another nodal

point. It follows from the method of adjusting the currents that $x_1 \pm y_1\sqrt{-1}$ gives a pair of conjugate roots of the given equation. The real roots, therefore, are all given by the abscissæ of the nodal points lying on the axis of X. The method is not rigorous, as Mr. Lucas has not considered the magnitude of the error introduced owing to the finite size of the conducting sheet. It would be very laborious to apply in practice. In solving a biquadratic, for instance, the currents in five wires would have to be adjusted to given values before the equipotential lines could be mapped out.

We shall now describe an electromagnetic method, in which the horizontal field due to the earth's magnetism is used in an analogous manner to the conducting sheet in Mr. Lucas's method. A drawing-board with a slit cut in it, a few pieces of bell-wire, any form of "charm" compass, ordinary ammeters and rheostats or lamp-resistance boards such as are found in every physical laboratory can be utilized at once for the experiment (see § 7).

2. *The Electromagnetic Method.*

Let us suppose that we have to find the roots for the equation

$$f(x) \equiv a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 = 0. \quad . \quad . \quad (1)$$

We first, by the ordinary methods given in books on algebra, resolve the expression

$$\frac{f(x)}{(x-b_1)(x-b_2)\ldots(x-b_n)}$$

into partial fractions. The numbers $b_1, b_2, \ldots b_n$ are any convenient numbers so chosen, however, that

$$b_1 + b_2 + \ldots + b_n = -a_{n-1}/a_n. \quad . \quad . \quad . \quad . \quad (2)$$

For example, if the second term of $f(x)$ be missing we must choose $b_1, b_2, \ldots$ so that $\Sigma b = 0$.

We thus obtain

$$\frac{f(x)}{(x-b_1)(x-b_2)\ldots(x-b_n)} = a_n + \frac{A_1}{x-b_1} + \frac{A_2}{x-b_2} + \ldots + \frac{A_n}{x-b_n}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where, by (2),

$$A_1 + A_2 + \ldots + A_n = 0, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and

$$\left.\begin{aligned} A_1 &= \frac{f(b_1)}{(b_1-b_2)(b_1-b_3)\ldots}, \quad A_2 = \frac{f(b_2)}{(b_2-b_1)(b_2-b_3)\ldots} \\ \text{and} \quad A_n &= \frac{f(b_n)}{(b_n-b_1)(b_n-b_2)\ldots(b_n-b_{n-1})} \end{aligned}\right\}, \quad (5)$$

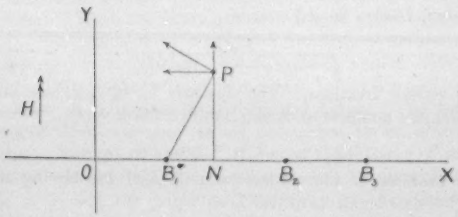
Let us now consider the magnetic field round a long vertical wire carrying a current of C amperes, and let us suppose that the earth's horizontal field in the neighbourhood is uniform, and that its horizontal intensity in C.G.S. units is H. The magnetic force at any point P at a perpendicular distance of r centimetres from the axis of the wire will be the resultant of a force $C/5r$ acting at right angles to the plane containing r and the axis of the wire and a force H directed to the magnetic pole. There is always a neutral point* on the line through the axis of the wire perpendicular to the magnetic meridian. If x be the distance of this point from the axis

$$x = C/(5H). \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This formula is utilized in a well-known rough laboratory method of measuring H when C is known or *vice versâ*.

Let us now suppose that we have n vertical wires arranged in a plane perpendicular to the magnetic meridian, and let

Fig. 1.



them cut the plane of the paper perpendicularly at B_1, B_2, .. B_n (fig. 1) which are at distances $b_1, b_2, \ldots b_n$ from O. If $C_1, C_2, \ldots C_n$ be the values in amperes of the currents in the wires, and H the horizontal intensity of the earth's

* A. Russell, 'The Electrician,' vol. xxxi. p. 282 (1893).

magnetic field, the components X and Y of the resultant magnetic force at $P(x_1, y_1)$ will be given by

$$-X = \frac{C_1}{5r_1}\cdot\frac{y_1}{r_1} + \frac{C_2}{5r_2}\cdot\frac{y_1}{r_2} + \frac{C_3}{5r_3}\cdot\frac{y_1}{r_3} + \dots$$

and

$$Y = H + \frac{C_1}{5r_1}\cdot\frac{x_1-b_1}{r_1} + \frac{C_2}{5r_2}\cdot\frac{x_1-b_2}{r_2} + \dots,$$

where $r_m^2 = (x_1-b_m)^2 + y_1^2$.

Hence multiplying X by ι and subtracting we get

$$Y + X\iota = H + \frac{C_1}{5r_1^2}(x_1-b_1-y_1\iota) + \frac{C_2}{5r_2^2}(x_1-b_2-y_1\iota) + \dots$$

$$= H + \frac{C_1/5}{x_1+y_1\iota-b_1} + \frac{C_2/5}{x_1+y_1\iota-b_2} + \dots.$$

At a neutral point the resultant magnetic force is zero, and therefore both X and Y are zero. Hence, if x_1 and y_1 are the coordinates of a neutral point, $x_1+y_1\iota$ is a root of the equation

$$0 = H + \frac{C_1/5}{x-b_1} + \frac{C_2/5}{x-b_2} + \dots \quad . \quad . \quad , \quad (7)$$

Comparing this with equation (3) we see that if we adjust the values of the currents so that

$$C_1 = 5H\,.\,A_1/a_n,\ C_2 = 5H\,.\,A_2/a_n, \dots$$

$$C_n = 5H\,.\,A_n/a_n,$$

then $x_1+y_1\iota$ is a root of the equation $f(x)=0$, and therefore $x_1-y_1\iota$ is also a root.

It follows from (4) that $\Sigma C = 0$, and therefore only $n-1$ ammeters and only $n-1$ rheostats are required.

As an introduction to the method let us consider the theory of quadratic equations.

3. *Quadratic Equations.*

Let us suppose that the equation is

$$x^2 + bx + c = 0.$$

In this case it is convenient to write

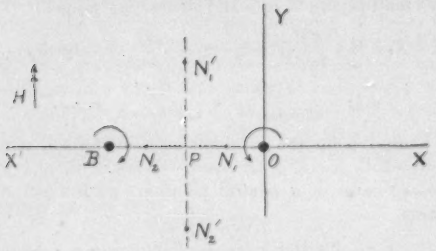
$$\frac{x^2+bx+c}{x(x+b)} = 1 + \frac{c/b}{x} - \frac{c/b}{x+b},$$

and, therefore,

$$H\frac{x^2+bx+c}{x(x+b)}=H+\frac{(5Hc/b)}{5x}-\frac{(5Hc/b)}{5(x+b)}. \quad . \quad . \quad (8)$$

In fig. 2 let us suppose that the plane of the paper represents a horizontal plane, and that H is the direction of the earth's magnetic force. Let us also suppose that two long

Fig. 2.



vertical wires cut the plane of the paper perpendicularly at O and B respectively, the length of BO being b centimetres and the magnitude of the current C, in amperes, in the O wire being $5Hc/b$, and the B wire carrying the return current $-5Hc/b$. If the direction of the current C be into the paper at B and out of it at O, the circular lines of force due to the currents in each wire will act in the directions of the curved arrow-heads shown in the figure.

We shall now consider how the numerical values of the roots of the quadratic equation alter as c, and therefore also C, increases from zero to infinity, b which we suppose to be positive remaining constant.

We have already shown that the coordinates of the points at which the resultant magnetic force is zero, that is, the neutral points, determine completely the numerical values of both the real and imaginary roots of the equation.

When c is very small and positive the neutral points lie between B and O. For a particular value of c, for instance, they are at N_1 and N_2 on the axis of X. We see at once from symmetry that $ON_1=N_2B$. If the length of ON_1 on

the same scale that BO is b be $-x_1$, the roots of the equation for this value of c are both real and equal to $-x_1$ and $-b+x_1$ respectively. As we increase the value of c, and therefore of the current in the wires, the neutral points N_1 and N_2 approach one another and coincide at P. In this case the roots are each equal to $-b/2$. For greater values of c the neutral points cannot possibly lie on BO as the resultant magnetic force due to the currents at all points of this line is greater than H. From symmetry the neutral points lie on the line N_1PN_2 bisecting OB at right angles. For a particular value of the current they are at the points N_1 and N_2 on this line. If $PN_1=y_1$ then $PN_2=-y_1$ and the corresponding roots of the equation are $-b/2+y_1\sqrt{-1}$ and $-b/2-y_1\sqrt{-1}$. The real part of the imaginary roots is therefore independent of c, but the coefficient of $\sqrt{-1}$ increases as c increases.

When c is negative the curved arrow-heads in fig. 2 must be drawn in the opposite directions. N_1 will now be at a distance x_1 from O along OX, and N_2 will be at a distance $-b-x_1$ from O along OX′. The roots in this case, therefore, are always real and of opposite sign, and continually increase numerically as c increases.

When b is negative the point B in fig. 2 will be to the right of OY, and the discussion of the roots in this case is equally simple.

4. *The Equation to Curves passing through the Neutral Points.*

From the preceding discussion it will be seen that the locus of the real roots of an equation determined in this way is the axis of x. Fig. 2 shows that for a quadratic equation when b is positive the locus of the points giving the imaginary roots is the straight line $N_1'N_2'$. Similarly when b is negative it is the parallel line given by equation $x=b/2$. It is important to know the equations to curves on which the neutral points must lie in the general case.

Let us suppose that $x+y\iota$ is a root of the equation $f(\xi)=0$. In this case we have

$$f(x+y\iota)=0,$$

and, therefore, by Taylor's theorem,

$$f(x)-\frac{y^2}{\underline{2}}f''(x)+\frac{y^4}{\underline{4}}f^{iv}(x)-\ldots$$

$$+iy\left\{f'(x)-\frac{y^2}{\underline{3}}f'''(x)+\ldots\right\}=0.$$

This equation can be satisfied either by

$$f(x)=0, \quad \text{and} \quad y=0, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

or by

$$f(x)-\frac{y^2}{\underline{2}}f''(x)+\frac{y^4}{\underline{4}}f^{iv}(x)-\ldots=0 \quad . \quad (10)$$

and

$$f'(x)-\frac{y^2}{\underline{3}}f'''(x)+\frac{y^4}{\underline{5}}f^{v}(x)-\ldots=0 \quad . \quad (11)$$

The equations (9) obviously give the real roots of the equation which must all lie on the line $y=0$, that is, on the axis of X. The points of intersection of the equations (10) and (11) determine the imaginary roots.

As equation (11) does not contain the constant term of the equation $f(\xi)=0$, it follows that it gives the curve locus of a series of imaginary roots of the equations formed by varying c. Where the curve represented by equation (11) cuts the axis of x we have $f'(x)=0$. At these points there are at least two equal roots.

For the quadratic equation $x^2+bx+c=0$, (11) becomes $2x+b=0$, and this is the equation to the line of neutral points $N_1'PN_2'$ shown in fig. 2.

5. *Cubic Equation.*

Let us suppose that the equation has been reduced to the form

$$x^3-b^2x+c=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

We easily find that

$$H\frac{x^3-b^2x+c}{x(x^2-b^2)}=H+\frac{C}{5(x-b)}-\frac{2C}{5x}+\frac{C}{5(x+b)}, \quad (13)$$

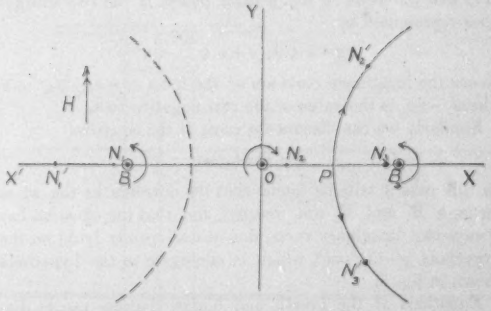
where $C=5Hc/2b^2$.

In this case (fig. 3) three long vertical wires are used. Let them cut the plane of the paper at right angles at B_1, O,

and B_3 respectively, and let $B_1O = OB_3 = b$. The currents in the wires passing through B_1 and B_3 must each be made equal to C, and the current in the wire passing through O should be the return current 2C. The arrow-heads indicate that the direction of the current flow is out of the paper at B_1 and B_3 and into it at O. H denotes the direction of the earth's magnetic field.

Let us now suppose that c increases uniformly from zero to infinity. When c is small, and therefore the currents are small, two neutral points N_2 and N_3 obviously lie between O and B_3 and a neutral point N_1 lies along B_1X'.

Fig. 3.



As the current increases the neutral points N_2 and N_3 approach one another and the neutral point N_1' moves along B_1X'. For a particular value of c, N_2 and N_3 coincide at P, and we have two equal roots. For greater values of c the neutral points N_2 and N_3 move uniformly along the curve $N_2'PN_2'$, one being as much above the line XX' as the other is below it. But the neutral point N_1 always moves along the axis of X.

The equation (12), therefore, has always three roots. When c is small they are all real, two of them being positive and one negative. For a certain value of c the two positive roots become equal, and for greater values of c we have two conjugate imaginary roots and one real negative root. Both

the real and imaginary parts of the conjugate roots continually increase as c increases.

The equation to the curve $N_2'PN_3'$ in fig. 3 can be found at once from (11); it is

$$3x^2-y^2=b^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The negative branch of this hyperbola indicated by the dotted curve in fig. 3 gives the locus of the neutral points when c is negative. In this case there is obviously always one real positive root. Increasing the value of b in (12) is equivalent to putting the wires B_1 and B_3 further apart. This obviously increases the limits of the values of the positive real roots. When b is negligibly small we see by (14) that the locus of the neutral points is the two straight lines represented by

$$(y-x\sqrt{3})(y+x\sqrt{3})=0.$$

Hence the imaginary roots are of the form $x_1 \pm x_1\sqrt{3}\sqrt{-1}$ where $-2x_1$ is the value of the real negative root.

Similarly we can discuss the roots of the equation

$$x^3+b^2x+c=0.$$

In this case it will be found that the currents in the wires through B_1 and B_3 are unequal, and that the equation has always two imaginary roots, the neutral points lying on the hyperbola $y^2-3x^2=b^2$, which is conjugate to the hyperbola shown in fig. 3.

Equations of the fourth and higher degrees can be discussed in like manner. To get the most instructive results care has to be taken to choose the distances between the wires so that the analytical expressions for the required currents may be as simple as possible. If this be not done analytical difficulties will often be encountered in interpreting the results.

6. *Finding the Roots of an Equation.*

The great and so far as we know the unique advantage of this method is that it enables us to find the imaginary as well as the real roots of an equation almost at once. In equations occurring in many physical problems it is the latter roots which we desire to find, and this method enables

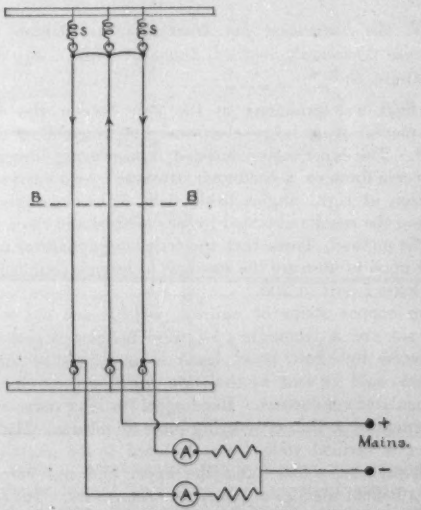
the physicist to find quickly approximate values of these roots.

If due precautions are taken the maximum inaccuracy of this method need not exceed one per cent. This is the accuracy obtainable by careful students, who need have no previous experimental training, in finding H by measuring the distance of the neutral point from a long vertical wire carrying a known current.

7. *Description of Apparatus.*

We shall now describe the simple apparatus we use for teaching purposes. In fig. 4 the arrangement of the apparatus

Fig. 4.



for solving a cubic equation is shown. S, S represent springs attached to insulators. These are necessary in order to keep the wires taut. The wires are No. 16 single cotton-covered

wire, and pass through a long slit cut in a drawing-board BB. The ammeters A are any of the ordinary ammeters used in the laboratory reading up to 10 amperes, and are in series with lamp-boards and rheostats. The currents are adjusted to the required values, and a sheet of sectional paper with a slit in it is put on the board. By means of an ordinary charm-compass the neutral points can then be readily found. It is advisable to make a little sketch of the lines of force near the neutral points, as this is a help in indicating their true position. The coordinates of these points read off from the sectional paper give the real and imaginary roots of the equation.

LXIV. *An Instrument for Measuring the Strength of an intense horizontal confined Magnetic Field.* *By* F. W. Jordan, *B.Sc.**

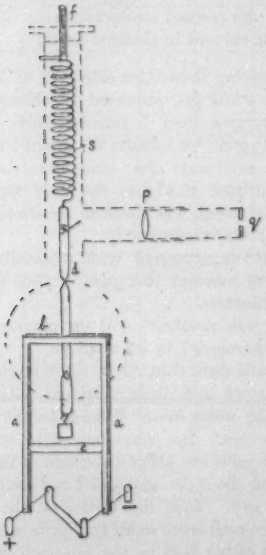
The field was produced in the gap betwen the conical pole pieces of a large electromagnet resembling a split toroid. The experiment consisted in measuring directly the transverse force on a conductor, traversed by a current in a direction at right angles to the field. The close agreement between the results obtained by this method and those by the ballistic method, shows that the following apparatus may be relied upon to measure the strength of intense confined fields to at least 1 part in 200.

Two copper strips of uniform width, each cut to form three sides of a rectangle $a\,b\,a$, were fastened together and connected by short tinsel leads to terminals, so that the current could be sent in the same direction through each of the insulated conductors. Bending of the long strips $a\,a$ was prevented by a thin connecting plate of mica e. The lower part of a vertical strip d was fastened to the centre of the horizontal conductors b and the upper end was suspended from a helical spring s of phosphor bronze wire. The tension of this spring was adjusted by a screw f to raise the con-

* Communicated by S. Skinner, M.A. Read June 25, 1909.

ductors to the sighted position, where a small circular hole in the vertical strip appeared to be bisected by a near horizontal wire. In the latter adjustment, parallax was avoided by using a single lens microscope p with a horizontal slit q for the eye aperture. The vertical motion of the conductors was limited by a projection moving between stops.

Fig. 1.



To make a measurement the apparatus was arranged so that the horizontal conductors were at right angles to the field at the centre of the gap. A known copper weight was suspended from the centre of the horizontal conductors and the strength of the current was adjusted to raise these to the

sighted position. At the same time the apparatus was moved, if necessary, to a position where the transverse forces on the vertical conductors were equal and opposite.

The strength of the field

$$F = \frac{mg}{lC} \text{ gausses}$$

where mg dynes is the force on the horizontal conductors b.

l cm. is the sum of the distances between the centres of the vertical conductors $a\,a$.

C is the current in absolute units.

The helical spring showed an extension of 1·2 cm. per gm. and a load of 0·005 gm. produced an observable extension. The weights ranged from 1 gm. for fields of about 1000 gausses to 8 gm. for fields of about 10,000 gausses. In a gap 1 cm. wide the conductors became slightly unstable and drifted slowly to the near face of the gap. This initial movement was checked by stretching two hairs on either side of the vertical strip.

The distance l as measured with a travelling microscope was found to be constant to 1 part in 1000 over the upper part of the conductors.

The current was measured with an ammeter to 1 part in 500 and ranged from 0·7 to 1·5 ampères.

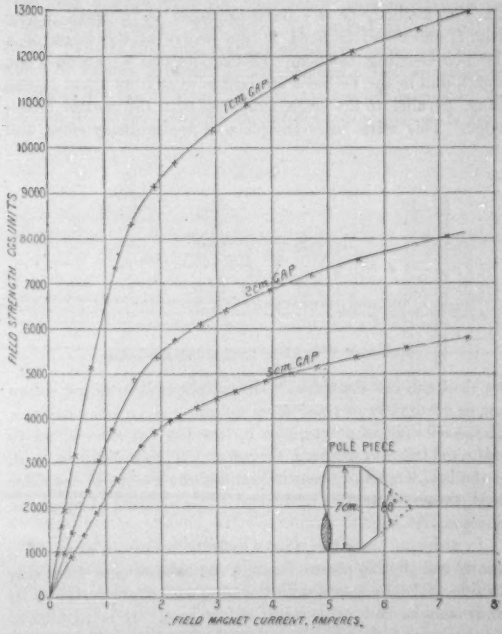
The tinsel leads were situated at right angles to the plane of the conductors and their terminal attachments were separated by the same mean distance as the centres of the vertical conductors. The maximum field strength in the direction of the leads was about 6 per cent of the field strength at the centre of the 3 cm. gap and 2 per cent at the centre of the 2 cm. gap. Thus the forces on the leads in this particular experiment were small and quite negligible.

The strength of the field in the gap itself was uniform to about 2 per cent., and since the diameter of the test coil in the ballistic method was made equal to the length of the horizontal conductor, it follows that no appreciable difference between the results of the two methods was to be expected.

The minimum field strength measured was about 1000

gausses and the maximum field strength attained in the 1 cm. gap was about 13,000 gausses. The results obtained by this apparatus agreed with those of the ballistic method

Fig. 2.



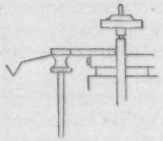
to about 1 part in 400. The field strength curves for gaps of three different widths were constructed from the results and are shown as an example of use of this instrument.

South Western Polytechnic Institute, Chelsea.
26 May, 1909.

LXV. *On a Method of Determining the Sensibility of a Balance.* *By* J. H. Poynting, *Sc.D., F.R.S., and* G. W. Todd, *M.Sc.**

In the method, as we have arranged it, a small frame (fig. 1, end view) is fixed at the centre of the beam of a 16-inch Oertling balance. This carries two Vs about 2 cm. apart, and in the Vs lies a straight wire or fibre about $3\frac{1}{2}$ cm. long, parallel to the beam and level with the central knife-edge. This wire takes the place of the ordinary rider, and

Fig. 1.



End view of V frame fixed to balance-beam.

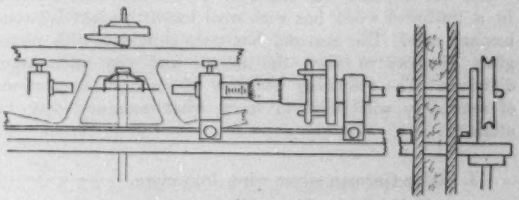
we shall call it "the rider." Its weight is determined before use as accurately as possible by weighing on an assay balance. The sensibility is determined by moving the rider either to right or left a measured distance. If this distance is d, if the half length of beam is b, and if the weight of the rider is R, the movement is equivalent to an addition of weight to one pan, Rd/b.

In order to move the rider a definite distance a stout horizontal rod (fig. 2) passes through the balance-case from side to side without contact with the case, and is supported at its ends outside, and independent of, the case. It is parallel to the beam and a little lower than the V frame. On the rod are fixed horizontally two Brown & Sharp micrometer-screws divided to ·01 mm. and allowing an estimate of ·001 mm. Their axes are in one line coinciding with the axis of the rider, and they are fixed so that one can bear against one end and the other against the other end of the rider. Their ends

* Read June 25, 1909.

are plane and the ends of the rider are bluntly pointed. Each micrometer screw-head has a cross piece fixed on it, and a fork which can be rotated about an axis in the continuation of the axis of the screw by a pulley outside the case, can engage with the cross piece, and so advance or withdraw the screw. The pulley is worked by an endless string passing to a pulley at the side of the observer, who is about 2 metres in front of the balance. The micrometer divisions are illuminated and each micrometer is viewed by its own telescope. The position of the balance-beam is read

Fig. 2.



Arrangement of right-hand micrometer-screw.

by a double-suspension mirror, telescope and scale. The scale is divided to millimetres and is about 3 metres from the mirror. The double-suspension mirror is fully described in the Phil. Trans. A, 1891, p. 572. It is of course not essential to the method, but was chosen because of the great magnification of the deflexion which it gives.

Let us suppose that the value of the scale-divisions of the deflexion is to be determined by a movement of the rider from right to left. The two micrometer-screws are withdrawn so that neither is in contact with the rider, that on the left so far that the rider will not touch it in its subsequent travel. The beam is lowered and allowed to swing. Then the right-hand screw is advanced till it bears against the end of the rider and pushes it some small distance. The contact is seen to have occurred by the interference with freedom of swing, as watched in the telescope. The micrometer is then read. Let its reading be m_1. It is then withdrawn a little

so as to leave the rider free, and the centre of swing C_1 is determined in the usual way from three successive turning points. Then the micrometer is advanced again so as to push the rod a little further, and its reading m_2 is taken. It is then withdrawn and the new centre of swing C_2 is taken. If $m_1 - m_2 = d$, $C_1 - C_2$ divisions deflexion are due to an addition of Rd/b to the left pan.

The right-hand micrometer may then be withdrawn and the left-hand micrometer may be brought into action in a similar manner, and so on, the two screws being used alternately.

The balance-case was fixed on a shelf and was enclosed in a tin-foiled wood box with wool loosely packed between box and case. The case and box were provided with plate-glass windows to view the mirror and the micrometer divisions. The following abstract of some determinations of sensibility will serve to show what accuracy may be attained :—

I. Rider German silver wire, 7·35 mgm.

Half length of beam, 20·272 cm.
10 determinations alternately left and right.
Mean travel of rider ..... 2·4850 mm.
Mean deflexion ..... 21·26 divisions.
Mean value for 20 divisions ... 0·0848 mgm.
The separate determinations range between
20 divisions = 0·0877
and 20 „ = 0·0824.

II. The same rider.

10 determinations alternately left and right.
Mean travel of rider ..... 5·2713 mm.
Mean deflexion ..... 45·47 divisions.
Mean value for 40 divisions ... 0·1681 mgm.
The separate determinations range between
40 divisions = 0·1722
and 40 „ = 0·1632.

III. Rider German silver wire, 189·05 mgm.

7 determinations alternately left and right.
Mean travel of rider ..... 0·1764 mm.
Mean deflexion ..... 38·70 divisions.
Mean value for 40 divisions ... 0·1691 mgm.
The separate determinations range between
40 divisions = 0·1709
and 40 „ = 0·1654.

IV. The same rider.

7 determinations alternately left and righ
Mean travel of rider ..... 0·3004 mm.
Mean deflexion ..... 64·84 divisions.
Mean value for 60 divisions ... 0·2578 mgm.
The separate determinations range between
60 divisions = 0·2612
and 60 „ = 0·2518.

LXVI. *The Balance as a Sensitive Barometer.* *By* G. W. Todd, *M.Sc.**

It occurred to the author while testing with Professor Poynting the accuracy of a new method of determining the sensibility of a balance, described on p. 926, in which a thin rod or fibre is used instead of the usual rider, that a balance might with proper precautions be converted into a very delicate barometer.

A difference in volume on the opposite sides of a balance will give rise to motions of the pointer when the density of the surrounding air changes. Small oscillations of the pointer, due chiefly to convection currents, become negligible if this difference in volume is sufficiently large. Alterations in the density of the air are produced by changes in pressure, in temperature, and in the percentage of aqueous vapour in it. By eliminating the effects of any two of these the variations in the other may be obtained. Thus the balance might be used either as a barometer, thermometer, or hygrometer.

* Read June 25, 1900.

Experiments were carried out with the balance mentioned in the previous paper. A sealed copper sphere, about 760 c.c., was suspended from one arm and counterpoised with brass weights placed in the opposite scale-pan. The movements of the pointer were observed by means of a small "doubly suspended" mirror which reflected a distant scale into a telescope. One division in this telescope corresponded to a rise or fall in the atmospheric pressure of the order of ·002 mm. of mercury.

If W, V are the weight and volume of the sphere and W′, v similar quantities for the counterpoise, we have, assuming that the difference in volumes of the opposite sides of the balance itself is nil and that the air is dry,

$$W-VDp\frac{273}{760}\frac{1}{A}=W'-vDp\frac{273}{760}\frac{1}{A},$$

where p = atm. press. in mm., D = density of air at 0° C. and 760 mm., A = abs. temp.

$$\therefore\quad W-W'=(V-v)Dp\frac{273}{760}\frac{1}{A}.$$

If the air is moist and the pressure of aqueous vapour is f mm.,

$$W-W'=(V-v)\frac{273}{760}\frac{1}{A}\{(p-f)D+fd\},$$

where d = dens. of wat. vap. at 0° C. and 760 mm.

Suppose there is a change ∂W when there are changes ∂p, ∂f, ∂A, then

$$\partial W=(V-v)\frac{273}{760}\left[\partial p\frac{D}{A}+\partial A\frac{(D-d)f-Dp}{A^2}+\partial f\frac{(d-D)}{A}\right].$$

With the balance-case dry,

$$\partial W=(V-v)\frac{273}{760}\left[\partial p\frac{D}{A}-\partial A\frac{Dp}{A^2}\right]$$

or

$$\partial p=\frac{760A}{(V-v)273D}\partial W+\frac{p}{A}\partial A.$$

In the micro-barograph, which is given later (fig. 1),

$$\partial W=4{\cdot}6\times10^{-6}\partial S,$$

where S is the scale-reading in the telescope. This ratio was the mean of several obtained by moving the rider a known distance and noting the deflexion in the telescope, taking into account the rate at which S was altering at the time. The quantity $(V-v)$ differed by less than 1 c.c. from 760 c.c., so that

$$\partial p = \frac{4{\cdot}6 \times 10^{-6} A}{273 D} \partial S + \frac{p}{A} \partial A.$$

With the temperature and pressure variations acting together the whole scale would often pass across the field of the telescope in less than half an hour, and it had to be brought into view again by moving the rider in the required direction by means of the micrometer-screws. Observations of S and A were taken alternately every thirty seconds, and the barograph was obtained by plotting the terms on the right of the above equation and adding.

The barographs showed many oscillations in pressure of the order of ·02 mm., the period of oscillation being from ten to fifteen minutes, a period quite different from that of the balance. The question arose whether these oscillations were real oscillations of pressure, or whether they were caused by oscillations in temperature not shown on the mercury thermometer suspended in the balance in consequence of the capacity for heat of its bulb.

To decide this question twelve thermojunctions were made with fine wires of copper and constantan, each wire being about six centimetres long. Six of these junctions were embedded in wax and placed in a box of brass filings, while the other six were left exposed to the air. The covered junctions having their capacity for heat made very large by the surrounding brass filings, would "lag" behind the exposed junctions and would not be affected by small oscillations in temperature. The ends of this thermopile were connected to a low resistance galvanometer. One division on the scale was approximately equal to a difference of ·009° C. between the exposed and covered junctions. When the thermopile was exposed in the room the scale-reading oscillated aimlessly to and fro, but when placed inside the balance no oscillations could be detected. The spot of light travelled slowly and uniformly in one direction, showing

that the temperature in the balance was slowly rising and that the covered junctions were "lagging."

After this the temperature of the air inside the balance was measured by thermojunctions to avoid errors due to lagging. One set of junctions was kept at a fairly constant temperature in a Dewar tube containing water, the temperature being indicated on a Beckmann thermometer.

An idea of the sensitiveness of such a balance-barometer may be obtained from the table given below and the corresponding micro-barograph. The observations given extended over half an hour. The Beckmann thermometer in the Dewar vessel indicated the same temperature throughout.

TABLE.

1° C. difference between the junctions in the balance and those in the Dewar vessel corresponded to 28·8 divisions on the galvanometer-scale.

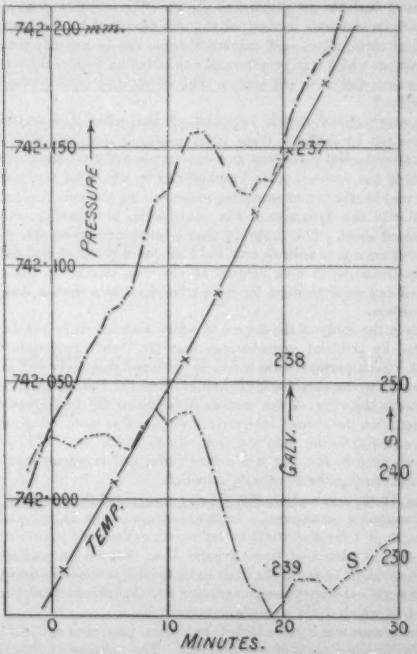
Galv. zero = 272·2. Room temp. = 13°·8 C. Rider at 13·152.

Time.	Telescope.	Galv.	Time.	Telescope.	Galv.
P.M.			P.M.		
4—0	243·7	239·1	4—16	229·2	
1	243·3		17	226·3	237·3
2	242·6	238·9	18	225·6	
3	243·6		19	223·1	
4	243·7		20	224·7	237·0
5	243·9	238·6	21	227·0	
6	242·8		22	227·0	
7	242·2		23	226·1	236·7
8	244·9	238·3	24	225·8	
9	246·7		25	226·1	
10	245·0		26	227·7	236·3
11	246·2	238·0	27	228·5	
12	246·3		28	230·9	
13	244·0		29	232·0	236·0
14	240·1	237·7	30	232·6	
15	234·6		31	232·1	

The second column in the table gives the telescope-readings of the balance-pointer, and the third gives the

galvanometer readings from which the temperature inside the balance is obtained. In the barograph the nearly straight curve represents the temperature, the lower irregular

Fig. 1.



curve the actual observations of the position of the balance-beam, and the upper curve the positions reduced to a constant temperature.

Discussion.

Mr. M. E. J. Gheury expressed his interest in Mr. Todd's successful attempt to apply the well-known influence of barometric pressure upon the indications of a sensitive balance to detect small variations in the

atmospheric pressure. Prof. Marvin has just described (Monthly Weather Review, 1908) a delicate recording barometer, which shows plainly variations of ·05 mm.; another type of recording barometer, the statoscope, records variations of the order of ·02 mm. Both instruments give traces which show, even in calm weather, a succession of irregular ripples of variable amplitude and period. The question presents itself: can such an elaborate method as the one described by the Author be of practical value, when such minute changes can be actually recorded by instruments which require practically no attention besides the adjustment of the zero, and now and again a time check, and need no temperature correction?

Another point of greater importance is that when such small changes as these can be detected, there is no means of discriminating between the meteorological variations and the purely artificial ones, due to the fact that the instrument is in a building in which for various reasons there may be eddies and ascending or descending currents having nothing to do with the dynamics of the atmosphere, but leaving traces upon the record sheet. It is probable that a large number of the variations observed are due to entirely artificial changes of pressure, and such sensitive instruments, to give records of any use, should be placed in an isolated and quiet position, far from a building, in a shelter designed for the purpose.

It is in the study of the degree to which such sensitive barometers are affected by artificial perturbations that Mr. Todd's apparatus becomes useful. Such perturbations could be produced purposely and at regular intervals, and the magnitude of their effect on the instrument ascertained. To decide, therefore, if the records obtained so far by similar sensitive apparatus are the correct interpretation of genuine meteorological phenomena seems to be the only practical use to which Mr. Todd's ingenious apparatus can be put, but it is a very useful and necessary investigation to which much interest remains attached.

The Author in reply to Mr. Gheury recognized the practical value of a self-recording instrument. In the telescope of his balance-barometer one-tenth of a division could be estimated, so that the sensitiveness was from fifty to a hundred times greater than that of the statoscope. It therefore might be worth the little extra trouble involved in compounding two records—the temperature variation and the balance pointer variation—each of which could be made automatically.

The balance was little affected by sudden jars, such as the slamming of doors in other parts of the building. The opening or shutting of a door in the same room only occasionally caused a small motion, of the order of half a division, in the telescope. During observations this door was of course kept shut. A wave in the barograph of the same period as that of the balance might be due to a momentary impulse transmitted through the building, but other waves would not be open to the same suspicion.

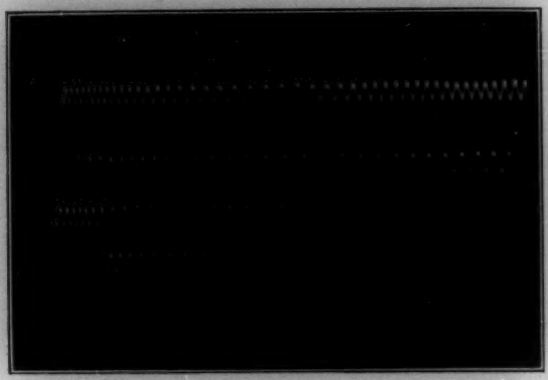


FIG. 7.

FIG. 8.

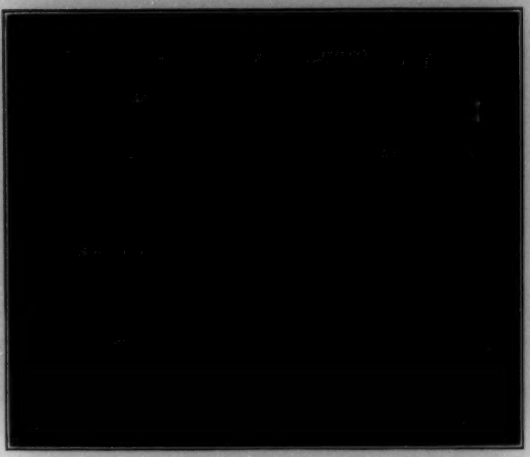


FIG. 7 *a*.

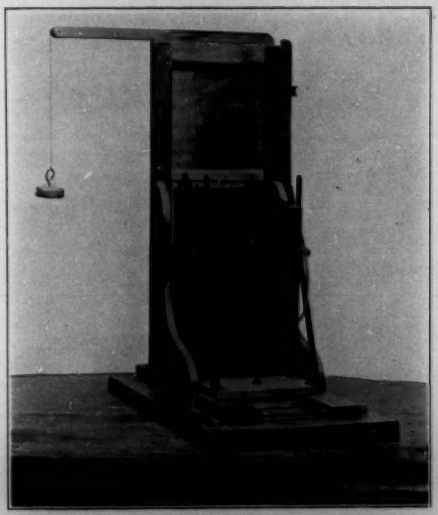
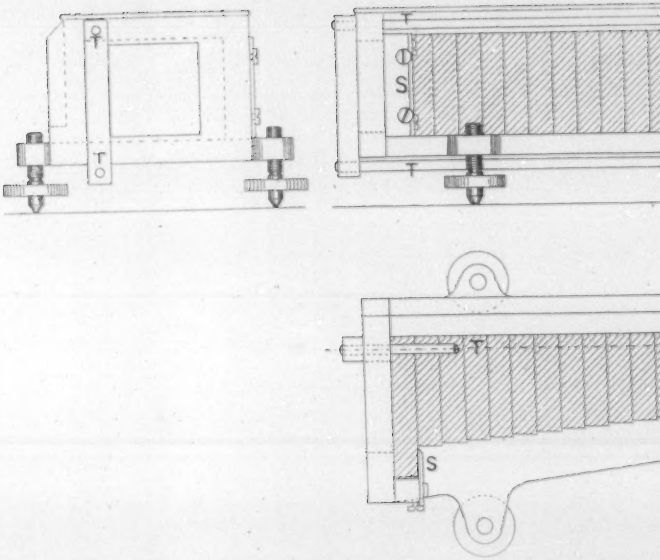


FIG. 9.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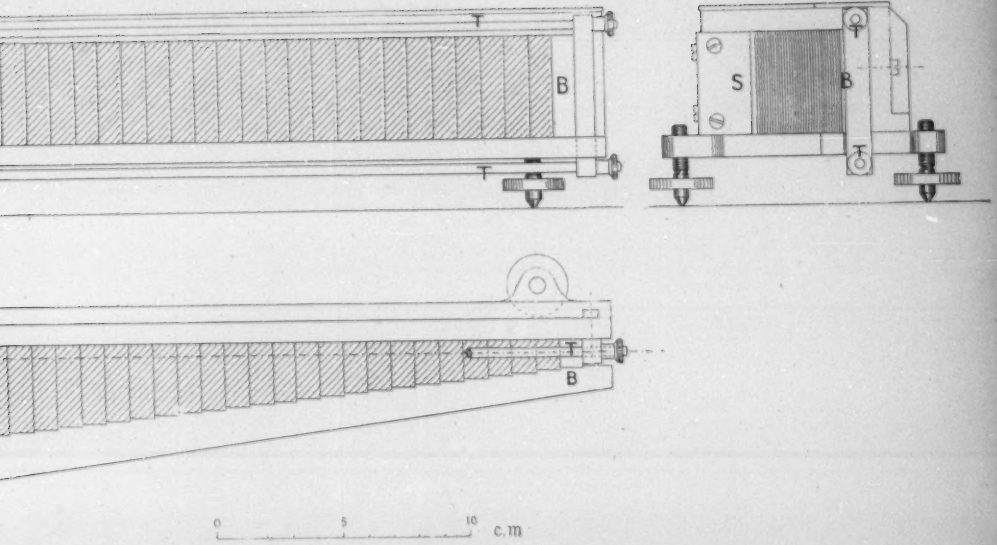
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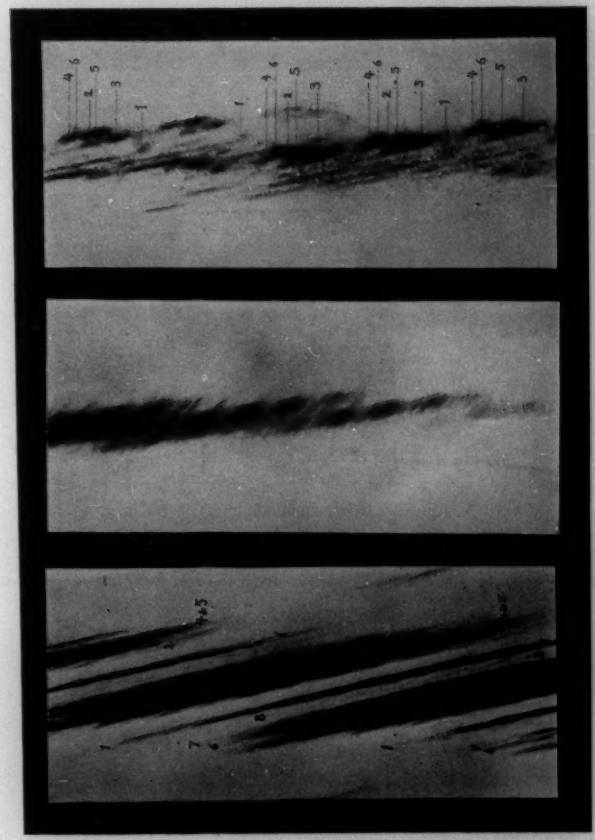
c.m


3.
2.
1.

4 5 6 6α 7α 8 8α 5α b₂ b₁ 4 5 f 2 3 3α 6 6α 7α 1
4 5 6 7 1 2 3 3α 4 5 5α 6 7
Diagram of Spectrum
1 2 3 4 5 6 7 8
Wave-length scale 0 50 100 150 200 250 300 350 400 450 500 Millimetre-Ångstrom units

FIG. 1.

FIG. 2.

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PROCEEDINGS

AT THE

MEETINGS OF THE PHYSICAL SOCIETY

OF LONDON.

SESSION 1907–1908.

February 8th, 1907.

Meeting held at the Royal College of Science.

Prof. J. Perry, F.R.S., President, in the Chair.

The following Candidates were elected Fellows of the Society:—
L. Bairstow, J. de G. Hunter, C. Jakeman, C. H. Lees, C. C. Paterson, E. H. Rayner, W. Rosenhain, and F. E. Smith.

The following Paper was read:—
The Magnetic Fields and Inductive Coefficients of Circular, Cylindrical, and Helical Currents. By Mr A. Russell.

February 22nd, 1907.

Meeting held at the Royal College of Science.

Prof. J. Perry, F.R.S., President, in the Chair.

The following Candidates were elected Fellows of the Society:—
W. H. Bragg and C. A. B. Garrett.

Prof. Perry announced that the Council had elected Dr W. H.

Eccles as a Member of Council to fill the vacancy caused by the resignation of Prof. H. A. Wilson, F.R.S.

The following Papers were read:—

1. Transformer Indicator Diagrams. By Prof. T. R. Lyle.

2. The Ionization of Gases by α Particles of Radium. By Prof. W. H. Bragg.

March 8th, 1907.

Meeting held at the Royal College of Science.

Prof. J. Perry, F.R.S., President, in the Chair.

The following Candidates were elected Fellows of the Society:—
E. Boardman, G. H. Martyn, and J. S. G. Thomas.

The following Paper was read:—
The Rate of Recovery of Residual Charge in Electric Condensers. By Prof. F. T. Trouton, F.R.S., and Mr S. Russ.

March 22nd, 1907.

Meeting held at the Royal College of Science.

Prof. J. Perry, F.R.S., President, in the Chair.

The following Candidates were elected Fellows of the Society:—
W. D. Butcher, E. Rutherford, and W. Thomson.

The following Papers were read:—

1. Experimental Mathematics. By Mr Pochin.

2. Logarithmic Lazytongs and Lattice-work. By Mr Blakesley.

Mr R. J. B. Roberts exhibited a Compensating Micro-manometer.

April 26th, 1907.

Meeting held at the Royal College of Science.

Prof. J. PERRY, F.R.S., President, in the Chair.

The following Candidates were elected Fellows of the Society:—
T. AYRES, A. BLACKIE, Jun., F. L. HOPWOOD, S. RUSS,
and W. A. SCOBLE.

The following Papers were read:—

1. Electrical Conduction produced by Heating Salts. By Mr A. E. GARRETT.

2. The Influence of Pressure upon Convection Currents. By Mr W. S. TUCKER.

Mr W. B. CROFT exhibited some Solenoids which are turned by the Earth's Magnetic Field, and some other Electrical Apparatus.

Mr J. A. TOMKINS exhibited a Simple Apparatus for mechanically illustrating the tangent and the sine laws.

May 10th, 1907.

Meeting held at the Royal College of Science.

Prof. J. PERRY, F.R.S., President, in the Chair.

The following Paper was read:—
Stereoscopy with long Base-line. By Dr T. C. PORTER.
The effects were illustrated by examples on the screen.

May 24th, 1907.

Meeting held at the Royal College of Science.

Prof. J. PERRY, F.R.S., President, in the Chair.

The following Candidates were elected Fellows of the Society:—
Mrs HERTHA AYRTON and A. E. HODGSON.

The following Papers were read:—

1. The Measurement of Mutual Inductance by the aid of a Vibration Galvanometer. By Mr A. CAMPBELL.

2. The Rate of Decay of the Active Deposit from Radium. By Messrs. W. Wilson and W. Makower.

Mr S. G. Brown exhibited an apparatus for Relay Working of long Submarine Telegraph Cables.

June 14th, 1907.

Meeting held at University College.

Mr H. M. Elder, Vice-President, in the Chair.

The following Paper was read:—

Observations on the Electric Arc. By Mr W. L. Upson.

Dr J. A. Fleming showed experiments upon the Poulsen Arc as a means of producing electrical oscillations.

Mr R. Appleyard exhibited a Direct-Reading Conductivity-Bridge for rods.

June 28th, 1907.

Meeting held at the Central Technical College.

Prof. W. E. Ayrton, F.R.S., Past-President, in the Chair.

The following Candidate was elected a Fellow of the Society:—

O. V. Thomas.

The following Papers were read:—

1. A Cosine Flicker Photometer. By Mr J. S. Dow.

2. Some Phenomena in Colour Vision. By Mr J. S. Dow.

3. The Production of Sand-Ripples on the Sea-shore. By Mrs Ayrton.

Mr J. T. Irwin gave a demonstration of his Hot Wire Oscillographs and Wattmeters.

October 25th, 1907.

Meeting held at the Royal College of Science.

Prof. J. Perry, F.R.S., President, in the Chair.

The following Papers were read:—

1. The Use of Variable Mutual Inductances. By Mr A. Campbell.

2. Magnetic Oscillators as Radiators in Wireless Telegraphy. By Dr J. A. Fleming, F.R.S.

November 8th, 1907.

Meeting held at the Royal College of Science.

Dr C. Chree, F.R.S., Vice-President, in the Chair.

The following Candidate was elected a Fellow of the Society:—
J. M. Gordon.

The following Papers were read:—

1. A Graphic Method for Stream Lines and Equipotential Surfaces. By Mr L. F. Richardson.

2. The Lateral Vibration of Bars supported at two points with one end overhanging. By Dr J. Morrow.

November 22nd, 1907.

Meeting held at the Royal College of Science.

Prof. J. Perry, F.R.S., President, in the Chair.

Mr Skinner exhibited specimens of Singing Sand.

Mr L. Bairstow exhibited a Micromanometer.

Prof. C. V. Boys, F.R.S., showed a Diabolo Experiment.

Prof. H. A. Wilson, F.R.S., exhibited a Gyroscope illustrating Brennan's Mono-railway.

December 13th, 1907.

Meeting (informal) held at the Royal College of Science.

An Exhibition of Apparatus was given by the following firms:—Messrs. R. & J. Beck, Ltd.; The Cambridge Scientific Instrument Co., Ltd.; Casella & Co.; Crompton & Co.; Elliott Bros.; Everett, Edgcumbe & Co.; Evershed & Vignoles; Clark Fisher & Wadsworth; Gambrell Bros.; J. J. Griffin & Sons; A. Gallenkamp & Co.; A. Hilger Ltd.; Leslie Miller; Nalder Bros. & Co.; Nalder Bros. & Thompson; Newton & Co.; R. W. Paul; James Pitkin & Co.; Ross Ltd. There was also an Exhibit by the Meteorological Office.

January 24th, 1908.

Meeting held at the Royal College of Science.

Prof. J. Perry, F.R.S., President, in the Chair.

The following Candidates were elected Fellows of the Society:—
J. G. Howarth and R. S. Smith.

The following Paper was read:—
Recalescence Curves. By Mr W. Rosenhain.

Annual General Meeting.

February 14th, 1908.

Meeting held at the Royal College of Science.

Prof. J. Perry, F.R.S., President, in the Chair.

The following Report of the Council was read by the Secretary:—

Since the last Annual General Meeting, thirteen ordinary Science Meetings and one informal Meeting of the Society have been held. Of these, twelve were held at the Royal College of Science; one, that on June 14th, was held at University College, by invitation of Prof. J. A. Fleming, F.R.S., and one, that on June 28th, at the Central Technical College, by invitation of Prof. W. E. Ayrton, F.R.S. The average attendance at the Meetings has been 58 as compared with 50 during the preceding year.

The third Annual Exhibition of Apparatus by Manufacturers was held on December 13th, when the attendance of Fellows and Visitors amounted to about 200.

Light refreshments are now provided before the Afternoon Meetings and after the Evening Meetings. This innovation appears to be appreciated, and it is hoped that the opportunity thus afforded will enable Fellows to become personally acquainted with each other, and to interchange opinions more readily than has been possible hitherto.

The number of ordinary Fellows now on the roll, as distinct from Honorary Fellows, is 437, an increase of 11 on the number last year; 27 new Fellows have been elected, two Fellows have been reinstated, and two Honorary Fellows, namely, Prof. Gabriel Lippmann and Prof. Simon Newcomb, have been elected. There have been 11 resignations, the names of two Fellows have been

struck off the List for non-payment of subscriptions; and the Society has to mourn the loss by death of 6 Fellows, namely, one Honorary Fellow, P. J. C. Jules Janssen, and 5 ordinary Fellows, namely, Lord Kelvin, Sir W. Perkin, J. R. Brittle, J. MacFarlane Gray, and J. W. Kearton.

The Report of the Council was adopted.

The Report of the Treasurer and the Balance Sheet were presented and adopted.

The election of Officers and other Members of Council then took place, the new Council being constituted as follows :—

President.—C. Chree, Sc.D., LL.D., F.R.S.

Vice-Presidents who have filled the Office of President.—Prof. G. C. Foster, F.R.S.; Prof. W. G. Adams, M.A., F.R.S.; Prof. R. B. Clifton, M.A., F.R.S.; Prof. A. W. Reinold, M.A., F.R.S.; Prof. W. E. Ayrton, F.R.S.; Prin. Sir Arthur W. Rücker, M.A., D.Sc., F.R.S.; Sir W. de W. Abney, R.E., K.C.B., D.C.L., F.R.S.; Shelford Bidwell, M.A., LL.B., F.R.S.; Prin. Sir Oliver J. Lodge, D.Sc., F.R.S.; Prof. S. P. Thompson, D.Sc., F.R.S.; R. T. Glazebrook, D.Sc., F.R.S.; Prof. J. H. Poynting, M.A., D.Sc., F.R.S.; Prof. J. Perry, F.R.S.

Vice-Presidents.—W. Duddell, F.R.S.; H. M. Elder, M.A.; Prof. J. A. Ewing, F.R.S.; W. Watson, D.Sc., F.R.S.

Secretaries.—W. R. Cooper, M.A.; Prof. W. Cassie, M.A.

Foreign Secretary.—Prof. S. P. Thompson, D.Sc., F.R.S.

Treasurer.—Prof. H. L. Callendar, M.A., LL.D., F.R.S.

Librarian.—W. Watson, D.Sc., F.R.S.

Other Members of Council.—A. Campbell, B.A.; W. H. Eccles, D.Sc.; A. Griffiths, D.Sc.; J. A. Harker, D.Sc.; Prof. C. H. Lees, D.Sc., F.R.S.; T. Mather, F.R.S.; A. Russell, M.A.; S. Skinner, M.A.; S. W. J. Smith, M.A.; Prof. L. R. Wilberforce, M.A.

Votes of thanks were passed to the Auditors, to the Officers and Council, and to the Board of Education.

Prof. J. Perry, F.R.S., the retiring President, then vacated the Chair in favour of Dr C. Chree, F.R.S., who delivered his Presidential Address.

Treasurer's Report.

The position of the Society this year shows a decided improvement. There are more new members than usual, and subscriptions have been paid more punctually. The balance in bank has increased by nearly £240, more than £100 of which, however, is due to composition-fees and recovered income-tax. The value of the securities has declined somewhat owing to the general depression, but this is probably of a temporary nature. There are no outstanding liabilities, and the income shows a healthy tendency to exceed the expenditure.

HUGH L. CALLENDAR,
Hon. Treasurer.

The Treasurer in Account with the Physical Society, from January 1st, 1907, to December 31st, 1907.

Dr.	£	s.	d.	£	s.	d.
Jan. 1, 1907. Balance in Bank				176	7	7
Entrance-Fees	28	7	0			
Subscriptions	443	3	5			
Extra Subscriptions	29	6	0			
Arrears	54	12	0			
Abstracts etc.	8	7	6			
				563	15	11
Composition-Fees				84	0	9
Dividends:—						
Furness Debenture Stock	15	3	10			
Metropolitan Board of Works	6	13	0			
Lancaster Corporation Stock	11	8	0			
New South Wales	8	9	0			
Midland Railway	38	0	0			
Great Eastern Railway	19	0	0			
London, Brighton, & South Coast Railway	24	18	10			
				123	12	8
Sales of Publications				37	15	8
Income Tax recovered (3 years)				19	8	6
				£1005	0	4

Cr.	£	s.	d.	£	s.	d.
Science Abstracts				279	6	0
Taylor and Francis:—						
Proceedings	113	12	9			
Abstracts, etc.	83	19	0			
Distribution	14	14	6			
Separate Copies	16	13	0			
				228	19	3
Periodicals				3	9	0
Binding				7	9	8
Reporting				35	0	0
Royal Asiatic Society				2	2	0
Attendance				7	10	0
Petty Cash (Treasurer and Secretaries)				11	13	6
Refreshments				8	14	6
Commissions and Postages					9	0
Apparatus				2	12	0
R. S. Subscription and Cheque unpaid				2	2	0
Balance in Bank				415	13	5
				£1005	0	4

Audited and found correct,

HUGH L. CALLENDAR, *Honorary Treasurer.*

20th January, 1908.

CHARLES SELBY WHITEHEAD,
HENRY R. J. BURSTALL, } *Auditors.*

PROPERTY ACCOUNT OF THE PHYSICAL SOCIETY, DECEMBER 31, 1907.

ASSETS.	£	s.	d.
Subscriptions due, Treasurer's estimate	50	0	0
£533 Furness Ry. Co. 3 per cent. Debenture Stock	421	0	0
£1600 Midland Railway 2½ per cent. Preference Stock	1088	0	0
£200 Metropolitan Board of Works 3½ per cent. Consolidated Stock	202	0	0
£400 Lancaster Corporation 3 per cent. Redeemable Stock	332	0	0
£284 2*s.* 9*d.* New South Wales 3½ per cent. Inscribed Stock	249	0	0
£500 London, Brighton, and South Coast Railway Ordinary Stock	510	0	0
£500 Great Eastern Railway 4 per cent. Debenture Stock	550	0	0
Balance in Bank	415	13	5
Publications in Stock, estimated	200	0	0
	£4017	13	5

LIABILITIES.	£	s.	d.
Balance	4017	13	5
	£4017	13	5

Audited and found correct,

HUGH L. CALLENDAR, *Honorary Treasurer.*

20*th January*, 1908.

CHARLES SELBY WHITEHEAD,
HENRY R. J. BURSTALL, } *Auditors.*

Presidential Address to THE PHYSICAL SOCIETY OF LONDON,

14th February, 1908.

C. CHREE, Sc.D., LL.D., F.R.S.

THE losses of the Society by death since the last general meeting have been few, but they include two most distinguished Fellows, Lord Kelvin and Sir W. H. Perkin. Both joined the Society in 1875, Perkin in January, Kelvin in June. The work with which Perkin's name is generally associated was chemical, and does not call for prolonged comment here. It must have been a great satisfaction to all who knew him, that he lived long enough to receive the remarkable international tribute which but shortly preceded his death. Lord Kelvin was one of our earliest presidents. During his lifetime British Physicists had the satisfaction of feeling assured that amongst their number was one who was universally recognised as the foremost man in his department of knowledge. There have already appeared several obituary notices of Lord Kelvin by old pupils and others having much more intimate knowledge of him personally than I possess. Especially would I refer to an article by Prof. Ayrton in the 'Times,' possessing that touch of the imagination which is essential to a life-like portrait. Lord Kelvin was active in a great variety of fields, appealing to very varied types of mind, and to different classes of the community. The fact that he, an eminent pure scientist, was the inventor of electrical and nautical instruments of the highest practical utility, has been, I think, a most potent cause in impressing the value of science on the average English mind—I say English advisedly. It is one of the chief reasons why English physicists are recognised by their countrymen and countrywomen as having some claim to be regarded as useful units in the population. Another side of Lord Kelvin's activity also appealed strongly to the public imagination. I refer to his fondness for formulating theories as to matters of general interest, such as the internal state or age of the earth, or the supply of solar heat. These I regard rather in the light of scientific poetry. It is considered the privilege of the young to dream dreams, and Lord Kelvin's mind was always young. Without imagination we

should lose most of the pleasures of life, and the imaginings of a mind like Lord Kelvin's, even in subjects where facts are still sadly deficient, are likely to possess interest for all future workers in the field. But apart from what appeals to the popular mind, Lord Kelvin's writings contain a vast amount of solid and most original work. He made substantial contributions to Hydrodynamics and Thermodynamics, to Heat, Electricity, Magnetism, Elasticity, and Optical Theory. Of the subjects he has dealt with, Elasticity is the one on which I am most qualified to speak. The prominent feature in his work in that department is the tendency to give theory a form leading to concrete practical applications. When it was absolutely necessary, Lord Kelvin carried out intricate mathematical calculations of considerable length; but whilst he was undoubtedly an able analyst, analysis seemed with him always a means to an end. Once the end was reached, the perfection of the analysis, or its extension so as to reach results of greater symmetry or more generality, did not seem to appeal to him. Not infrequently, indeed, he did not take the trouble to publish the analysis at the time. The reason was certainly not mental inertia, quite the contrary. It was due, I think, to his intense mental activity. He had a constant flow of new ideas, one treading on the heels of another; and before any one idea had received a long exposure on the stage of his mind, another was pressing to take its place. Lord Kelvin almost at once obtained the recognition which his powers merited. He was not exposed when young to a lack of appreciation such as sometimes serves to sour those to whom distinction comes late in life. But allowing for all this, the generosity with which he recognised the value of the work of younger and unknown men was a very pleasing feature of his character.

Turning to the question of the progress of the Society, I learn from Mr. Cooper that since 1903 our membership has increased by 30, or fully 7 per cent. With the increased number of persons concerned directly or indirectly with Science, one would naturally expect an increase in the membership of Societies such as ours; but this tendency is opposed by increased specialisation and the consequent rise of new Societies. Considering modern tendencies, one class of the community from which I think we might reasonably have expected substantial support has accorded it but slightly. Ladies are freely admitted to our membership, but for some reason they have not hitherto shown much eagerness to

avail themselves of this opportunity. As an additional encouragement to them, I might refer to a recent departure, viz., the supply of tea and other light refreshments at the meetings. Tea at the Royal Society affords an opportunity for many valuable conversations, and I see no reason why tea at the Physical Society should not prove similarly useful. This departure is due, I think, in considerable measure to the more optimistic attitude which our Treasurer finds himself able to take with regard to our financial position.

Another domestic occurrence I wish to refer to is a resolution of the Council intended to encourage speakers, especially the authors of papers, to be reasonable in the demands which they make upon the bank of time. The resolution runs: "That with the object of facilitating discussion, the reader of a paper already printed and distributed should be invited to limit himself to 20 minutes or to such time as the Chairman may think desirable, and speakers on the discussion to 10 minutes."

When there are several papers of some length down for reading, it is obviously impossible for all the authors to go into minute details unless the duration of the meeting is unduly prolonged. Papers of importance are usually in type and proofs are obtainable before the meeting. Those who are specially interested in the subject would most naturally derive *details* from the proof. The course most likely to be generally useful is for the author, after a brief *resumé* of previous knowledge, to give an outline of the leading new principles or results which he has reached, in a form intelligible to those who are not specialists. Such a statement, especially when accompanied by a few experiments or lantern-slides, will supply the average hearer with all he is likely to assimilate. The discussion after the paper is read should afford specialists the opportunity of criticising or of seeking explanations on specific points. It is obvious that if many Fellows wished to speak for 10 minutes, the time available for discussion would soon be exhausted. But all that most speakers have to say on a paper can usually be said in 5 minutes or less. It will be my object when in the chair to give effect to the spirit of the resolution; but its success must depend largely on the reasonableness of individuals, and I trust that Fellows will not be found lacking in this possession. Before leaving the subject, I would express a hope that more of our ex-presidents may find it possible to put in an occasional appearance. Their intervention in discussions would,

I am sure, be generally appreciated, and prove an encouragement to those contributing papers.

I propose now to give a brief *resumé* of the results of some work which has occupied a good deal of my attention during the last two or three years. The National Antarctic Expedition of 1901–4 was supplied with magnetographs, and the reduction and discussion of the curves was entrusted to the National Physical Laboratory. The work is still in progress, and when finally completed the results will appear in the official volumes of the Expedition. It occurred to me, however, that a preliminary account of some of the outstanding phenomena, on an occasion such as this, might serve a useful purpose; and this view, I am glad to say, recommended itself to the authorities of the Royal Society and the Director of the National Physical Laboratory.

When referring to any British national undertaking, such as a war or a scientific expedition, one is expected to apologise for a greater or less amount of preliminary muddle. Perhaps as a variant I may be allowed to say a few words as to what should in my opinion be done when the next National Scientific Expedition is being prepared. Supposing the physical programme as extensive as in the late expedition, comprising subjects so large and diverse as Terrestrial Magnetism, Atmospheric Electricity, Auroral observations and determinations of Gravity, there ought to be at least two physical observers. These ought to be men of first-class University attainments, having a thorough knowledge of general physics, assuming that such men can be got who are at the same time strong, hardy, and persevering. Before the expedition sets out they should devote at least 6 months—better 12—to a special study of the subjects they will have to deal with, so as to obtain at least a general knowledge of the leading phenomena on which it is hoped that the expedition will throw further light. They should also make themselves thoroughly competent as observers. All the apparatus for the expedition should be ready and thoroughly tested at least three months before the expedition sets out, and the observers should use this apparatus sufficiently to become entirely at home with it. A programme should be drawn up, and the observers practised in its execution. To give confidence to the public, and represent different interests, it is probably necessary on such an occasion that committees of a somewhat unwieldy size should be nominated; but these committees should delegate the construction of the programme for each

subject to one or two men who have an intimate knowledge of its recent developments, the general body contenting itself with seeing that the programme is based on common-sense as well as on special technical knowledge. The programme should be drawn up in ample time to admit of the observers being practised in its execution. Such a programme is not to be regarded as a cast-iron statute, because no one can foresee exactly what is going to happen; but it ought to make absolutely clear what are the physical aims of the expedition, and what are the most likely means of attaining them. The necessity for a programme would no doubt be lessened if the expedition were under the command of a Physicist of resource and ripe experience. But if—as is most likely—it is under the command of some one whose knowledge of physical science is very limited, it is eminently desirable that some authoritative instructions should exist which will ensure that opportunities for promoting the scientific objects of the expedition shall be utilised to the utmost. It might also serve a useful purpose—though in so hurried an era as ours the idea may appear quixotic—if, on the return of the expedition and after a general examination of its results, something equivalent to a scientific court-martial were held by a competent judicial body whose expressions of approval or blame would carry weight. As matters stand, the observer who confines himself to purely scientific work—however efficient—can have no assurance that he will not be entirely overshadowed by the doers of exploits which appeal to the popular imagination.

Returning now to the 'Discovery' 1901–4 Expedition: at the start the fates were most unpropitious. First one and then a second physicist was laid aside, and within a very short time of the date when the Expedition was due to sail there was no physical observer; whilst the magnetographs, which had been ordered in Germany, had not yet reached this country. Fortune then began to change. The Expedition secured the services of Mr. Bernacchi, who had but recently returned from the 'Southern Cross' Expedition, which was the first to spend a winter in the Antarctic. The 'Discovery' sailed, leaving Mr. Bernacchi to follow with the pendulum apparatus and the magnetographs, after obtaining such practice as time might allow. The magnetographs then turned up at Kew, though not until I had left on my annual holiday. They proved to be of a type absolutely strange to all at Kew, and requiring more delicate

handling than the ordinary Kew pattern. Fortunately the services of Dr. Harker and Mr. F. E. Smith were available, and they succeeded in erecting the instruments and getting them to work. Mr. Bernacchi, however, had only a few days' experience of them before he had to set sail. On returning from my holiday, I was somewhat perturbed to be met with high encomiums of the magnetographs on the ground of their sensitiveness. They had shown the wickedness of the electric trams, then a burning topic, in a particularly effective manner. I should have been still more perturbed if I had known that the quartz-fibre which carried the horizontal-force magnet when at Kew was the stoutest of the large number supplied. Such, however, proved to be the case, and as that fibre was broken ere the Antarctic was reached, it fell to Mr. Bernacchi to replace it by a succession of fibres giving the instrument a sensitiveness four or five times that customary in Europe. The natural result was a not infrequent loss of trace, and a lot of extra trouble to the observer and to those who have had to reduce the curves. The sensitiveness of the declination magnetograph—regarded as a measurer of angles—was little over half that of the Kew magnetograph; but in the Antarctic, where the horizontal force was only about a third of that in England, it had as a measurer of force a sensitiveness some 50 per cent. greater than that of the Kew magnetograph at Kew. If the sensitiveness of these two instruments had been a fifth of what it actually was, it would have been a great improvement. The third instrument—recording vertical force—as used in the Antarctic did not err in the direction of over-sensitiveness—rather the opposite, but it had unfortunately a very large temperature coefficient. This coefficient can be determined satisfactorily only under the conditions of actual use. To have determined the coefficient at Kew, where the vertical force is opposite in sign to that in the Antarctic, and only three-fifths as large, would have been futile. In the Antarctic, unfortunately, a direct determination was not found feasible, and not the easiest of my labours during the past year has been to extract the necessary information from the Antarctic curves themselves. Thanks to the excellent behaviour of a sensitive self-recording metallic thermometer inside the vertical-force magnet box, and to the nature of the temperature changes experienced, the problem has I think been successfully dealt with; but the labour of reducing the vertical-force records has been at least trebled. It will, I hope, be clearly

understood that the previous remarks have no bearing on the question of the suitability of the type of magnetograph for use under normal conditions. They are directed solely to explaining wherein the exceptional conditions of the Antarctic had not been provided for. Without some explanation of this kind, it would have been impossible for you adequately to realise the difficulties with which Mr. Bernacchi had to contend, difficulties for which he was in no way responsible, and which no one I think could have encountered with unvarying success. I would take this opportunity of acknowledging the valuable assistance given by Mr. Bernacchi throughout the whole reduction of the magnetic curves. At the start he spent some months at Kew arranging the material, and he tabulated a large proportion of the declination curves. Subsequently the tabulations were carried on by two junior assistants, Mr. Maudling and Mr. Gendle, who both proved most expert and discriminating tabulators, and did excellent work.

I shall now show some lantern-slides, for which I am indebted to Mr. Hugo, of the National Physical Laboratory, which will I hope give a general idea of the principal results derived from the magnetic curves. As a preliminary, I shall show a slide of Kew declination curves (fig. 1), as being of a type more suited to explain the general character of magnetic records. To economise photographic paper, two successive days' records are taken on the same sheet. There are thus two curved lines, each with its corresponding straight base line. The upper curved and the upper base line refer to the first day. The paper is wound round a drum turned by clockwork. The base line is produced by light reflected from a fixed mirror, the pencil of light being interrupted for a few minutes at two-hour intervals, so as to mark the time. The curved line arises from light reflected from a mirror carried by a freely suspended horizontal magnet. The magnet alters its direction as the declination alters, the mirror turning with it, and the reflected light moves across the paper, thus altering the ordinate of the curved line, *i. e.*, its distance from the base line.

In this case, the positive direction answers to increasing westerly declination, a change of 1 cm. in ordinate meaning a change of 8'·6 in declination*. The equivalent of 1 cm. depends on the distance between the photographic paper and the magnet. If the

* The curve is reproduced here on a reduced scale, 1 cm. representing approximately 18' of declination.

Fig. 1.—KEW DECLINATION, 1895 { September 6 10 13 to 7 10 13 A.M. G.M.T.
„ 7 10 15 „ 8 10 8 „ „

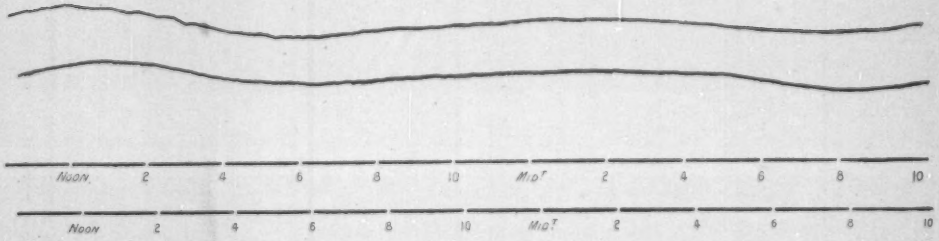
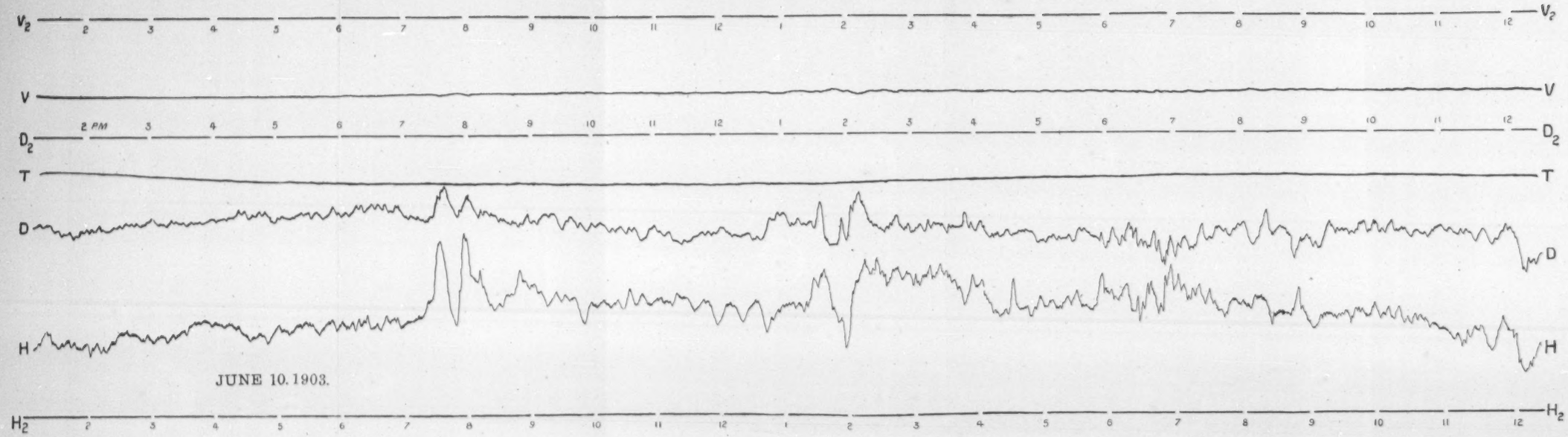


Fig. 2.—ANTARCTIC CURVES.



instrument remains untouched, the base line should answer to a certain constant declination, whose value is determined by reference to the absolute observations made with a unifilar magnetometer. Suppose, for instance, that the absolute declination observed at a certain time is 16° 30′ W., and that the curve ordinate at this hour is + 5 cm. Then the base-line value is less than the observed value by the equivalent of 5 cm., or 43′, and so is 15° 47′ W. From this known value of the base line, and the known equivalent of 1 cm. of ordinate, we can now derive the declination at any specified instant. Fig. 1 represents very quiet magnetic conditions. The large regular increase of ordinate at the start answers to the swing of the magnet to the west in the forenoon and early afternoon, which is the most conspicuous feature of the diurnal variation in England at all seasons of the year.

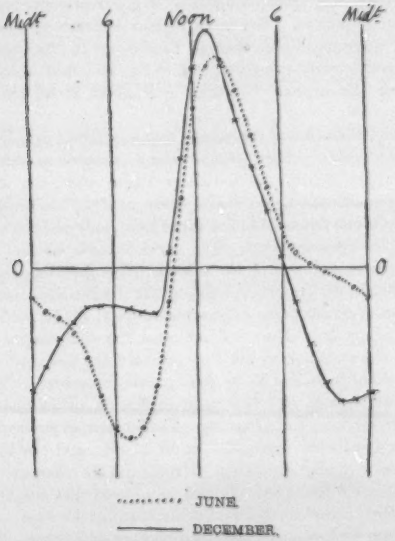
The next slide (fig. 2) represents magnetic curves actually taken in the Antarctic. They present many conspicuous differences from the Kew curves. In the Eschenhagen instrument all three elements—declination, horizontal force, and vertical force—are recorded on one drum. On the other hand, only one day's curves are recorded on each sheet. The curve for each element has its own base line, and in each the trace is interrupted once an hour. In the figure, D, H, and V indicate the Declination, Horizontal Force, and Vertical Force curves respectively, D_2, H_2, and V_2 the corresponding base lines. T indicates the temperature of the metallic thermometer inside the vertical-force magnet-box. It has the same base line V_2 as the vertical-force curve. The day selected, strange as it may appear, represents an unusually quiet state of matters for the Antarctic. The declination and horizontal-force magnets were practically never at rest, and the incessant disturbances render it difficult to recognise the character of the regular diurnal variation. It will be noticed that the D and H curves nearly interfere, and at first one may feel disposed to reflect on the observer for having the traces so near each other. Keeping the traces separate was, however, only one of the objects he had to aim at; a still more important thing was to keep all on the sheet. It was thus prudent to have the traces near the centre of the sheet during quiet times.

The next slide (not reproduced) shows a day of by no means exceptional disturbance. The full width of the sheet answered to nearly 5° of declination, but the declination trace on this occasion

actually crossed from side to side in about 20 minutes. A curious interrelation is visible between the vertical-force and temperature traces, the one coming down the sheet when the other went up. This really arose from the large temperature coefficient of the vertical-force magnet.

When driven at the normal speed, the drum carrying the photographic paper rotated completely in 25 hours. If the observer

Fig. 3.



allowed a longer interval than this to elapse between starting and stopping, a second day's trace began to superpose itself on the first. A margin of one hour in the day proved somewhat narrow for the exigencies of the Antarctic. In particular there were occasional wind storms of such violence that it was too dangerous to venture as far as the magnetic hut, and the observer had to wait until the worst of the storm was over. Exactly the same

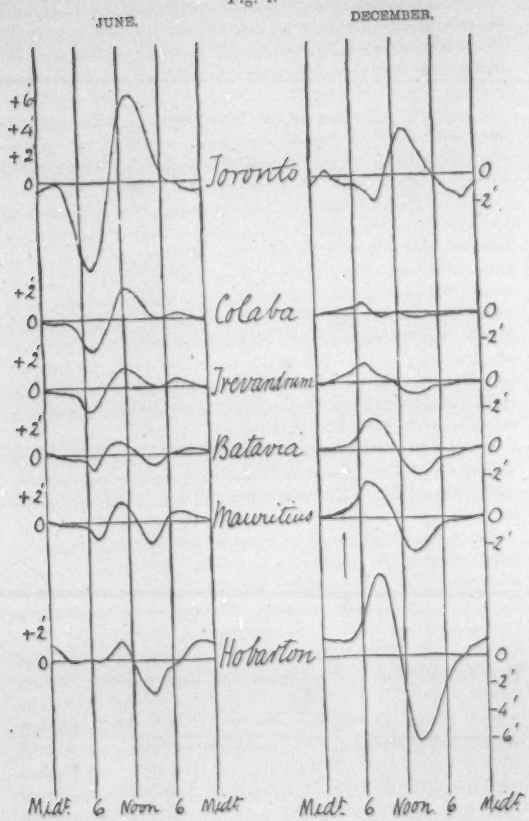
trouble, it may be mentioned, was experienced by the recent American (Ziegler) Polar Expedition in Franz Josef Land. On the rare occasions when a blizzard delayed Mr. Bernacchi's visit to the magnetograph, there was considerable superposition of the traces, and their disentanglement has not always been found possible.

I pass now to the data for the diurnal variation of the magnetic elements, as derived from the curves. To deduce the regular diurnal inequality for a given month or season, one measures the curve ordinate at every hour, finds the mean value for each hour of the 24, and finds the difference between this and the corresponding mean for the entire day. It will again be convenient to illustrate this first by reference to Kew results. Fig. 3 shows the nature of the diurnal inequality of declination at Kew, at midsummer (June) and midwinter (December). The horizontal line OO answers to the mean value for the day. The positive direction in the ordinate signifies that the north end of the magnet is pointing to the west of its mean position. The curves are really drawn to different scales, so chosen as to give the same mean length of ordinate for each. If drawn to the same scale, the June curve would have a range nearly $2\frac{1}{2}$ times that of the December curve.

It will be seen that the curves, though differing slightly in the time of the westerly maximum, still closely resemble one another during those hours of the day when the changes of declination are most rapid, but they differ notably in the early morning.

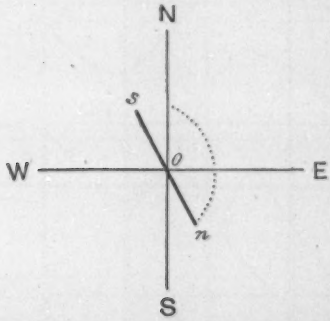
The next slide (fig. 4) is intended to assist in trasferring the imagination to the Southern hemisphere. The curves all show declination, and in all a positive ordinate represents movement of the north pole towards the west. The stations are selected so that pairs possess nearly equal north and south latitudes. Thus Toronto and Hobarton have latitudes about 43°, Colaba and Mauritius about 20°, Trevandrum and Batavia about 7°. As regards climate, the difference between December and June is more marked in high than in low latitudes, but this is not the case as regards the *type* of the declination curve. Looking at either the June or the December curves, one can see a gradual transition from Toronto to Hobarton. Certain inconspicuous features of the midwinter curve in one hemisphere are cases apparently of arrested development, answering to prominent midsummer features in the other hemisphere. But the principal point calling for attention is the nature of the difference between

Fig. 4.


JUNE.
DECEMBER.
+6
+4
+2
0
Toronto
0
-2'
+2'
0
Colaba
0
-2'
+2'
0
Trevandrum
0
-2'
+2'
0
Batavia
0
-2'
+2'
0
Mauritius
0
-2'
+2'
0
Hobarton
0
-2'
-4'
-6'
Midt. 6 Noon 6 Midt.
Midt. 6 Noon 6 Midt.

the June curve at Toronto and the December curve at Hobarton; *i. e.*, the midsummer curves at the extreme stations. They are almost the exact opposites of one another. When the needle at Toronto moves to the west, that at Hobarton moves to the east. In other words, if we drew these two curves so that up the sheet meant movement to the west at Toronto, but movement to the east at Hobarton, they would closely resemble one another. It might thus be expected that in dealing with declination in the Antarctic, the positive direction of the ordinate would be taken as answering to movement of the north end of the magnet to the east. There is here however a little difficulty, which figure 5 will explain.

Fig. 5.



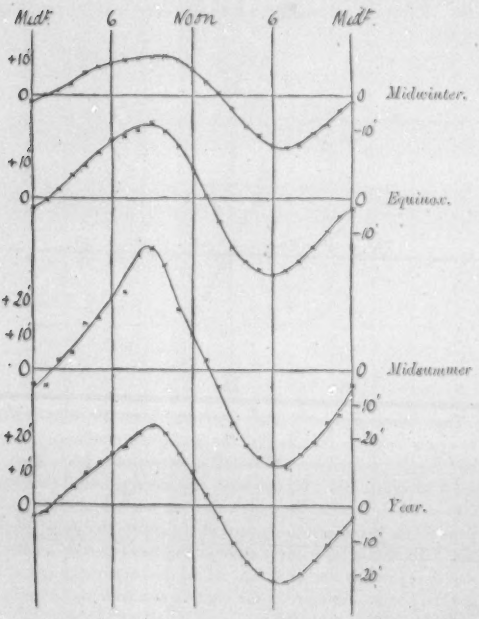
The 'Discovery's' "Winter Quarters" were to south-east of the south magnetic pole, so that the declination needle *ns* (*n* being the "north" pole) oriented itself as shown relative to the geographical meridian NS. If, following the usual practice, we define NO*n* as the declination (easterly), then an increase of easterly declination really means a movement of *n* towards the geographical west. In such a topsy-turvy state of matters one does not know what to expect the magnet to do. It has thus appeared simplest to take the positive direction of the ordinate as answering to increase in the declination angle NO*n*.

The two slides now to be shown (not reproduced) represent the regular diurnal variation of declination at Winter Quarters for each month of the year. To get smooth curves one would of

course require a good many years' data, but in every case the general character is quite clear. The range is 5 or 6 times as large as that at Kew. From May to July the sun was continuously below the horizon, but the diurnal range remained large, at least double that at Kew at midsummer. Another remarkable fact, the small seasonal change in the type of the diurnal variation, will be better seen in the next slide (fig. 6), which gives mean

Fig. 6.

DECLINATION.

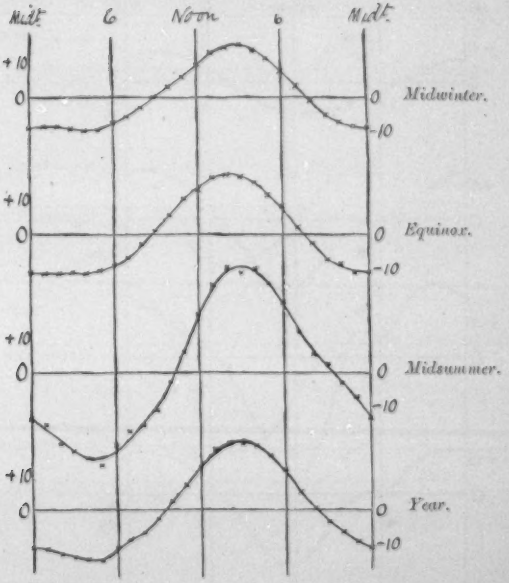


results for 3 seasons :—Midwinter (May to July), Equinox (March, April, September, October), and Midsummer (November to January), as well as for the whole year.

The curves pass through or close to all the observation points, and yet they are remarkably smooth. The way in which the "accidental" irregularities—*i. e.*, the disturbances which either have no period or a very long period—cut out is truly extraordinary. At some Arctic stations, observers have described a fundamental difference of type between the diurnal inequalities derived from

Fig. 7.

HORIZONTAL FORCE.



all and from quieter days. At "Winter Quarters" the regular diurnal inequality shows a decidedly reduced range on the quieter days, but the difference in type appears small. At Kew, and in temperate latitudes generally, as figs. 3 and 4 show, the changes of declination during the day are conspicuously larger

and more rapid than those during the night. At "Winter Quarters" there is no such conspicuous difference. Also the hours at which the extreme positions are reached, about 9 A.M. and 7 P.M., are so totally different from those customary, that it is difficult to say exactly wherein the Antarctic declination curves resemble and wherein they differ from those of temperate latitudes.

Fig. 8.

VERTICAL FORCE.

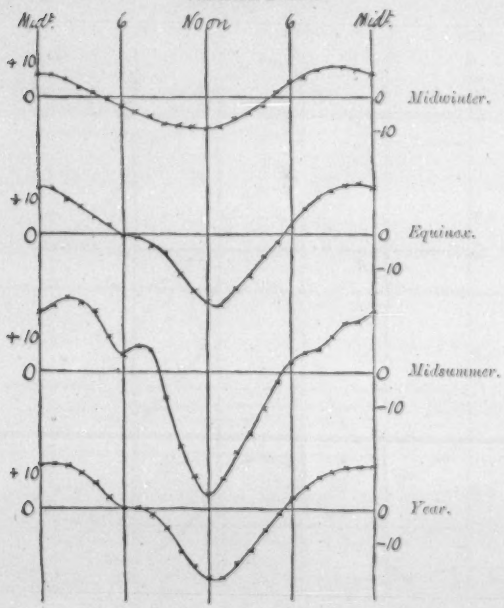


Fig. 7 illustrates the diurnal inequality of the horizontal force, *i. e.*, the variation in the intensity of the force directed along the local magnetic meridian. The positive direction of the ordinate means an increased value of the force; the unit, 1 γ, represents ·00001 C.G.S. The horizontal force at "Winter Quarters" was

little over a third of that at Kew, but the range of the diurnal inequality at Winter Quarters is at least 50 per cent. greater than that at Kew, whether at midwinter or midsummer. The type of the inequality again varies but little with the season, and is quite unlike that at any temperate station.

Fig. 9.

INCLINATION.

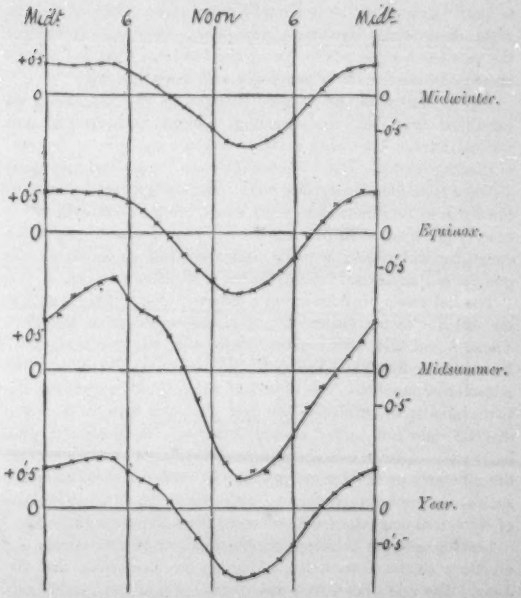


Fig. 8 illustrates the diurnal inequality of the vertical force. The unit again is 1 γ, and the positive direction of the ordinate answers to increasing force. The vertical force of course in the Antarctic pulls the *south* pole of a dipping needle downwards. The range is on the whole slightly less than in the case of the horizontal force, but quite double that at temperate stations in

Europe. It should, however, be mentioned that at "Winter Quarters" the vertical force is fully 10 times the local horizontal force, and more than 60 per cent. larger than that at Kew, so that compared to the size of the element the range of the vertical force at "Winter Quarters" is not so noteworthy.

There is some little apparent difference in type in the vertical force diurnal inequalities at different seasons. The kink, however, in the early morning part of the midsummer curve may be at least in part "accidental," as it occurs during those hours of the day when disturbances were most prominent. For most months of the year one had records for two years 1902 and 1903, but for the three midsummer months practically only from one year.

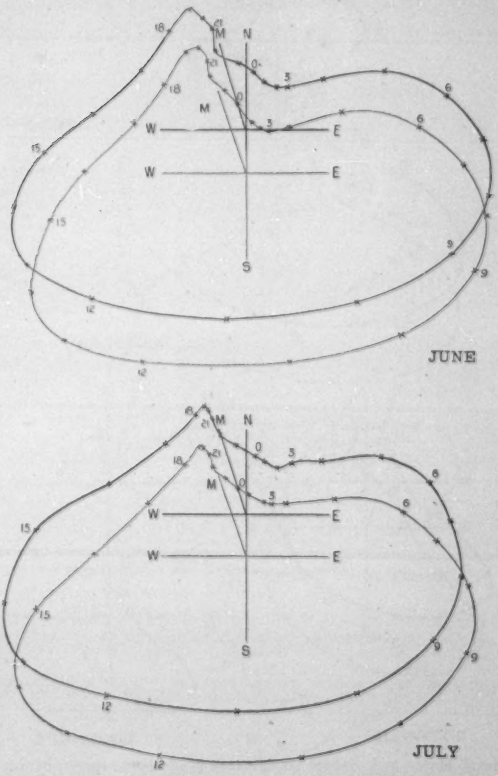
Fig. 9 illustrates the diurnal changes in the inclination as calculated from the corresponding changes in horizontal and vertical force. Increasing positive ordinate answers to increase of (Southerly) dip. The "Winter Quarters" was at no very great distance from the S. magnetic pole. The daily range however in the dip is by no means abnormally small, being in fact over 50 per cent. larger than at Kew. Whether the bend near 6 A.M. in the curve for midsummer is really representative, or is merely the product of "accidental" disturbances, it is difficult to say.

The last group of slides are of a different type. They are what are called "vector diagrams." Fig. 10—copied from the Phil. Trans. A, vol. 204, 1905—gives side by side vector diagrams for Kew (thicker lines) and Falmouth (thinner lines), for each of two midsummer months. The object of showing the curves for the two places is to emphasise the fact that the type of a vector diagram does not under normal conditions vary rapidly with longitude. Kew an inland station and Falmouth on the sea-coast differ notably in the diurnal variation of meteorological elements such as temperature (near the earth's surface), but so far as changes of terrestrial magnetism are concerned the differences are small.

Leaving now the Falmouth curves alone, let us concentrate our attention on the Kew curves, taking say for illustration that for June. The line drawn from the origin—where the north-south and east-west coordinate axes cross—to say the point 12 on the curve represents in magnitude and direction a disturbing force which would account for the observed departures in the horizontal components of the magnetic force at the hour 12 (noon) from their mean values for the day. Supposing, for instance, that the diurnal inequality were entirely due to electric currents in the

atmosphere, this line represents the horizontal component of the magnetic force exerted by these currents at noon at the earth's

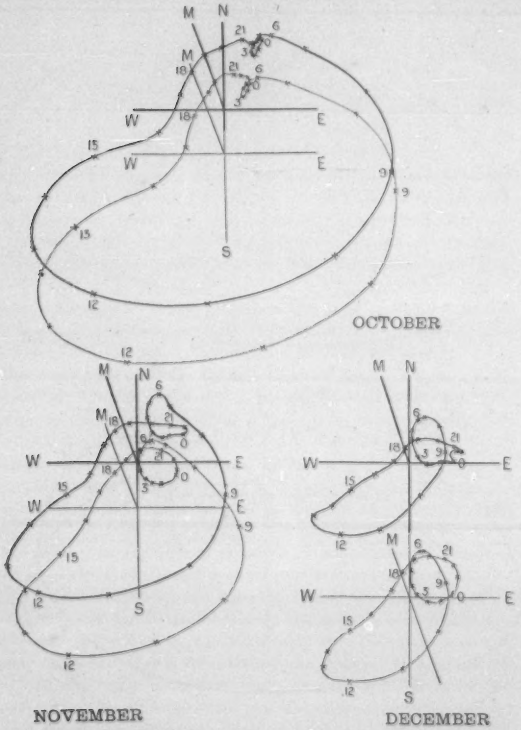
Fig. 10.



surface. N.S. and E.W. represent the geographical north-south and east-west directions, the line to M the local magnetic meridian.

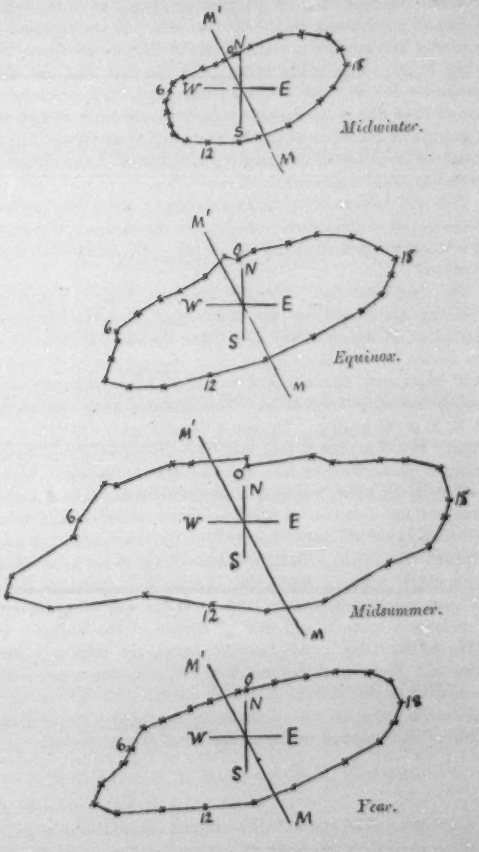
The hours are counted from 0 (midnight) to 23. The disturbing force, it will be seen, goes right round the compass in 24 hours, travelling in the direction of the hands of a watch. The force is

Fig. 11.



much larger, and changes its direction much more rapidly, by day than by night. The season chosen is that when the curve is most symmetrical, but even at midsummer its asymmetry is considerable. The next slide (fig. 11) shows the October, November, and

Fig. 12.


M'
N
18
6
W
E
S
12
M
Midwinter.
M'
N
18
W
E
6
S
12
M
Equinox.
M'
N
W
E
6
18
S
12
M
Midsummer.
M'
N
18
W
E
6
S
12
M
Year.

December vector diagrams, again for Kew and Falmouth. If the Falmouth diagram did not serve to corroborate, one might well hesitate to accept the Kew diagram for December as bearing even a distant resemblance to the actual facts. As we approach midwinter, a loop appears in the part of the diagram answering to the night hours. This really arises from the fact that the diurnal inequality has at most places an important 12-hour component, and at Kew this is able to assert itself prominently at midwinter. The scale of force is exactly the same in figures 10 and 11. The distances of the points marked N, E, S, and W from their corresponding origin represent in all cases 10 γ.

This will help to bring home the fact of the great increase in the intensity of the forces to which the regular diurnal inequality at a temperate station is due as we pass from midwinter to midsummer.

The next and last slide (fig. 12) gives vector diagrams for "Winter Quarters" for the whole year and the same three seasons as in figs. 6 to 9. As in figs. 10 and 11, NS and EW are drawn in and perpendicular to the geographical meridian. MM' represents the magnetic meridian, the north pole of the magnet pointing towards M. The distance from the origin to N, E, S, or W is 10 γ. The hours 0 to 23 are counted from midnight. It will at once be seen that there are remarkable differences from the phenomena at Kew and Falmouth. Seeing that we are dealing in fig. 12 with a station in the Southern hemisphere, the fact that the direction of motion is anti-clockwise is not to be regarded as an anomaly, though it might have been an unsafe *à priori* conclusion. But, for one thing, there is much less a-symmetry than in England. Again, the diagrams for the different seasons differ but little in type; and the differences, as regards amplitude and rate of motion of the radius vector, between the "day" and "night" hours are relatively inconspicuous. From a mathematical standpoint the most striking peculiarity in the diurnal inequality in the Antarctic in all the elements is the large size of the fundamental or 24-hour Fourier "Wave" as compared to the harmonics of shorter period.

PROCEEDINGS

AT THE

MEETINGS OF THE PHYSICAL SOCIETY

OF LONDON.

SESSION 1908–1909.

February 28th, 1908.

Meeting held at the Imperial College of Science and Technology.

Dr. C. Chree, F.R.S., President, in the Chair.

The following Papers were read:—

1. Contact Potential Differences determined by Null Solutions. By Messrs. S. W. J. Smith and H. Moss.

2. An Experimental Examination of Gibbs's Theory of Surface Tension as the Basis of Adsorption, with an Application to the Theory of Dyeing. By Mr. W. C. M. Lewis.

March 13th, 1908.

Meeting held at the Imperial College of Science and Technology.

Dr. C. Chree, F.R.S., President, in the Chair.

The following Candidates were elected Fellows of the Society:—
A. E. Garrett, A. E. Hall, and W. R. Williams.

The following Papers were read:—

1. The Distribution in Electric Fields of the Active Deposits of Radium, Thorium, and Actinium. By Mr. S. Russ.

2. A Note on certain Dynamical Analogues of Temperature Equilibrium. By Prof. G. H. Bryan.

March 27th, 1908.

Meeting held at the Northampton Institute, Clerkenwell, by the kind invitation of the Governing Body and the Principal.

Dr. C. Chree, F.R.S., President, in the Chair.

The following Papers were read:—

1. Notes on the Plug Permeameter. By Dr. C. V. Drysdale.

2. The Use of Shunts and Transformers with alternate-current measuring Instruments. By Dr. C. V. Drysdale.

3. Wattmeters. By Dr. C. V. Drysdale.

An Experimental Demonstration of Alternate Current Wave Propagation in a Helix was given by Dr. C. V. Drysdale.

April 10th, 1908.

Meeting held at the Imperial College of Science and Technology.

Dr. C. Chree, F.R.S., President, in the Chair.

The following Candidates were elected Fellows of the Society:—
S. D. Chalmers and E. N. Ladler.

The following Papers were read:—

1. An Experimental Investigation of the nature of Gamma Rays. By Prof. W. H. Bragg and Mr. Madsen.

2. Experiments on Artificial Fulgurites. By Miss D. D. Butcher.

3. Short-Spark Phenomena. By Mr. W. Duddell.

May 8th, 1908.

Meeting held at the Imperial College of Science and Technology.

Dr. C. CHREE, F.R.S., President, in the Chair.

The following Papers were read:—

1. A Modified Theory of Gravitation. By Dr. C. V. BURTON.

2. An examination of the Formulæ for Grading of Cables. By Mr. C. S. WHITEHEAD.

3. Illustrations of Geometrical Optics. By Mr. R. M. ARCHER.

May 22nd, 1908.

Meetings held at the Imperial College of Science and Technology.

Dr. C. CHREE, F.R.S., President, in the Chair.

A Special General Meeting was held at which the following Resolution was passed:—

Resolution:—"That the Articles of Association contained in the printed document submitted to the Meeting, and, for the purpose of identification, subscribed by the Chairman thereof, be, and the same are hereby adopted as the Articles of Association of the Society."

An Ordinary Meeting of the Society was then held.

The following Papers were read:—

1. The Spectrum Top. By Mr. F. P. SEXTON.

2. The Coefficient of Diffusion. By Mr. B. W. CLACK.

3. The Production of Small Alternating Currents of Variable Frequency. By Mr. B. S. COHEN.

June 12th, 1908.

Meetings held at the Imperial College of Science and Technology.

Dr. C. CHREE, F.R.S., President, in the Chair.

A Special General Meeting was held at which the following Resolution was confirmed:—

Resolution:—"That the Articles of Association contained in the printed document submitted to the Meeting, and for the

purpose of identification, subscribed by the Chairman thereof, be, and the same are hereby approved; and that such Articles of Association be, and they are hereby, adopted as the Articles of Association of the Society."

An Ordinary Meeting of the Society was then held.

The following Papers were read:—

1. Experiments on a Directive System of Wireless Telegraphy. By Messrs. E. Bellini and A. Tosi.

2. The Lateral Vibration and Deflection of Clamped-Directed Bars. By Dr. J. Morrow.

3. The Resistance of a Conductor of Uniform Thickness whose breadth suddenly changes, and on the shape of the Stream-lines. By Prof. C. H. Lees.

4. The Inductance of two Parallel Wires. By Dr. J. W. Nicholson.

5. Homogeneous Secondary Radiation. By Dr. C. G. Barkla and Mr. Sadler.

6. Note on the amount of Water in a Cloud formed by expansion of moist air. By Prof. W. B. Morton.

7. An Elementary Treatment of the Motion of a Charged Particle in a Combined Electric and Magnetic Field. By Prof. W. B. Morton.

October 23rd, 1908.

Meeting (informal) held at the National Physical Laboratory, by the kind invitation of the Director, Dr. R. T. Glazebrook, F.R.S.

The laboratories were thrown open for inspection, and demonstrations were given by Mr. Bairstow, Mr. Campbell, Dr. Harker, Mr. Jeffcott, Mr. Paterson, Dr. Rayner, Mr. Rosenhain, Mr. Seaton, and Mr. Stanton.

A large number of Fellows and Visitors were present.

November 13th, 1908.

Meeting held at University College by the kind invitation of Prof. F. T. Trouton.

Dr C. Chree, F.R.S., President, in the Chair.

The following Candidate was elected a Fellow of the Society:—
B. J. Palmer.

The following resolution of condolence on the death of Prof. W. E. Ayrton was moved by the President, supported by Prof. Carey Foster and Prof. S. P. Thompson, and unanimously adopted:—

> "This Meeting of the Physical Society desires to record its sense of the great loss sustained by Physical Science and by the Society through the death of their Past-President, Professor Ayrton, and to express their sincere sympathy with Mrs. Ayrton and the other members of Professor Ayrton's family in their bereavement."

The following Papers were read:—

1. The Photo-electric Properties of Potassium-sodium Alloys. By Prof. J. A. Fleming.

2. The Recombination of Ions in Air. By Dr. P. Phillips.

3. The Influence of a Magnetic Field on the Photographic Patterns produced by the Electric Spark. By Prof. A. W. Porter.

4. The Rate of Growth of Viscosity in Congealing Solutions. By Mr. A. O. Rankine.

5. The Isoneres of Adsorption by Solid Surfaces from Solutions. By Prof. F. T. Trouton.

Prof. A. W. Porter showed an Experiment illustrating the Anomalous Effect of Lagging thin Wires.

November 27th, 1908.

Meeting held at the Imperial College of Science and Technology.

Dr. C. Chree, F.R.S., President, in the Chair.

The following Candidate was elected a Fellow of the Society:—
R. L. Jones.

The following Papers were read:—

1. A Graphic Method of Dealing with Refracting Surfaces. By Mr. H. S. Allen.

2. An Accurate Method of Measuring Moments of Inertia. By the late Prof. W. CASSIE.

3. The Diffusion of Actinium and Thorium Emanations. By Mr. S. RUSS.

4. Elliptic Polarization. By Mr. J. WALKER.

5. An Experimental Investigation of Gibbs's Theory of Surface-Concentration. By Mr. W. C. M. LEWIS.

December 11th, 1908.

Meeting (informal) held at the Imperial College of Science and Technology.

An Exhibition of Apparatus was given by the following firms :— Messrs. The Cambridge Scientific Instrument Co.; C. F. Casella & Co.; A. C. Cossor Ltd.; G. Cussons Ltd.; J. H. Dallmeyer Ltd.; Elliott Bros.; Everett, Edgcumbe & Co. Ltd.; A. Gallenkamp & Co. Ltd.; Gambrell Bros.; J. J. Griffin & Sons; A. Hilger Ltd.; India Rubber, Gutta Percha & Telegraph Works Co., Ltd.; Hans Knudsen; Marconi's Wireless Telegraph Co. Ltd.; Nalder Bros. & Thompson Ltd.; Newton & Co.; R. W. Paul; Pitkin & Co.; Siemens Bros. & Co. Ltd.; Snell & Tinsley; Strange & Graham Ltd.; Swift & Son; Alexander Wright & Co. Ltd.; Carl Zeiss Ltd.

January 22nd, 1909.

Meeting held at the Imperial College of Science and Technology.

Dr. C. CHREE, F.R.S., President, in the Chair.

The following Candidates were elected Fellows of the Society :— H. BANISTER, H. BRUNSKILL, F. P. FULLER, T. HARRIS, W. WILLINGS.

The following Papers were read :—

1. The Effective Resistance and Inductance of a Concentric Main. By Dr. A. RUSSELL.

2. Luminous Efficiency of a Black Body. By Dr. C. V. DRYSDALE.

3. The Use of the Potentiometer on Alternate Current Circuits. By Dr. C. V. DRYSDALE.

Annual General Meeting.

February 12th, 1909.

Meeting held at the Imperial College of Science and Technology.

Dr. C. Chree, F.R.S., President, in the Chair.

The following Report of the Council was read by the Secretary:—

Since the last Annual General Meeting, ten ordinary Science Meetings, two informal Meetings, and two Special General Meetings of the Society have been held. Of these, eleven were held at the Royal College of Science; one, that on March 27th, at the Northampton Institute, by invitation of Dr. R. M. Walmsley; one, that on Oct. 23rd, at the National Physical Laboratory, by invitation of Dr. R. T. Glazebrook, F.R.S.; and one, that on Nov. 13th, at University College, by invitation of Prof. F. T. Trouton, F.R.S. The average attendance at the Meetings has been 61 as compared with 58 during the preceding year.

The Fourth Annual Exhibition of Apparatus by Manufacturers was held on December 11th, when the attendance of Fellows and Visitors amounted to about 300. Greater advantage is now being taken of this event by exhibitors, and the attendance this year was much larger than hitherto.

Certain alterations to the Articles of Association have been introduced, more particularly in regard to the election of the Council. A copy of the revised Articles has been forwarded to every Fellow of the Society.

The number of Ordinary Fellows now on the roll, as distinct from Honorary Fellows, is 430, a decrease of 7 on the number last year; 12 new Fellows have been elected. There have been five resignations; the names of three Fellows have been struck off the List for non-payment of subscriptions; and the Society has to mourn the loss by death of two Honorary Fellows, namely, H. Becquerel and E. Mascart; also one of their Past Presidents, Prof. W. E. Ayrton, one of their Secretaries, Prof. W. Cassie, and nine other Ordinary Fellows, namely, Lord Blythswood, Mr. S. Hall, Prof. A. S. Herschel, Mr. S. Joyce, Mr. F. J. M. Page, the Earl of Rosse, Capt. J. H. Thomson, Mr. J. F. Walker, and Dr. W. E. Wilson.

The Report of the Council was adopted.

The Report of the Treasurer and the Balance Sheet were presented and adopted.

The election of Officers and other Members of Council then took place, the new Council being constituted as follows :—

President.—C. CHREE, Sc.D., LL.D., F.R.S.

Vice-Presidents who have filled the Office of President.—Prof. G. C. FOSTER, F.R.S.; Prof. W. G. ADAMS, M.A., F.R.S.; Prof. R. B. CLIFTON, M.A., F.R.S.; Prof. A. W. REINOLD, M.A., F.R.S.; Sir ARTHUR W. RÜCKER, M.A., D.Sc., F.R.S.; Sir W. DE W. ABNEY, R.E., K.C.B., D.C.L., F.R.S.; SHELFORD BIDWELL, M.A., LL.B., F.R.S.; Prin. Sir OLIVER J. LODGE, D.Sc., F.R.S.; Prof. S. P. THOMPSON, D.Sc., F.R.S.; R. T. GLAZEBROOK, D.Sc., F.R.S.; Prof. J. H. POYNTING, M.A., D.Sc., F.R.S.; Prof. J. PERRY, D.Sc., F.R.S.

Vice-Presidents.—W. DUDDELL, F.R.S.; Prof. A. SCHUSTER, Ph.D., F.R.S.; S. SKINNER, M.A.; W. WATSON, D.Sc., F.R.S.

Secretaries.—W. R. COOPER, M.A.; S. W. J. SMITH, M.A., D.Sc.

Foreign Secretary.—Prof. S. P. THOMPSON, D.Sc., F.R.S.

Treasurer.—Prof. H. L. CALLENDAR, M.A., LL.D., F.R.S.

Librarian.—W. WATSON, D.Sc., F.R.S.

Other Members of Council.—A. CAMPBELL, B.A.; W. H. ECCLES, D.Sc.; A. GRIFFITHS, D.Sc.; J. A. HARKER, D.Sc.; Prof. C. H. LEES, D.Sc., F.R.S.; T. MATHER, F.R.S.; A. RUSSELL, M.A., D.Sc.; Prof. E. RUTHERFORD, D.Sc., F.R.S.; F. E. SMITH; R. S. WHIPPLE.

Votes of thanks were passed to the Auditors, to the Officers and Council, and to the Governors of the Imperial College of Science and Technology.

The President then delivered an Address.

Treasurer's Report.

After the temporary boom of 1907 has succeeded another period of depression. It has been almost impossible, in spite of strenuous efforts, to collect any arrears of subscriptions, the amount under this head standing at £4 4*s.* 0*d.* as compared with £54 12*s.* 0*d.* in 1907. Only nine new Fellows have paid entrance fees as compared with 27 in 1907, and the amount from composition fees is unusually small. In other respects there is little change in the position or expenditure of the Society, except that last year's Printing Bill was unusually heavy, and the balance, allowing for liabilities, is slightly reduced in consequence. The provision of refreshments at the Meetings has entailed a small additional outlay, but the catering has been carried out most economically, and the total expense for 1908 is little more than that for the exhibition of apparatus alone in 1907.

HUGH L. CALLENDAR,
Hon. Treasurer.

The Treasurer in Account with the Physical Society, from January 1st, 1908, to December 31st, 1908.

Dr.	£	s.	d.	£	s.	d.
Jan. 1, 1908. Balance in Bank				415	13	5
Entrance-Fees	9	9	0			
Subscriptions	388	9	1			
Extra Subscriptions	28	4	0			
Arrears	4	4	0			
Abstracts etc.	4	16	0			
Students	1	11	0			
				436	13	1
Composition-Fees				21	0	0
Dividends:—						
Furness Debenture Stock	15	3	10			
Metropolitan Board of Works	6	15	0			
Lancaster Corporation Stock	11	8	0			
New South Wales	8	9	0			
Midland Railway	38	0	0			
Great Eastern Railway	19	0	0			
London, Brighton, & South Coast Railway	21	7	7			
				120	1	5
Sales of Publications				43	6	8
Interest on £400 deposit				4	17	6
				£1041	12	1

Cr.	£	s.	d.	£	s.	d.
Science Abstracts				277	3	6
Taylor and Francis:—						
Proceedings	102	0	0			
Abstracts, etc.	91	8	6			
Distribution	13	1	6			
Separate Copies	14	3	0			
				220	13	6
Periodicals				3	9	0
Binding				3	0	0
Reporting				35	0	0
Attendance				9	0	0
Refreshments				9	12	0
International Catalogue (2 years)				11	12	4
Petty Cash (Secretaries)				10	0	0
Dinner				5	12	0
Commissions				3	9	
To Deposit Account				400	0	0
Balance in Bank				56	6	6
				£1041	12	1

Audited and found correct,

HUGH L. CALLENDAR, *Honorary Treasurer.*

21*st January*, 1909.

HENRY R. J. BURSTALL,
A. A. C. SWINTON. } *Auditors.*

PROPERTY ACCOUNT OF THE PHYSICAL SOCIETY, DECEMBER 31, 1908.

ASSETS.	£	s.	d.
Subscriptions due, Treasurer's estimate	70	0	0
£533 Furness Ry. Co. 3 per cent. Debenture Stock	405	0	0
£1600 Midland Railway 2½ per cent. Preference Stock	1104	0	0
£200 Metropolitan Board of Works 3½ per cent. Consolidated Stock	205	0	0
£400 Lancaster Corporation 3 per cent. Redeemable Stock	332	0	0
£284 2s. 9d. New South Wales 3½ per cent. Inscribed Stock	252	15	0
£500 London, Brighton, and South Coast Railway Ordinary Stock	530	0	0
£500 Great Eastern Railway 4 per cent. Debenture Stock	555	0	0
Balance in Bank	56	6	6
On deposit	400	0	0
Publications in Stock, estimated	200	0	0
	£4110	1	6

LIABILITIES.	£	s.	d.
Printing	94	13	1
Cheque not presented	2	2	0
Balance	4013	6	5
	£4110	1	6

HUGH L. CALLENDAR, *Honorary Treasurer.*

21st January, 1909.

Audited and found correct,

HENRY R. J. BURSTALL,
A. A. C. SWINTON, } *Auditors.*

Presidential Address to THE PHYSICAL SOCIETY OF LONDON,

12th February, 1909.

C. CHREE, ScD., LL.D., F.R.S.

§ 1. During the past year our losses have been heavy. In Prof. Ayrton we have lost an ex-president whose services to technical education have been widely recognised. Several notices of Prof. Ayrton have appeared by those whose personal knowledge of him was much closer than mine; in particular I would refer to the article by Prof. Perry in "Nature." Prof. Ayrton was a not-infrequent attendant at meetings of the Society, and when he came he had usually something interesting to say on the papers before the meeting. Prof. Cassie, our late Secretary, was a personal friend of my own, our acquaintance extending back to 1882 when we were undergraduates at Cambridge. Prof. Cassie graduated at Aberdeen University in 1881, taking the first place in the examination for honours in Mathematics and Natural Philosophy (Physics). He had the distinction of winning the Ferguson Scholarship, open to graduates of the four Scottish Universities. At Cambridge he was 5th Wrangler and obtained a 1st Division, 1st Class, in the final part of the Mathematical Tripos. He was also the first to be elected to the Clerk-Maxwell Studentship. After engaging for some time in University Extension lecturing and private tuition, he was appointed to the Professorship of Physics at the Royal Holloway College, previously held by Prof. Callendar. His energy and efficiency when acting as our Secretary will I think be recognised by all who came in contact with him in that capacity.

Professor Cassie's death was exceptionally sudden, and happening when it did at the end of the Spring Session, it might have led to confusion in the secretarial work, but for the public spirited action of Mr. H. M. Elder, who put his long experience as Secretary at the disposal of the Council. Mr. Elder continued to act as Secretary until the appointment in November of Dr. S. W. J. Smith.

§ 2. During the year the Council passed in review the Articles of Association, and a variety of small changes were introduced, which were approved by statutory meetings of the Society.

Probably the most important change related to procedure intended to prevent Fellows' annual contributions falling into arrear. Our Treasurer had found that even his powers of persuasion did not invariably meet with the success which they deserved.

As has now been the practice for some years, an exhibition of instruments was held in the Autumn Session. Whether regarded from the number and interest of the exhibits, or from the number of Fellows and Visitors who attended, the exhibition was eminently successful. For this the Society is mainly indebted to our Senior Secretary, Mr. W. R. Cooper, and to the active co-operation of Professor Callendar and the Staff of the Physics Department of the Imperial College of Science and Technology.

§ 3. As you have only just confirmed the election of two new Honorary Fellows, it may perhaps appear premature to refer to the subject. Unless, however, the matter is referred to to-night, it may be overlooked. I am afraid that eminent unquestionably as all our Honorary Fellows are, it is not every Fellow, perhaps not even every member of the Council, who would welcome a *viva voce* examination as to the exact sources of their distinction. Of Prof. Julius Thomsen I must confess my knowledge is mainly second-hand, as his subject, physical chemistry, is one with which I can hardly claim an intimate personal acquaintance. His contributions to thermo-chemistry are, however, very numerous, and his reputation in that domain is world wide. Dr. René Benoit is the Director of the International Bureau of Weights and Measures at Sèvres. The work of that institution relates primarily to the standards of length and mass, but the study of standards of length entails minute study of temperature, and has thus brought thermometry within the domain of the Bureau International. From the point of view of accuracy, the work at Sèvres has probably never been excelled. The Committee in supreme control of the Bureau International contains representatives from all the leading scientific countries, and though these gentlemen are usually selected principally with reference to their official position, their number usually includes several of the most capable metrologists of the day. The staff at Sèvres has also included in addition to Dr. Benoit other eminent physicists, notably Dr. Guillaume and Dr. Chappuis. Thus the excellence of the work done at Sèvres must be ascribed to various sources, but much is due unquestionably to Dr. Benoit himself, to his skill as an observer and his insight in discussing experimental results.

§ 4. Leaving now historical matters, I propose giving an account of some work in which I have recently been engaged. Last year I described some of the phenomena observed in the regular diurnal variation of Terrestrial Magnetism, dealing more particularly with the results of the National Antarctic Expedition of 1902–4. To-night I intend to discuss some phenomena relating to the irregular variations in Terrestrial Magnetism.

When the National Antarctic Expedition set out, one of the items in the programme was a certain scheme of observations in which it was hoped that a large number of observatories would co-operate. According to this scheme, which originated I believe in Germany, there were two pre-arranged term-days a month, throughout which hourly readings at exact hours G. M. T. should be taken, and in each term-day there was a pre-arranged term-hour, during which it was hoped that the drums of magnetographs would be rotated at twelve times the usual speed, so as to secure a very open time scale. The principal object, presumably, was to facilitate the intercomparison of the records of the disturbances, of which in the usual course a considerable number would naturally present themselves during the term-hours.

For the comparison of the records from different stations of disturbances of considerable size an open time scale is most valuable, as the times of occurrence of individual phenomena can be determined with so much higher precision than in the ordinary slow-run curves. When, however, there are no large disturbances, quick-run curves lose their advantage to a large extent, and in some respects they are even inferior to ordinary curves. A difference of 1 mm. in the ordinates of two points catches the eye much more readily when their difference of abscissa is 1 cm. than when it is 12 cm. 1902 and 1903, especially the former, were years of few sun-spots, and as is usually the case near sun-spot minimum magnetic disturbances were relatively few, and with one or two exceptions of minor size. As it so happened, the selected term-days were all quiet days, and there was not a single term-hour in which any but the most trifling disturbance presented itself at the co-operating stations.

§ 5. The term-days having thus failed to supply material for a study of disturbances, I made, with the sanction of the Royal Society's Magnetic Committee, an addition to the pre-arranged programme. A list was prepared of the days of largest magnetic disturbance in 1902 and 1903 for which Antarctic records existed, and this

was sent to the directors of the observatories at Christchurch (New Zealand), Mauritius, and Colaba (Bombay), with the request that they would supply copies of their magnetic curves for these days. To the three directors of the above mentioned observatories, Dr. Coleridge Farr, Mr. C. T. F. Claxton, and Mr. N. A. F. Moos, I am much indebted for their generous response to the request. I am similarly indebted to Mr. E. Kitto and the Committee of the Falmouth Observatory for the loan of a number of Falmouth magnetic curves. Even in 1902 the Kew records suffered perceptibly from electric tram currents, so that records from a second and undisturbed English Station were highly useful. The results to which reference will be made to-night are derived from the Christchurch, Mauritius, Colaba, Falmouth and Kew records, and from those at the 'Discovery's' Winter Quarters in the Antarctic (lat. 77° 51′ S., long. 166° 37′ E.). The full discussion will form part of the volume still to be published on the Antarctic magnetic results. For permission to anticipate to-night to some extent the publication of that volume I am indebted to the President of the Royal Society and the Director of the National Physical Laboratory.

§ 6. Immediately on the discovery by Prof. Hale of the Zeeman effect in light from sun-spot areas, indicating an intense magnetic field, several astronomers suggested that the mystery of magnetic storms had received its final solution. They seemed unaware of the fact that magneticians had not waited for any such visible sign before endowing the Sun with electric currents and magnetic fields, and that the possibilities of direct magnetic action exerted by the Sun had been considered by a succession of physicists. It is many years since it was recognised that the size of magnetic disturbances constituted a serious difficulty. Lord Kelvin, for instance, made a rough estimate of the expenditure of energy which such direct action would necessitate during a large magnetic storm, and came to the conclusion that it was inadmissibly large. A few weeks ago, Dr. A. Schuster published in "Nature" an estimate of the possible size of the effect on the Earth of the largest of sun-spot areas, endowed with what he considered the largest magnetic moment reasonably attributable, and found it practically negligible.

Some years ago Mr. E. W. Maunder in an important series of papers advanced evidence from a list of Greenwich magnetic storms that these tend to follow one another at intervals of about twenty-seven and a quarter days, answering to the revolution

period of solar meridians relative to the earth. He postulated the ejection from spot areas on the sun of some kind of discharge, travelling like a water jet, which on reaching the Earth set up magnetic storms, the discharge from a single area of the Sun persisting sometimes for several rotation periods. Mr. Maunder suggested that as the discharge formed only a narrow stream, this removed the difficulty raised by Lord Kelvin on the ground of the large expenditure of energy required. As I pointed out, however, this apparently overlooks the fact that on Mr. Maunder's theory the disturbance on the Sun continues for many days or even months, while Kelvin supposed it to last only while the magnetic storm was experienced on the Earth. Further, as severe magnetic storms usually last for many hours, sometimes for several days, the cross section of one of Mr. Maunder's hypothetical jets must be very considerable. It is thus by no means obvious that his theory postulates less expenditure of energy than Kelvin's estimate. How much energy it entails must depend on the nature of the solar discharge, and as to that Mr. Maunder makes no definite suggestion. The discharge might be electrified particles, as Arrhenius has suggested, Röntgen rays, as Nordmann postulates, or kathode rays, as Prof. Birkeland advocates, or some entity not yet recognised.

§ 7. If magnetic storms are due to a discharge of particles or ions travelling from the Sun, the paths of these when near the Earth are largely determined by the Earth's magnetic field. According to Störmer's elaborate mathematical calculations, the electric carriers will be confined to spaces near the Earth's magnetic poles. The law of variation of the resulting disturbances at different parts of the Earth's surface will naturally be very complicated. All that can very well be inferred is that magnetic disturbances will be less in equatorial regions than elsewhere. If, on the other hand, the Sun behaves as a magnet, one would expect the phenomena to be relatively simple. The distance of the Earth from the Sun is so large compared to the Sun's diameter, and still more so when compared to the Earth's diameter, that a magnetic force originating in the Sun would be practically constant throughout the Earth and its atmosphere if the Earth itself were non-magnetic. The Earth's magnetism of course is a complication, but according to the Gaussian theory the magnetism is internal, and in the expression for the potential terms of higher order are small compared to those of the first. It is thus natural to suppose that

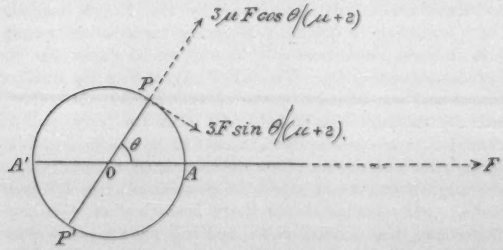
magnetically the Earth's material must in the main be fairly uniform, or that the magnetic quality may to a first approximation be treated as a function only of the distance from the centre.

In the present state of our knowledge I do not think we are justified in regarding direct action from the Sun as an impossibility. We do find that in at least some storms magnetic changes of similar size and character present themselves at very distant stations, and in some instances they arise suddenly at times of profound quiet. The phenomena unquestionably do suggest the entrance of the Earth into a stray magnetic field. It is quite conceivable, as I have remarked before, that such fields might arise from some source quite other than the Sun. It seems thus worth while considering what the consequences would be of the existence of an external field, uniform but for the presence of the Earth.

§ 8. If we regard the Earth as a sphere of radius a, having uniform permeability μ, and if F denote the strength which the disturbing field would possess if the Earth were not magnetic, the potential V is given by

$$V = -Fr\cos\theta + \frac{\mu-1}{\mu+2}\,F\frac{a^3}{r^2}\cos\theta. \quad . \quad . \quad . \quad (1)$$

Here r and θ are polar co-ordinates, the origin being at the Earth's centre, and the initial line $\theta=0$ answering to the direction of the field F.



In the figure, A P A P' is a section through O the centre of the sphere, A' A being the direction of the impressed field F.

The force at any point P (a, θ) just outside the sphere may be regarded as composed of

a normal component $3\mu F\cos\theta \div (\mu+2)$,
and a tangential component $3F\sin\theta \div (\mu+2)$.

If ψ be the inclination to OA of the resultant force R at P, then

$$\tan\psi=(\mu-1)\sin 2\theta\div\{(\mu+1)+(\mu-1)\cos 2\theta\}, \quad . \quad . \quad . \quad . \quad (2)$$

and

$$R/F=\{3/(\mu+2)\}\{\mu^2-(\mu^2-1)\sin^2\theta\}^{\frac{1}{2}}\equiv\{3/(\mu+2)\}\{1+(\mu^2-1)\cos^2\theta\}^{\frac{1}{2}}. \quad (3)$$

If $(\mu-1)/1$ be small, then ψ is necessarily small, and R/F can differ but little from unity.

Over the surface, R obviously has its maximum value where $\theta=0$, and its minimum where $\theta=\pi/2$, and

$$\left.\begin{aligned} R_{max.}/F&=3\mu/(\mu+2),\\ R_{min.}/F&=3/(\mu+2)\end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

so that

$$R_{max.}/R_{min.}=\mu. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The maximum value occurs at the ends of the diameter $\theta=0$, where the force is entirely vertical, the minimum at all points on the perpendicular great circle, where the force is entirely tangential (*i. e.* horizontal).

Clearly, everything turns on the size of μ. If μ were large, R would have a wide range of values, and the disturbing forces at different parts of the surface would vary widely in direction. It is, however, difficult to believe that μ is large. For one thing, the earth's field itself is only of the order 0·5 C.G.S. and a disturbance as large as ·01 C.G.S. is very exceptional. Even in the best magnetic steel μ is low in small fields. Apart from mere theoretical considerations, there is the fact that if μ were large the horizontal (*i. e.* tangential) component $3F\sin\theta/(\mu+2)$ would tend to be negligible compared to the vertical $3\mu F\cos\theta/(\mu+2)$ except at places for which θ is nearly $\pi/2$. Now the tendency is all the other way. Vertical force curves at most places are conspicuously quieter than horizontal force curves.

A point worthy of special remark is that R has the same value, and the same absolute direction in space, for places whose angular co-ordinates are θ and $\pi+\theta$; *i. e.*, the disturbing forces are identical at any two places on the surface which are diametrically opposite to one another.

§9. As the Earth's crust is on the whole non-magnetic, it is clear that the above simple case cannot be an exact representation of the facts. It may thus be worth while glancing at the next simplest hypothetical case, in which we have an earth consisting

of a nucleus of radius a and permeability μ, with a surrounding crust of lesser permeability μ' and external radius a'. In this case, supposing the impressed field of strength F as before, we find for the potential external to the Earth

$$V = -Fr\cos\theta + F\frac{a'^3}{r^2}\cos\theta\,\frac{(\mu+2\mu')(\mu'-1)+(\mu-\mu')(2\mu'+1)(a/a')^3}{(\mu+\mu2')(\mu'+2)+2(\mu-\mu')(\mu'-1)(a/a')^3}. \quad . \quad . \quad (6)$$

The only sub-case of practical interest is that in which $\mu'-1$ is small, *i. e.* in which the material of the surface layer or crust is only slightly magnetic. In this case, assuming a/a' not to be very small, we find the potential approximately given by

$$V = -Fr\cos\theta + F\{(\mu-1)/(\mu+2)\}a^3\cos\theta/r^2. \quad . \quad (7)$$

This is identical in form with (1), only whilst a in (1) represents the Earth's radius, in (7) it represents the radius of the highly magnetic nucleus. If the external slightly magnetic layer be thin, the phenomena are much the same as in the case of uniform permeability μ. If $(a'-a)/a'$ be considerable, the variation of R over the surface is materially reduced. Whether $(a'-a)/a'$ be small or not, the values of R at diametral points are necessarily equal.

§10. Before dealing with the magnetic curves, there is one point I had better explain. The most usual procedure is that followed at Kew, Falmouth, and Colaba, where movement of the photographic trace up the sheet signifies increase in the magnetic elements, declination, horizontal force and vertical force. At Mauritius, however, movement up the sheet answers to increase only in the declination; in the case of the horizontal and vertical forces it signifies decrease. At Christchurch movement up the sheet denotes decrease in all three elements. In the Antarctic curves it denotes increase for the vertical force, but decrease for the two other elements. As the question naturally suggests itself why the original curves were not inverted when necessary during the reproduction, so as to secure that in all movement up the sheet should answer to increase in the element considered, I had better mention the reasons for not doing so. The three elements in the Antarctic were all recorded on one sheet, and to have inverted the declination and horizontal force traces, while leaving the vertical force traces unchanged, would have been troublesome. Then, it must be remembered that increase and decrease in declination are purely relative terms. Declination is westerly at Kew, Falmouth, and Mauritius, but easterly at Colaba and Christchurch. Thus increase of declination at the two last-

mentioned stations means exactly the opposite of what it does at the first three. At the 'Discovery's' Winter Quarters declination was easterly, but about $152\frac{1}{2}°$—*i. e.* the north pole of the magnet pointed about $27\frac{1}{2}°$ to the east of south—thus an increase of declination answered to a westerly directed disturbing force.

At Kew an increase in the horizontal force H means a disturbing force inclined at $16\frac{1}{2}°$ to the west of north; at Christchurch it means a force inclined at 16° to the east of north; in the Antarctic it means a force inclined at $27\frac{1}{2}°$ to the east of south. Then at Winter Quarters and Christchurch, where the longitude is nearly 180° E., the west has nearly the same absolute direction in space as has the east at Kew and Falmouth.

Finally, increase in the vertical force means at Kew, Falmouth, and Colaba a force directing the north pole of a magnet towards the Earth's centre, whereas at Mauritius, Christchurch, and Winter Quarters it means a force outwardly directed. Taking these points into consideration, I decided to invert none of the curves.

At Falmouth, Colaba, Mauritius, and Christchurch, the magnetographs are of the Kew pattern, with nearly equal time scales. The sensitiveness is also nearly the same for declination and horizontal force at the four stations. This greatly facilitates intercomparison of the curves.

§11. I now pass to some lantern-slides illustrating various magnetic disturbances recorded in 1902 and 1903. The first slide (fig. 1) shows the records of horizontal force on April 5–6, 1903, at Falmouth, Colaba, Mauritius, and Christchurch, with the corresponding declination records from the first and last of these stations. The four central curves are the horizontal force ones. Corresponding times (G.M.T.) at the various stations come at least very approximately in the same vertical line. For some hours previous to 11 P.M. on April 5 the curves were very quiet. Then suddenly the horizontal force trace begins to go up the sheet at Falmouth and Colaba, and down the sheet at Mauritius and Christchurch, thus indicating an increase of force at all four stations. The time scale of the original curves * is about 1 hour to 15 m.m., thus, rapid as the initial change of force is compared to the ordinary standard of magnetic changes, it is by no means instantaneous. It occupied in fact about 4 minutes.

With magnetic curves of the type, time can hardly be measured to nearer than 1 minute, so that all one can say is that the

* The curves are all shown here on a reduced scale.

initial movements appeared to commence simultaneously at about 11.25 P.M., G.M.T., on April 5. Such sudden movements are a fairly common prelude to large magnetic disturbances, and at a particular station they usually have a characteristic type. Thus at Kew, the initial change of horizontal force is usually an increase. In the present case the initial movement though not peculiar to the horizontal force is less marked in the declination. Though the Colaba and Mauritius horizontal force curves are, as explained, inverted relative to one another, their close resemblance will be readily recognised.

Though there is a fairly close general resemblance between later parts of the records from the different stations, the difference in details renders it difficult to identify corresponding movements. Even at the same station the turning points in the curves of the different elements did not appear to occur simultaneously.

The storm of April 5–6 was hardly worthy of so conspicuous an initial movement. At Kew and Falmouth the total range of declination was only about 25′.

§ 12. As already stated, the curves at the co-operating stations were very quiet for some hours prior to 11 P.M. on April 5, but earlier in the day between 11 A.M. and 5 P.M. there were a few rapid though small movements.

As our next slide (fig. 2) shows, these earlier disturbances synchronised with very much larger disturbances in the Antarctic. To bring all the more striking of the Antarctic disturbances of April 5–6 within the compass of one figure, five or six hours trace have been omitted, being represented only by a small gap. It is the portion subsequent to the gap that answers to the disturbances shown in fig. 1. It should be noticed that whilst Greenwich time is used in fig. 1 and the other illustrations of disturbances at the co-operating stations, the time employed in all the Antarctic curves is the local time (L.T.) of Winter Quarters, which was 11 hours 7 minutes fast on Greenwich.

In fig. 2 and the other reproductions of Antarctic curves the letters D, H, V denote the declination, horizontal force, and vertical force traces, D_0, H_0, V_0 the corresponding base lines. T denotes the temperature inside the vertical force box. The scales and base-line values are shown at the margins.

The Antarctic curves were practically never quiet, so it is difficult to assign a definite beginning or end to a disturbance

FIG. 1.

April 5-6, 1903.

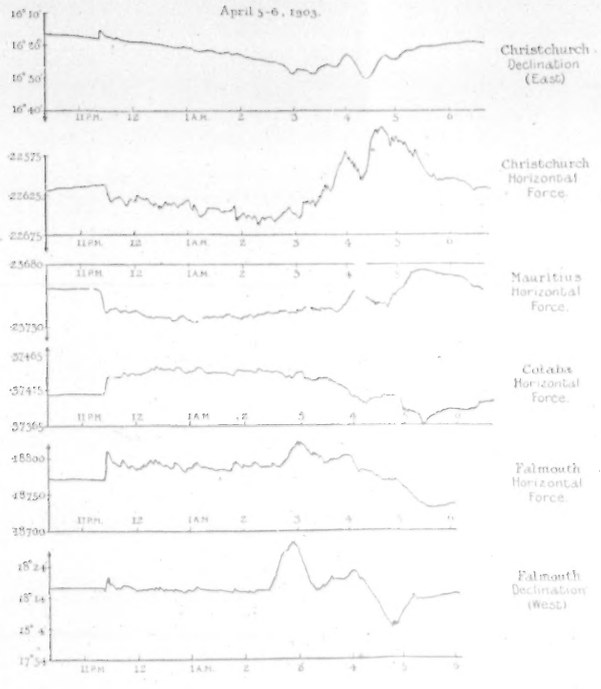


FIG. 3.

Horizontal force December 13, 1903.

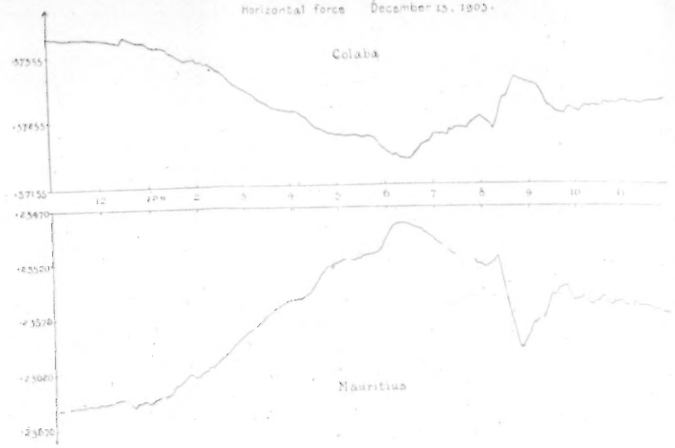


FIG. 5.

May 8, 1902.

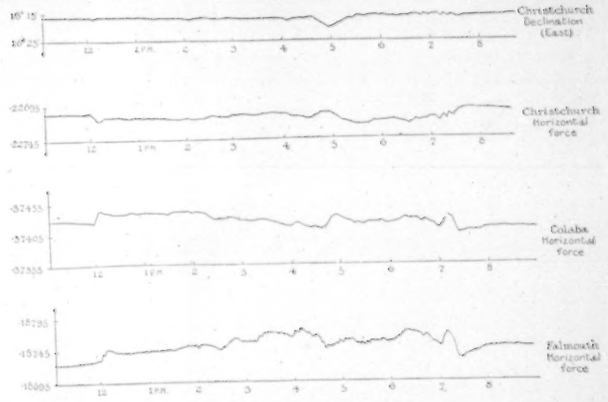


FIG. 2.—Antarctic, April 6, 1903.

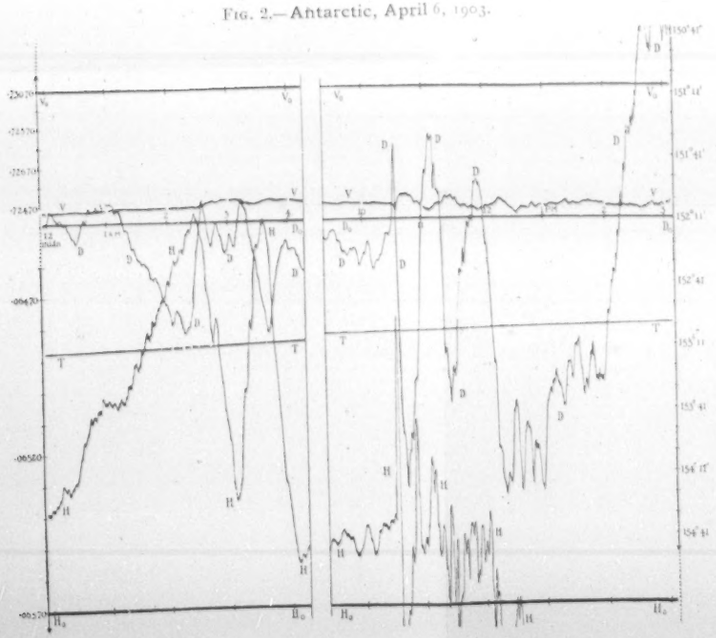


FIG. 4.—Antarctic, December 13–14, 1903.

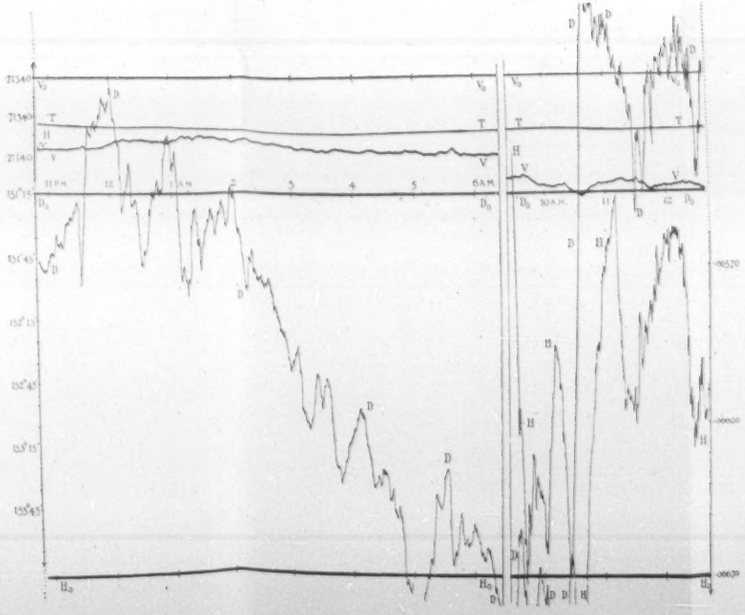
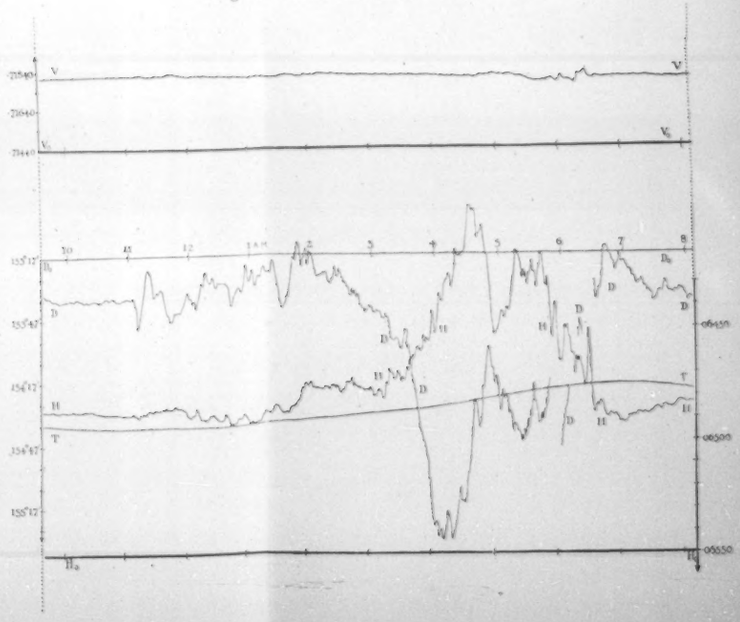


Fig. 6.—Antarctic, May 8–9, 1902.



there. What answers to the sudden commencement at the co-operating stations on April 5 is apparently the large sudden movement a little to the right of the gap in fig. 2. Commencing at the same time, 11.25 P.M. G.M.T. on the 5th (10.32 A.M. L.T. on the 6th) it also lasted 4 minutes. But whilst at the co-operating stations the initial movement was much more prominent than the movements immediately following it, the converse was true of the Antarctic. The initial movement there was immediately followed by an equally rapid and notably larger movement in the opposite direction. A further peculiarity of the Antarctic was that the initial movement was a *decrease* in all three elements.

§ 13. In this instance I have calculated, from the curve measurements, the components relative to 3 fixed axes of the disturbing force to which the initial movement may be ascribed. The axes were taken one parallel to the Earth's axis of rotation, and the others perpendicular, one being in the meridian containing Kew (or to the present degree of approximation Greenwich) and the other in the meridian 90° east. The north being taken as the initial direction, the inclinations to it of the disturbance vector were found to be 68° at Kew, 61° at Winter Quarters, 40° at Christchurch, and 11° at Colaba. The azimuths of the meridian planes containing these vectors were 224° E. at Kew, 189° E. at Winter Quarters, 167° E. at Colaba, and 148° E. at Christchurch. There is no very great degree of accuracy possible in the measurements of the disturbances, as they are small when considered absolutely; thus the departure from parallelism between the vectors at the several stations is not very striking. On the whole these results are by no means unfavourable to the hypothesis of direct solar action. The fact that the vertical force nearly vanished at Christchurch, where the disturbing force was least, is distinctly favourable; but some of the phenomena point to the opposite conclusion. The values found for the resultant disturbing force were 23 γ at Christchurch, 27 γ at Colaba, and 45 γ at Kew, but 102 γ at Winter Quarters. On the hypothesis of direct solar action this would imply a value of μ greater than 4.

Further, it will be remembered that on that hypothesis the vertical force ought to be largely in the ascendant at stations where the disturbing force is largest. But at Winter Quarters the vertical component was less than a third of the horizontal.

§ 14. The next few slides afford further examples of suddenly

commencing disturbances. The first (not reproduced) shows the declination and horizontal force disturbances at Falmouth on December 13, 1903. The main disturbance on this date exceeded that on April 5–6, 1903, but though it also possessed a sudden commencement, the initial movement was of a somewhat confused character. The next slide (fig. 3) shows the horizontal force changes at Colaba and Mauritius on December 13, 1903, corresponding to those at Falmouth just referred to. The initial movement, an increase of force—increase in the Mauritius curve answering to movement down the sheet—was much more conspicuous than at Falmouth. The initial rise was followed by an almost continual fall continuing for about 5 hours, the total decrease at either station amounting to nearly 200 γ.

In the Antarctic, as the next slide (fig. 4) shows, December 13 was highly disturbed. To bring the main movements within the limits of one figure, about three hours trace are omitted between 6 and 10 A.M. L.T. on the 14th. The sharp oscillation in the declination trace between 11 P.M. and midnight L.T. on the 13th answers to the commencing movement in fig. 3. The largest and most sudden movements, however, are those shown on the 14th, subsequent to the gap in fig. 4. The declination trace will be seen to traverse the full width of the sheet, representing nearly 5°, in about 20 minutes. It did not, however, prove possible to identify any of these large Antarctic movements with movements elsewhere.

§ 15. The next slide (not reproduced) shows sudden commencing movements in horizontal force which occurred on August 20 and November 6, 1902, at the co-operating stations. At Winter Quarters on August 20, 1902, the declination shows a sudden change lasting 4 minutes and apparently synchronous with the sudden commencement elsewhere; but it was immediately followed, as on April 5, 1903, by an equally rapid and larger movement in the opposite direction. If we take the initial movement, lasting from 9.6 to 9.10 P.M. G.M.T. on August 20, 1902, we find for the amplitude of the resultant disturbing force 38 γ at Winter Quarters, 14 γ at Kew, 11 γ at Colaba, and 6 γ at Mauritius and Christchurch. Thus the stations are in the same order, as regards magnitude, as on April 5, 1903. Further, the vectors were fairly similarly directed on the two occasions. Thus at Kew, the resultant vector on August 20 lay in the meridian 214° E., and made an angle of 61° with the Earth's axis, while on April 5 the corresponding angles were respectively 224° and 68°.

§ 16. Attention may be directed for a moment to a phenomenon presented by the Falmouth horizontal force curve of November 6, 1902 (not reproduced). A sudden commencement is recognisable at 3.52 P.M., but on looking closely at the initial movement it is seen to be distinctly double. There is first a small decrease, lasting only about 1½ minute, and the subsequent rise to a maximum at 3.58 is markedly checked after it has been in progress about 1½ minute. The different stations, it may be remarked, agree apparently as to the time of commencement, but not as to the time of reaching the summit.

Turning to the Antarctic curves of November 6, 1902 (not reproduced), we find a conspicuous oscillation in the horizontal force. An increase lasting about 1½ minute appears to synchronise with the short decrease at Falmouth; it is immediately followed by a larger increase which went on for six minutes after the summit appeared in the Falmouth curve. The other elements in the Antarctic were also disturbed. The vertical force, like the horizontal, shows a to and fro movement, but the second turning point seems slightly to precede that in the horizontal force. Owing presumably to the rapidity of the movement, and consequent faintness of the trace, details of the change of declination are partly lost.

§ 17. The next slide, fig. 5, shows a disturbance having a sudden commencement, which though comparatively small is in some ways classic. At Falmouth, Colaba, Christchurch and Mauritius (the last-mentioned curve is not reproduced) a sudden rise of horizontal force began at 11.59 A.M. G.M.T. on May 8, 1902, introducing a disturbance whose termination a few minutes after 8 P.M. is unusually well marked. There is a remarkable parallelism between several of the movements experienced at the different stations, notably from 2.10 to 2.28 P.M. and from 7.4 to 7.11 P.M. The commencement, as already remarked, is at or very close to 11.59 A.M., and the great eruption of Mont Pelée in Martinique which overwhelmed St. Pierre began at about 11.55 A.M. G.M.T. Dr. L. A. Bauer, now of the Carnegie Institution, Washington, noticed the practical agreement between the time of the Mont Pelée eruption and the commencement of the magnetic storm recorded at American observatories, and he made the suggestion that the coincidence might not be accidental, but that the magnetic movement might be a consequence of what happened at Martinique. It was clearly not a case of mere earthquake

tremors affecting the magnetographs, because the interval between the two events was too short to admit of even the rapidly travelling "preliminary tremors" reaching the observatories. Further, if it had been a direct seismic effect, it would have reached different stations at times growing later as the distance from Martinique increased. Thus what Dr. Bauer suggested the possibility of was a magnetic effect, arising presumably from displacement of magnetic matter at Mont Pelée, or from some direct influence of the eruption on the Earth's magnetism. Any suggestion of Dr. Bauer's naturally commands attention. I have thus gone into the question more fully than I should otherwise have done; but I may say at once that the conclusion I have reached is that the Mont Pelée eruption had probably no direct connection with the magnetic storm.

Before explaining the reasons for this view, I would first direct attention to one feature of the Falmouth horizontal force trace in fig. 5. It has been perhaps a little exaggerated by the Assistant who traced the curve from which the photograph was derived. What the original curve shows is rather a temporary hesitation to rise than a distinct decrease occurring at 12.3. There was, I think, a real decrease, but it was somewhat obscured in the original curve owing to the size of the spot of light which produced the photographic trace. The phenomenon is seen rather more clearly in the Kew traces, both horizontal force and declination. At Colaba there is only a slight indication of a pause in the horizontal force movement. No pause is visible at Mauritius or Christchurch, but the photographic copy available of the Christchurch curve was somewhat fuzzy, and the copy of the Mauritius curve was a tracing. Thus the absence of the phenomenon at these two stations is by no means certain.

In the Antarctic, as fig. 6 shows, we have in the declination curve a conspicuous double movement, the first peak occurring about 12.3 G.M.T. (11.10 P.M. L.T.) on May 8, *i. e.* at the time when the Falmouth curve shows a halt. The second peak in fig. 6 occurs 2 or 3 minutes later than the prominent peaks at Falmouth and Colaba; but the summits are all somewhat rounded, the main part of the movement being over by 12.8 or 12.9 G.M.T.

In the Antarctic, as elsewhere, there is more than usual definiteness both in the beginning and end of the 8-hours disturbance. It will be noticed that the movements during the

first half of the storm are considerably smaller than those during the second half, which occurred hours after St. Pierre was overwhelmed. The movements about 7 P.M. G.M.T. (6 A.M. L.T. on the 9th) were at all the stations decidedly the most rapid, so that the activity of the operating cause would seem to have been greatest about seven hours after the eruption. The rapidity of the Antarctic movements at this period led in fact to some loss of trace.

§ 18. Curiously enough, Colaba, Mauritius and Christchurch are pretty nearly equidistant from Mont Pelée, and Winter Quarters though somewhat nearer is not very much. Kew and Falmouth on the other hand are at only about half the angular distance of the first-mentioned three stations. One would expect any magnetic disturbance due to a volcanic eruption to be more or less symmetrical round the vertical at the spot where the eruption occurred, and to decrease rapidly as the distance from this spot increases. Thus if the source of the magnetic disturbance were Mont Pelée, one would expect the magnetic disturbances at Colaba, Mauritius, Christchurch and Winter Quarters to be small compared with those at Kew or Falmouth, and to be roughly equal amongst themselves. The vector, however, answering to the disturbance from 11.59 to 12.9 was only 19·2 γ at Kew as compared with 21·6 γ at Colaba; while its value at the latter station was much larger than the values at Mauritius and Christchurch, which were respectively only 11·1 γ and 9·1 γ. Again, if we calculate the vectors answering to the magnetic disturbances between 11.59 and 12.3 and between 12.3 and 12.9 separately, we obtain at Winter Quarters 28 γ for the former and 52 γ for the latter, as against 9·3 γ and 10·2 γ respectively at Kew. The commencing disturbance was in short much larger at Winter Quarters than at any of the other stations.

For the vectors answering to the movement later in the day from 7.4 to 7.11 P.M. G.M.T. we find for the amplitudes 23·1 γ at Kew, 19·8 γ at Colaba, 7·0 γ at Mauritius, and 9·8 γ at Christchurch, indicating a very similar distribution of force intensity to that of the commencing movement. The vectors moreover for the two movements were nearly parallel to one another at Kew, and still more so at Colaba.

If the commencing movement on May 8 had been due to so exceptional a cause as the Mont Pelée eruption, one would naturally have expected it to take an unusual form. But it was

of a type which showed a close resemblance not merely to the movement from 7.4 to 7.11 P.M. on the same day, but also to the commencing movements of several other storms which occurred in 1902 and 1903. Thus denoting the magnitude of the resultant vector by ΔR, its inclination to the Earth's rotation axis by θ, and the east longitude of the meridian plane containing it by ϕ, we have the following results for commencing disturbances :—

Date of Storm.	Kew.			Colaba.			Mauritius.			Christchurch.		
	ΔR	θ	ϕ	ΔR	θ	ϕ	ΔR	θ	ϕ	ΔR	θ	ϕ
	γ	°	°	γ	°	°	γ	°	°	γ	°	°
May 8, 1902 ...	19·2	61	225	21·6	21	139	11·1	58	17	9·1	46	208
Aug. 20, 1902...	14·4	61	214	10·6	16	203	5·9	21	32	5·7	49	135
April 5, 1903 ...	45·0	68	224	26·8	11	167	...			22·6	40	148
Aug. 25, 1903...	34·6	56	214	17·7	11	272						

These results point to the conclusion that the causes in operation during the commencement of the magnetic storm of May 8, 1902, were not of an exceptional kind.

§ 19. I now pass to what appears at first sight a remarkable example of the recurrence of a magnetic storm after Mr. Maunder's interval of 27 days. Fig. 7 shows an Antarctic disturbance occurring about 6 P.M. (L.T.) on July 26, 1903. I would specially direct attention to the vertical force and declination traces. The sensitiveness of the horizontal force magnet was so excessive that one is apt to form an exaggerated idea of the relative importance of the disturbances in that element. They were in reality usually notably less than those in the declination. In fig. 7 the declination curve shows a nearly symmetrical movement up and down the sheet. During the downward movement in D (*i. e.* increase of the element) the vertical force curve runs up the sheet (also increase of the element). During the first movement in D the V curve appears nearly level, but indicates in fact a very small decrease in force. The to and fro movement in D occupied about an hour. The H curve describes a movement similar in character to that in D but opposite in direction. Both the D and H curves went beyond the limit of registration, but apparently not very far.

FIG. 7.—Antarctic, July 26, 1903.

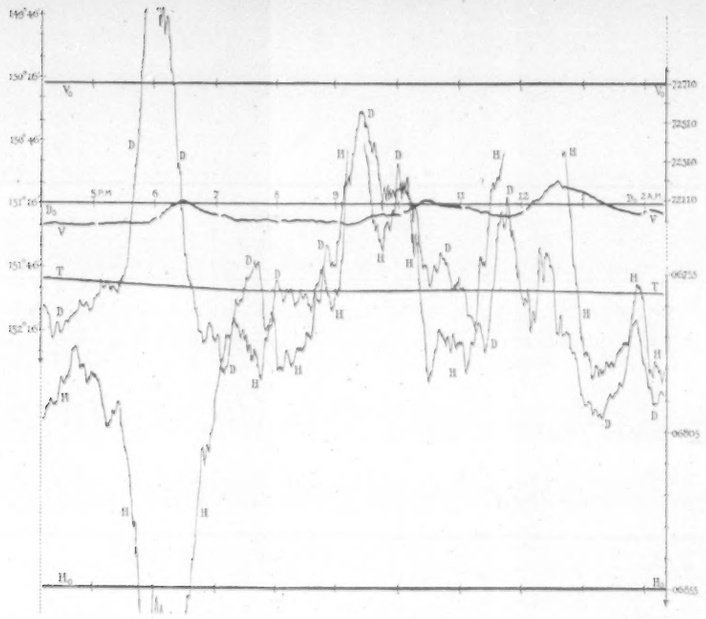


FIG. 9.—Antarctic, June 28, 1903.

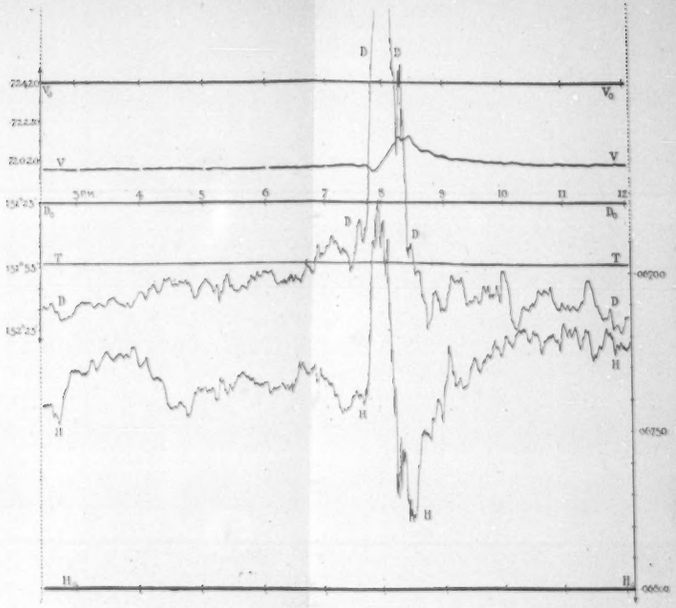


FIG. 8.—Antarctic, June 29, 1903.

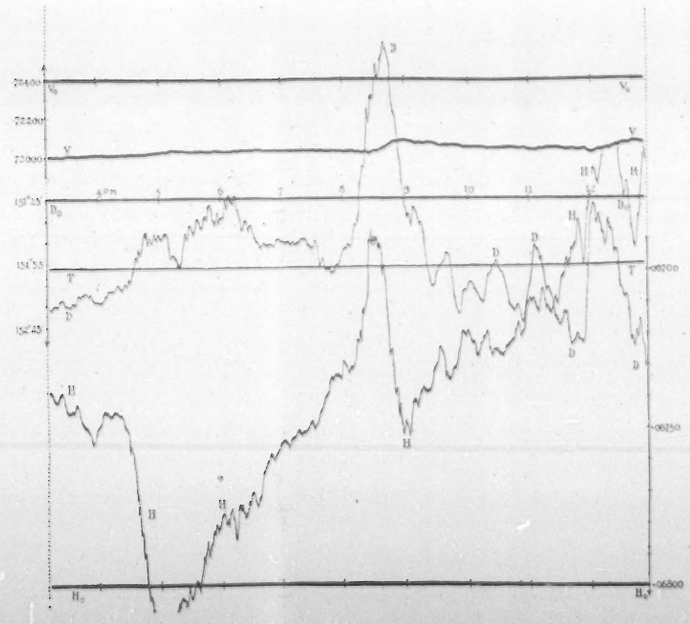
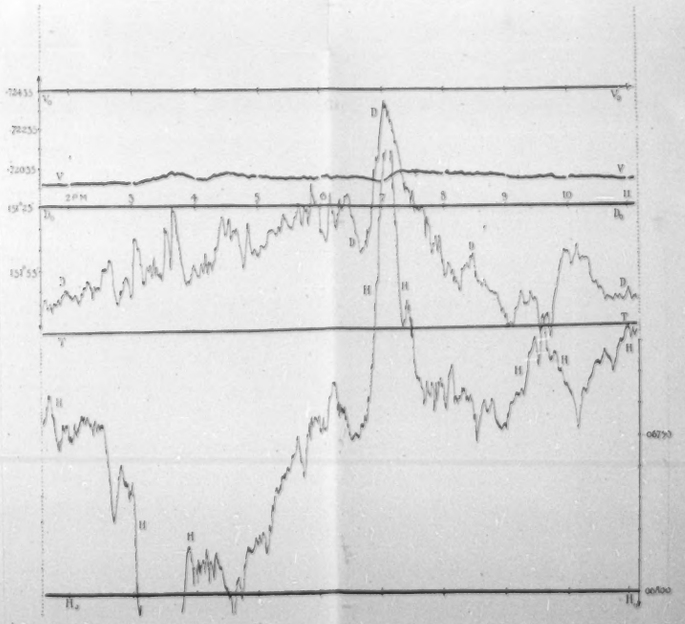


FIG. 10.—Antarctic, June 19, 1903.



The next curve, fig. 8, shows what was happening at Winter Quarters on June 29th, 1903, *i. e.* 27 days earlier. It will be observed that the D and V movements between 8 and 9 P.M. L.T. in fig. 8, and those culminating about 6 P.M. in fig. 7, present a remarkable resemblance, the chief difference being that the amplitude of movement was greater on the later occasion. The H movement, however, on June 29th was practically in phase with that in, instead of being in opposite phase as on July 26th. The exact interval between the two disturbances was very approximately 26 days, 21½ hours.

The next curve, fig. 9, shows the events of June 28th, 1903, *i. e.* the day immediately preceding that to which fig. 8 applies. The D and V movements culminating about 8 P.M. L.T. in fig. 9 present again a remarkable resemblance to those in fig. 7. We have thus a choice between a 28- and a 27-day period. Nor is our choice even thus limited, as the next curve, fig. 10, shows. It represents a disturbance culminating about 7 P.M. L.T. on June 19, or nine days earlier than the last.

§ 20. We can hardly suppose that the resemblance between these four curves means nothing more than accident. What first caught my attention was the resemblance between the curves of June 28 and 29 (figs. 9 and 8). It recalled the fact that more than one previous observer has remarked on the occasional resemblance between small declination disturbances on successive days. Resemblances of a kind are practically certain to occur in a long series of records, and I am not aware that in any previous case satisfactory evidence has been advanced that anything more than chance is *necessarily* involved. I have seen a good many instances of apparent repetitions at intervals of nearly 24 hours in the Kew curves, but none I think so striking as that afforded by figs. 8 and 9.

Perhaps the earliest occasion when the phenomenon was noticed was in a discussion of Lisbon curves by Senor Capello. This is referred to by Dr. Balfour Stewart in the '*Encyclopædia Britannica*.'* Balfour Stewart had himself noticed analogous cases at Kew, and whilst he apparently reserved his opinion as to the real significance of the phenomenon, he suggested a possible explanation, if a true physical cause should prove to be involved. The suggested explanation—which has a surprisingly modern flavour—was to the effect

* 9th Edition, 'Terrestrial Magnetism' (under '*Meteorology*'), Art. 87.

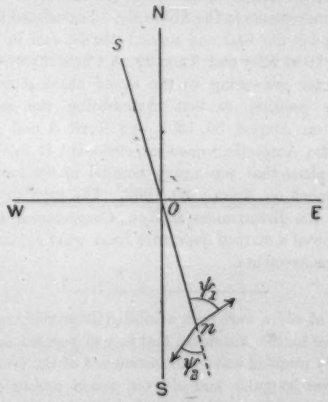
that an area near some meridian on the Earth's surface or in the superincumbent atmosphere might in some way become for several days at a time especially sensitive to solar action, so that magnetic disturbance would arise on successive presentations of that meridian to the sun.

The fact that the disturbances on June 19, 28, 29 and July 26 occurred not far from the same hour in the evening suggested to me, I must admit, for a moment an explanation of a much more prosaic character than that of Dr. Balfour Stewart; viz., that all the disturbances might have been artificial and due to misplaced activities on the part of the crew of the 'Discovery.' An instant's reflection convinced me that this could not be the true explanation, but it may occur to others. I have thus mentioned it so as to give it its *quietus*. When artificial disturbances are set up through the proximity of individuals with iron in their possession, rapid oscillations are set up in the magnets, which appear as blurred portions of trace in the records. Now the movements in figs. 7 to 10, though rapid, are in no ways blurred, but are quite sharp. Apart from these and other general considerations, I have positive proof of the genuineness of the disturbances on June 19 and July 26. These were days for which I had copies of the Christchurch curves, and on both occasions there were *prominent* disturbances there synchronising with those at Winter Quarters. Nay, further, it was possible to detect synchronous movements even at the other co-operating stations, including Kew itself.

§ 21. Before indulging in speculations as to the real nature of the phenomenon, I examined all the 600 Antarctic curves, and found no less than 82 clear examples of a disturbance showing a general resemblance to the disturbances of June 19, 28, 29 and July 26, 1903. In addition, there were a good many cases in which a lesser degree of resemblance presented itself, and others in which but for the absence of a vertical force trace one would probably have decided that a close resemblance existed. In more than half of the 82 occurrences the disturbance culminated between 7 and 9 P.M. L.T., and all except 18 took place between May and August (*i. e.* during Antarctic midwinter). The typical disturbance may be regarded as possessing two phases. During the first the three magnetic elements diminish, during the second they increase. In some individual cases, as in fig. 7, the H curve behaves oppositely to the other two, but the exceptions are comparatively few. On the average, the to and fro movements

are nearly equal in D and in H, but the increase in V during the second phase is on the average about 3½ times the decrease during the first phase. The average duration of the second phase slightly exceeds 20 minutes, and is about 3 minutes in excess of that of the first phase. Taking into account all the examples of the phenomenon in which there was no loss of trace, and grouping them under the months in which they occurred, I calculated for each group the mean amplitudes of the movements during the two phases. The mean derived from each group was regarded as answering to a single disturbance, and a calculation was made of the amplitude and direction of the disturbance vector to which each phase of the movement might be supposed due.

In the following table ψ denotes the inclination of the horizontal component of the vector to the magnetic meridian at Winter Quarters (about 152½° E), and χ the inclination of the vector itself to the horizontal plane. The suffixes 1 and 2 relate to the first and second phases respectively, the ψ angles being measured



as shown in the figure, in which *n* represents the north pole of the compass-needle *ns*, N geographical north. A + sign attached to χ means that the disturbing force acting on the north pole is directed upwards, the − sign denoting the contrary.

	ψ_1	χ_1	ψ_2	χ_2
April & May	71° 22′	−7° 13′	57° 22′	+21° 45′
June	81 25	−6 33	63 40	+26 6
July	77 22	−5 0	73 22	+19 46
August	82 32	−5 6	78 41	+18 21
September	73 29	−4 59	68 16	+19 35
All	75 53	−5 35	70 48	+20 47

The similarity of the results from the different groups points to a fairly uniform set of conditions and so presumably to a community of origin.

Examining the Kew and Christchurch curves for June 19, 28 and 29 and July 26, 1903, and the Colaba curves for the last-mentioned date, I found disturbances answering in time to the conspicuous movements in the Antarctic. I calculated the vectorial disturbances for the first and second phases, and in all cases—except June 19 at Kew and June 29 at Christchurch—it proved that the vector answering to the *second* phase showed a close approach in position to that representing the mean initial disturbance on August 20, 1902, and April 5 and August 25, 1903. In the Antarctic, somewhat curiously, it was the vector for the *first* phase that was nearly parallel to the vector for the initial movement on August 20, 1902. The relative amplitudes, however, of the disturbances at Kew, Christchurch and Winter Quarters showed a marked departure from what obtained for the sudden commencements.

A variety of slides were then exhibited illustrating various types of disturbance in the Antarctic, and it was pointed out how the comparatively rounded wave-like movements of the evening passed into the more irregular and shorter period oscillations of the morning. These and a variety of other disturbance curves will be included in the volume now under preparation, which deals with the magnetic records obtained at the Winter Quarters of the 'Discovery' between March, 1902 and January, 1904.